level **F**

Teacher's Guide

MATH WORKSHOP

REVISED EDITION

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INTRODUCTION

I. OUR POINT OF VIEW

Loose talk about reform in the teaching of elementary mathematics has spread confusion and shattered the confidence of many teachers. Uneasiness and fear have developed. This mood has damaging effects in the classroom.

Teachers have been told they are teaching horse-and-buggy mathematics in a space age. New and esoteric language has begun to flood the literature and to be introduced into discussions. Some innovators have pointed the finger of scorn at traditional subject matter, pronouncing it "bad mathematics."

Relax! The mathematics our pupils have learned in traditional classrooms is good, sound, useful mathematics. It is good enough to play a central role in any moon shot. Traditional content in elementary school arithmetic is sound. It is at the core of man's advance, including his past and future bursts into space.

Our real concern is with finding solutions to two interrelated problems. How can we help pupils (a) to learn *more* mathematics, and (b) to develop a *deep appreciation* for this increasingly important part of our culture? All proposals and discussions should be evaluated in terms of their contributions to the solution of these fundamental problems.

The traditional content taught to pupils in elementary schools is completely valid. Our elementary schools must teach pupils how to carry out the fundamental operations with whole numbers and fractions and how to use these skills in solving practical problems. But the truth of the matter is that pupils simply have not mastered the traditional content. Many have too weak a grasp on facts such as that $7+2=9$ and that $8\times 9=72$. (Most of these pupils will drop mathematics as soon as they can, losing contact with a tool that is essential in coping with the problems of the times.) The lack of mastery cannot be remedied just by giving more drill. Pupils will be able to master more of the traditional content only if we give them freedom to *invent* ways of finding answers, if we take steps to make sure that they *understand* what they are doing, and if they have ample opportunity to *use* what they learn.

The job of teaching more mathematics will become easier to the extent that pupils develop an appreciation for the subject. This appreciation can develop only when mathematics is learned with understanding and through active participation—it will not take place when mathematics is learned by rote. We believe that even inspired teaching of isolated fragments of mathematics will fail to produce pupils who understand and can use what they have learned.

Mathematics is a world of endless fascination. It is a world brimming with surprises, patterns, and enticing vistas. To see a pattern emerge from a conglomeration of facts, to organize seemingly disconnected ideas—these

are some of the joys of learning mathematics. It is this sense of excitement in mathematics that MATH WORKSHOP wants pupils to experience in elementary school.

To accomplish this, MATH WORKSHOP puts its *major subject matter emphasis* on *basic principles* and *unifying ideas* which tie together the facts and manipulative procedures of arithmetic. This emphasis simplifies the subject, makes it understandable, insures recall, and increases interest. Our *major pedagogical emphasis* is on teaching for discovery. We present content to pupils so that they are challenged to think about relationships, to look for patterns and clues, to draw logical conclusions for themselves, and to invent their own ways of finding answers.

II. THE SUBJECT MATTER OF MATHEMATICS

A. The Spiral Curriculum Concept in MATH WORKSHOP

The *24th Yearbook* of the *National Council of Teachers of Mathematics* talks about "the basic mathematical understandings which should be continuously developed and extended throughout the entire mathematics curriculum, grades K-12 and beyond." The concept of the spiral curriculum has been used in designing MATH WORKSHOP. At each level of the MATH WORKSHOP the main ideas of mathematics are introduced. Each successive year's activities deal with these unifying ideas with increasing sophistication.

However, pupils are not expected to master such ideas as, say, multiplication of fractions or addition of positive and negative numbers the first time they meet them. They encounter such ideas again and again, building always more depth of understanding.

The spiral curriculum rejects the "building block" approach to arithmetic that has been used in the past. Rather, the spiral curriculum calls for the early introduction of fundamental and unifying ideas with repeated reconsideration at later times. It calls for continual nurturing of important concepts year after year as understanding gradually develops.

In the early grades the hope for absolute mastery must be limited to the basic facts of arithmetic. Here, complete mastery can and must be achieved. But where big ideas are involved, mastery does not come quickly. It develops slowly and is based completely on understanding.

B. The Unifying Ideas or Strands in MATH WORKSHOP

MATH WORKSHOP emphasizes nine unifying ideas in mathematics:

1. *Structure*
2. *Sets*
3. *Number and counting*
4. *Numeration*

5. *Addition and subtraction*
6. *Multiplication and division*
7. *Functions and relations*
8. *Geometry*
9. *Measurement*

The coverage of these ideas on the pupil pages, in the Teacher's Guide, and in the practice sheets is summarized in the chart on pages 50-51. Each Teacher's Guide page lists the unifying ideas considered on the corresponding pupil page and practice sheet and contains many suggestions for extending these ideas beyond what is called for on the pupil page. Let us consider each of these unifying ideas.

1. Structure

Suppose that you came upon some sentences from a Martian arithmetic book:

1) ❘ ☆ □ = □ ☆ ❘
2) ⊓ ☆ ⧄ = ⧄ ☆ ⊓
3) (⊠ ☆ ┌) ☆ □ = ⊠ ☆ (┌ ☆ □)
4) (□ ☆ ⧄) ☆ ❘ = □ ☆ (⧄ ☆ ❘)
5) □ ☆ ⊠ = □
6) ⧄ ☆ ⊠ = ⧄

Somehow you have found out that the symbols:

❘, □, ⊓, ⧄, ⊠, ┌

represent different cardinal numbers. You guess that the symbol ☆ is an operation sign, and you wonder which operation it stands for. Sentences 1 and 2 convince you that ☆ stands for neither subtraction nor division, for these sentences suggest that ☆ stands for an operation for which order does not affect the results. Sentences 3 and 4 also support this conviction because they suggest that grouping does not affect the results. However, there is nothing about sentences 1-4 which would help you choose between addition and multiplication as the operation denoted by ☆. Perhaps the other sentences will help.

Sentences 5 and 6 turn out to be of little help. In fact, both of them suggest that ☆ could denote addition, multiplication, subtraction, or division. Fortunately, sentences 1-4 rule out both subtraction and division. If ☆ stands for addition then ⊠ denotes the number 0; if ☆ stands for multiplication then ⊠ denotes the number 1.

It turns out that, based on what can be learned from sentences 1-6, the operation sign ☆ could stand for addition or it could stand for multiplication.

We can't decide from sentences 1-6 whether it is addition or multiplication that is denoted by ☆ because these operations have certain properties in common and these properties are reflected in sentences 1-6. The *additive*

structure of the number system is similar to its *multiplicative structure.* Both operations are commutative (sentences 1 and 2) and both are associative (sentences 3 and 4). Also, each operation has an identity element (sentences 5 and 6).

Whenever you provide exercises for pupils that bring any of these properties to the surface, you are paying attention to the structural properties of the number system. For example, exercises such as the ones which ask pupils to fill the frames to make the following into true sentences:

$$8\times 7=\bigcirc\times 8$$
$$7+(3+5)=(7+\triangle)+5$$

are exercises which help pupils develop an awareness of commutativity and associativity. Another such exercise is one which asks pupils to complete the addition *Cross Number Puzzle:*

4	3	
5	1	

Notice that the cell in the lower right-hand corner gets filled with the same number whether the pupil adds the numbers in the rows and then adds the sums:

4	3	7
5	1	6
		13

or adds the numbers in the columns and then adds the sums:

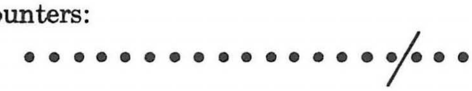
4	3	
5	1	
9	4	13

This invariance of the total highlights the rearrangement properties of addition.

Of course, the advice that one should give pupils exercises which bring structural properties to the surface does not necessarily mean that pupils need to be told the names for these properties. A pupil who has discovered these properties by working with displays of physical objects has a sound understanding of the properties. He thinks of them almost as laws of nature. A young pupil who tells you that $7+3$ is the same number as $3+7$ *because* "addition is commutative" has surely been coerced into using these words. He knows that $7+3=3+7$ because any display of objects which represents $7+3$ can be rearranged (without changing the total number of objects) into a display which represents $3+7$.

A pupil can be aware of and can use the structural properties without knowing their names. Exercises can be found which encourage pupils to make use of the properties. For example, a number line exercise which asks pupils to find the landing point when the starting point is 3 and the jump is 16 to the right could be solved by starting at 16 and jumping 3 to the right. The fact that this alternative procedure gives the correct answer strengthens a pupil's awareness of the commutative property of addition and helps to convince him of its usefulness.

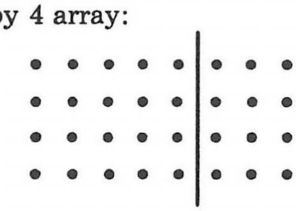
If a pupil finds the sum of 16 and 3 by finding the sum of 6 and 3 and then finding the sum of 10 and 9, he has made use of the associative property of addition in a helpful way. (Of course, he has also made use of the fact that the numbers are expressed in decimal notation.) He does not become convinced of the correctness of his technique by being told that it is based on the fact that addition is an associative operation. Rather he becomes convinced by observing that this display of counters:



can be rearranged into this display:

In addition to the structural properties of commutativity, associativity, and the existence of an identity element for addition, the operation of adding a number has the property that it can be "undone" by subtracting the same number. Similarly, the operation of multiplying by a (nonzero) number can be "undone" by dividing by that number.

Pupils also learn that certain multiplication facts can be discovered by using other multiplication facts and addition. For example, pupils can find the product of 8 by 4 by adding the products of 5 by 4 and 3 by 4. That this procedure is valid is based on the "physical" fact that an 8 by 4 array of counters can be partitioned into a 5 by 4 array and a 3 by 4 array:



This experience helps pupils become aware of the fact that multiplication distributes over addition—a property of multiplication and addition. The term "distributive property" is not used on the pupil pages and need not be used in class. The important thing is that pupils use this property in deriving certain products from others.

Levels A through E provide many opportunities for pupils to make use of the structural properties of the number system. In Level F we attack the problem of structure in a head-on manner. In a fanciful interchange between a resident of Earth and a visiting Martian, the Earth resident describes arithmetic to the visitor. Naturally, the most economical description is one which highlights the structural properties.

2. Sets

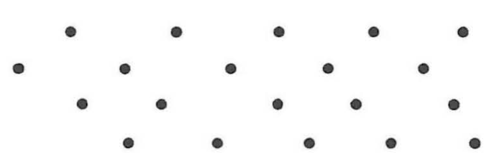
The principal contribution of sets to elementary arithmetic is that they can be used to clarify the notion of cardinal number. For a long time people thought that the word "two" in the expression "two apples" played the same kind of role as the word "green" in the expression "green apples." This was the cause of much confusion about the nature of cardinal numbers. An individual apple can have the property of "green-ness" but it can't have the property of "two-ness." Only sets can have the property of number. A set consisting of two green apples is a set whose cardinal number is two. Also, any set whose members can be matched with those of a set of two apples is a set whose cardinal number is two. From here it is an easy step to say that the cardinal number two is the class of all such sets.

Now, a set is an abstract entity just as a number is abstract. You can see two apples but you cannot see a set of two apples any more than you can see the number two or justice or liberty. For example, it is complete nonsense to ask the *set* of boys in the classroom to stand or to ask pupils to look at the *set* of counters on the table. There will come a time in the pupil's mathematical education when he will need to use the word "set" but we do not think it is a good idea for him to develop in elementary school the notion that a set is a tangible object. Therefore, we have rarely used the word *set* in the pupil book, and have used it sparingly in the Teacher's Guide. Instead, we have used nontechnical terms such as *display*, *group*, and *array*.

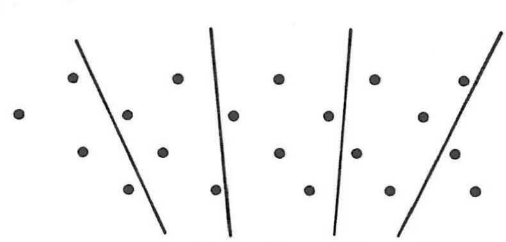
Teachers who wish to read a discussion of set language and notation as used in mathematics can refer to the Appendix of this book as a supplementary unit of study.

3. Number and Counting

In the primary grades the basic mathematical idea is that of counting. The operations of addition and multiplication are presented as shortcuts to counting. Given a display of objects:

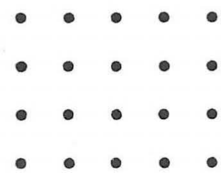


and asked to tell how many objects are in the display, a pupil might proceed to count them one by one. This is the most primitive approach. A more sophisticated approach is to partition the display, obtain the count for each of the smaller displays, and add the counts:



4 + 4 + 5 + 5 + 2

An even more sophisticated approach is to rearrange the objects in the display into a rectangular array:



If one is familiar with the count for a 4 by 5 array, he knows immediately that the original display contained 20 objects.

The study of fractions and negative numbers is considered an extension of the study of number and counting in this organization of unifying ideas.

A word about our treatment of fractions in Level E is in order. We present fractions as pairs of multiplication and division operators. Thus, to find $2/3$ of 6 feet, one can either first multiply 6 feet by 2 and then divide the result by 3 or first divide 6 feet by 3 and then multiply the result by 2. This approach to fractions as pairs of operators is not only mathematically sound but is very much in consonance with the way fractions are used in everyday life.

4. Numeration

The idea of numeration grows out of the need to report a count, and the mode of reporting is culturally determined. For example, if the objects in the display shown above represent glasses of milk, we might report "1 gallon and 1 quart" in answer to the question: How many? If the objects represent eggs, we might report "1 dozen and 8 more" or "4 less than 2 dozen."

Each of these reports is a description of the result of some partitioning procedure. When decimal partitioning is used, the English language provides us with words for giving brief reports. For example, the report "3 ten-groups and 7 one-groups" or "3 tens and 7 ones" can be abbreviated to "thirty-seven" and can be recorded in positional notation by the symbol "37."

Much activity in the study of denominate (measurement) numbers is provided at all levels of MATH WORKSHOP to help pupils understand that the results of a count may be recorded in other bases than base ten. They come to realize that systems of numeration are arbitrary, but that each has a pattern which can be used to keep matters straight.

5. Addition and Subtraction

The operation of addition is developed as a shortcut to counting. Given enough opportunities to find the sum of 5 and 4 by counting, for example, a pupil soon commits to memory the fact that $5+4$ is 9. He makes use of

this fact as well as of positional notation and associativity when finding the sum of 25 and 4:

$$25+4=(20+5)+4=20+(5+4)=20+9=29$$

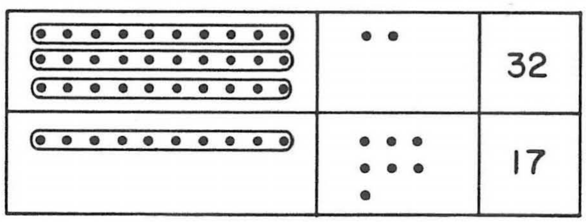
The addition facts from "$0+0=0$" through "$9+9=18$" should be memorized during the primary grades.

We use the word *subtraction* to denote the process used to undo the result of adding. So, for example, to compute $8-3$ amounts to finding the number whose sum with 3 is 8. From this point of view, it is natural to learn subtraction facts at the same time one is learning addition facts. Thus, we have *fact teams*, groups of four sentences, such as:

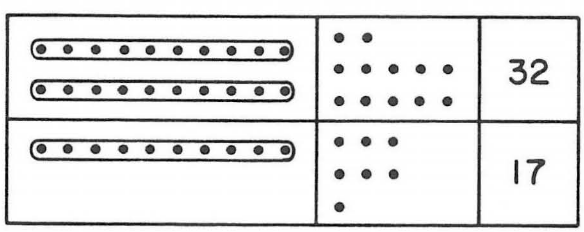
$$\begin{array}{ll} 5+3=8 & \quad 8-3=5 \\ 3+5=8 & \quad 8-5=3 \end{array}$$

which are derived from one addition fact with the help of commutativity and the fact that subtracting a number is the *inverse* of adding that number.

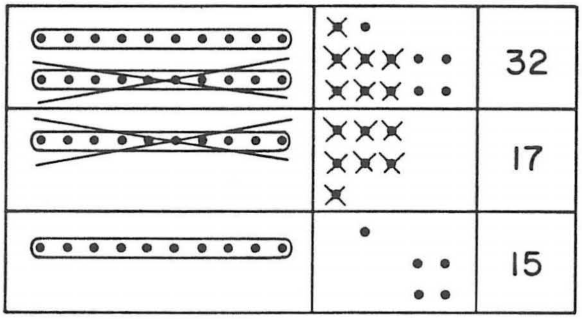
Addition and subtraction algorithms are developed as the end products of activities in which pupils manipulate physical objects and keep records of the manipulations. An algorithm becomes a record-keeping device for imagined manipulations. Thus, from the viewpoint of the pupil, an algorithm works because of laws which govern the manipulation of physical objects. For example, consider an algorithm used in subtracting 17 from 32. In terms of physical objects, one can find the difference by counting what remains when 17 objects are removed from a display of 32 objects. We start with decimally partitioned displays of 32 objects and 17 objects:



In order to carry out the take-away step, we need to break into one of the ten-groups. So, we rearrange the display for 32 by exchanging 1 ten-group for 10 one-groups:



and then take away 1 ten-group and 7 one-groups:

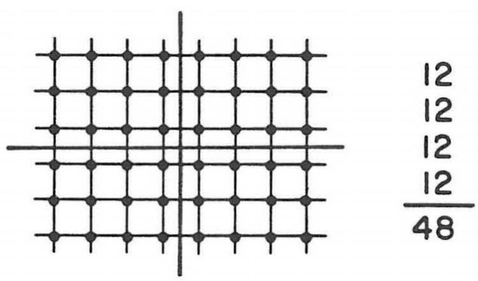


These three stages are recorded in the steps of the algorithm:

$$\begin{array}{r} 32 \\ \underline{-17} \end{array} \qquad \begin{array}{r} \overset{2}{\cancel{3}}{}^{1}2 \\ \underline{-17} \end{array} \qquad \begin{array}{r} \overset{2}{\cancel{3}}{}^{1}2 \\ \underline{-17} \\ 15 \end{array}$$

6. Multiplication and Division

Like addition, multiplication is developed as a shortcut to counting. In this case the counting involves objects arranged in rectangular arrays or otherwise grouped to encourage counting by groups. For example, counting the spaces in an egg carton and making 5 number line jumps of 2 to the right, starting at 0, are multiplicative situations. By continual practice in the primary years a pupil should commit to memory the multiplication facts from "$0 \times 0 = 0$" to "$9 \times 9 = 81$," and each of these facts should be discovered by the pupil. Some of the facts should be discovered by direct counting, but the discovery of others should involve partitioning of arrays and adding the counts of the parts. For example, a pupil might learn that 3×4 is 12 by counting the spaces in a certain type of egg carton. He can use this fact to learn that 6×8 is 48 by drawing 6 horizontal lines and 8 vertical lines and then partitioning the resulting display of crossing points:



Division is the process used to undo multiplication. So, for example, to compute $24 \div 8$ amounts to finding the number whose product with 8 is 24. From this point of view, it is natural to learn division facts at the same time one is learning multiplication facts. Thus, as in the case of addition and subtraction, we have facts teams for multiplication and division such as:

$$\begin{array}{ll} 8 \times 3 = 24 & \quad 24 \div 3 = 8 \\ 3 \times 8 = 24 & \quad 24 \div 8 = 3 \end{array}$$

which are derived from one multiplication fact with the help of commutativity and the fact that dividing by a (nonzero) number is the *inverse* of multiplying by that number.

7. Functions and Relations

An example of a function is the operation addition-of-2. This function assigns a unique number to each number. For example, it assigns 5 to 3, 19 to 17, and 202 to 200. A common way to exhibit a function is to show a table which lists some of the pairings. For example, to exhibit the function addition-of-2, we might show the table:

3	4	9	12	18	30
5	6	11	14	20	32

The numbers listed in the top row are the numbers on which the function operates. Those listed in the bottom row are the numbers we obtain from applying the function. The use of two colors helps to distinguish "input" from "output." By reversing input and output, this same table shows another function—subtraction-of-2—which is the inverse of addition-of-2.

All functions are examples of mathematical ideas known as relations. Numerical relations which are not functions are those which assign more than one number to some given number. For example, the greater-than relation assigns 4, 5, 6, etc., to the number 3.

8. Geometry

Under this general heading, one finds in MATH WORKSHOP a variety of topics—classification of common figures, comparisons of area and perimeter, partitioning of regions, angles, triangles, circles, parallel and perpendicular lines, congruence and similarity of triangles, coordinate systems, and straightedge and compass constructions.

9. Measurement

Under this heading is found work with denominate numbers and the conversion of units. Further activity is provided in measurement of geometric shapes and formulas for areas and volumes of plane and solid figures.

C. Another View of Mathematics: Counting and Shortcuts to Counting

You probably have noticed that a basic and simple notion recurs as a central theme in the previous discussion of the nine unifying ideas. It is that, to a large extent, elementary school mathematics is essentially *counting and shortcuts to counting.*

The authors of MATH WORKSHOP hope that all pupils will see arithmetic as a very simple subject. Pupils will develop basic confidence in their ability to move ahead if they are helped to realize that they have but a few easy tasks:

1. Learn how to COUNT
2. Learn how to RECORD RESULTS of counting
3. SOLVE PROBLEMS involving counting
4. Find STRATEGIES to avoid one-by-one counting

That's all there is to it!

Every pupil in elementary school ought to adopt the slogan: When in doubt, COUNT. He ought not look to the teacher, or the text, or his friends for assurance about his statements in arithmetic. Rather, he ought to look at the world of things . . . What happens there?

He ought to learn that there are many reliable ways to get to the solution to any exercise or problem—that he ought to use the way that seems most reasonable. You can help him understand this important freedom to find his own way and to use it. We try to give the pupil the equipment he needs to solve any problem so that, given enough time, he can, if all else fails, count his way to the answer.

Thus, our central purpose has been to give every pupil, from the fastest to the slowest, the confidence that he can always get right answers in computations. He must learn the strategies that are available to make the job easier, but he must understand them. It is the central purpose of MATH WORKSHOP to help pupils develop the confidence and ability to perform the four tasks enumerated above.

Lack of confidence is usually the real stumbling block to those who are called "slow learners." They are convinced they can't remember or they are afraid they will forget, so they try to avoid doing arithmetic. It never occurs to many of them that they can *always be right* if they count, and that they need use only those shortcuts or strategies they understand.

Slow learners will be given a new lease on their mathematical lives as we help them realize that there is less to memorize in this subject than any other. It all begins with counting and with thinking about what there is to be counted. Beyond that, if they learn the 390 basic facts of addition, subtraction, multiplication, and division, they can develop really effective strategies to avoid the hard work of one-by-one counting.

Fast learners need to learn about the other side of the coin. Since there is little that needs memorization and lots to think about, it is impossible to memorize your way through the subject. At some point along the way, the mind will rebel at all the memorization required—and the person, regardless of his ability to memorize, will become a "mathematical drop-out." The fast learner needs to realize that, since the subject is basically so simple, he can make important side-trips on his own, unafraid of pitfalls, confident he won't get lost.

III. HOW TO TEACH MATHEMATICS

A. Adopt a Sound Learning Theory

Every conscientious teacher has a theory of how pupils learn arithmetic. It is the sum of all the reasons behind his behavior in the classroom. If he asks the pupils to open the book to page 42 and do the even-numbered examples, he feels that activity will promote learning. If he tells Billy to write "$4 \times 7 = 28$" twenty-five times, he is convinced that the repetition will help Billy remember that fact the next time he is asked for the product of 4 and 7. If, instead, he asks Billy to count the number of days in 4 weeks, he feels that relating multiplication facts to recognizable events in the world he lives in will help Billy remember that $4 \times 7 = 28$.

If a teacher could get outside himself and analyze his teaching of just one mathematics class, he could then write down the theory of learning that guides him.

The authors of MATH WORKSHOP have a viewpoint on learning in mathematics. They recognize three different levels of learning experiences:

1. Rote Learning
2. Problem Solving
3. Independent Investigation

They believe that most failures and difficulties in arithmetic are the result of a conscientious, but misguided, emphasis by teachers and textbooks on rote memorization. The authors believe that the central task before us is to present arithmetic as a problem-solving activity and with unlimited opportunities for independent investigation.

1. Rote Learning

We share the ability to learn by rote memory with all animals. Pavlov's dogs learned by endless repetitions. Horses have been trained to point a hoof at 5 when the trainer calls out "two plus three." A psychologist, half in fun, pointed out that a stick also learns in this way—put a notch in a stick today and it will be there tomorrow.

Pages of random repetitions hope to educate students on the level of rote memory. If one page doesn't do the job, then let's try two. If pupils persist in forgetting, drill some more!

Actually, when pupils rebel openly or in any of the less obvious ways they are adept at inventing, that rebellion is usually a sign of an active mind. The human mind, after all, has a sense of dignity that is offended by long stretches of drill bent on rote memory as the objective. We should not be surprised when pupils turn away from assignments that recognize only the lowest level on which the human mind can operate.

Efforts to teach by rote memory have always been discouraging. Pupils don't respond "properly," so they are given larger doses of the same. In

desperation, the driller sooner or later reaches the conclusion that the pupil can't learn. The situation is further darkened by the mistaken theory that slow learners can only learn on the level of rote memory.

2. Problem Solving

The human mind is unique in its ability to solve problems—to face a question and announce: I don't know, but I can find out!

First graders are protected from questions such as "$19+3=$____" because the rules one needs to solve this problem by rote memory are not yet available. However, if a first grader can count as far as 22 and if he understands what is meant by addition, while he doesn't know the answer, he can find it. A consequence of teaching arithmetic at the level of rote memory in first grade was revealed when one of the authors asked a large group of beginning second graders to answer this question, asked in both forms they might have encountered:

$$\begin{array}{r} 19 \\ +\ 3 \\ \hline \end{array} \quad \text{and} \quad 19+3=\underline{\qquad}$$

None were familiar with the horizontal form. All wrote the same result . . . without exception:

$$\begin{array}{r} 19 \\ +\ \ 3 \\ \hline 112 \end{array}$$

They read this statement as: Nineteen plus three equals one hundred twelve. They were asked: If you had nineteen cookies and I gave you three more, would you have one hundred twelve cookies? Most answered: No! I would have twenty-two cookies.

These second graders had been deprived of the opportunity to solve problems and were already victims of their most unreliable faculty—the ability to learn by rote.

First graders can understand and solve problems designed to give them opportunities to use the simple facts of addition. Here is a question that almost every adult has to think about:

> How many ways do you know to complete the following as a true statement?

$$\underline{\qquad}+\underline{\qquad}=3$$

(We assume the pupil's acquaintance with numbers is limited to zero and the counting numbers.)

The pupil does not know the answer to the question . . . but he can find out—unless he has been trained to limit his response to "I don't know" when

he doesn't remember. A pupil who has often been encouraged to find out when he can't remember will make a list. He may write:

$$2+1=3$$
$$0+3=3$$
$$1+2=3$$
$$3+0=3$$

Is that all there are?

In response to this question, one first grade pupil simply reorganized the list without saying a word and took her seat.

$$3+0=3$$
$$2+1=3$$
$$1+2=3$$
$$0+3=3$$

This pattern was certainly an eloquent argument that there are no more!

Here is another first grade question that almost no adult can answer from memory. Yet it is a question that every adult and more first graders can work out if they make the effort:

> How many ways are there to complete the following as a true statement?
>
> $$\underline{\qquad}+\underline{\qquad}+\underline{\qquad}=3$$

(Again assuming the problem solver is limited to zero and the counting numbers.)

Notice that in "finding out" one uses or practices basic combinations in addition. Other similar problems could require use of or practice in all basic combinations. All the random exercises in memory that constitute "drill" have now been encountered incidentally along the way toward the solution of problems.

> How many numbers can be expressed by one of the following or as the sum of two or more of them—none to be used more than once in any sum?
>
> 4, 2, 16, 1, 8

Only those who are already familiar with this list have any idea of the surprises that lie ahead. We begin a list, expressing each number as one from this list or as the sum of some combination of numbers from the list:

$1=1$	$6=4+2$	$11=$ ______
$2=2$	$7=4+2+1$	$12=$ ______
$3=2+1$	$8=8$	$13=$ ______
$4=4$	$9=8+1$	$14=$ ______
$5=4+1$	$10=$ ______	$15=$ ______

As we extend the list, we find no gaps until we come to 32. But that is one more than the sum of the entire list.

Now, we go back to see how many can be expressed in more than one way (not counting changes in order). We write:

$7 = 4 + 2 + 1$ ("$7 = 2 + 4 + 1$" is considered the same combination)

Is there another combination from our list we might have used? One way to indicate 22 is "$16 + 4 + 2$." Is there another combination whose sum is 22?

We try many combinations—using or practicing addition—and eventually develop a suspicion that each can be expressed in only one way as one of the numbers listed or as a sum of the numbers listed.

The authors of MATH WORKSHOP know that there is no end to such interesting problems in mathematics—problems that can be solved by pupils who are encouraged to say: I don't know, but I can find out.

We have assumed the responsibility of building a Pupil Book and designing practice sheets that will help teachers and pupils approach the subject of mathematics as being basically concerned with solving problems. When the pupil recognizes his ability to solve problems, he responds with enthusiasm, and his power to recall basic facts and processes he has encountered flourishes.

3. *Independent Investigation*

Once the pupil has developed a confidence that he can find things out for himself, he may be so delighted with his accomplishments that he will look around for problems to solve, and may invent problems of his own. He will not be satisfied with the list 4, 2, 16, 1, 8, which he used to write all the numbers 1 through 31. He will want to expand the list so he can go beyond 31. What number or numbers should he add to the list? Or, he might like to try an entirely different list such as 1, 4, 7, 9, 18, which behaves similarly but with important differences.

When pupils begin posing problems for themselves and solving them, they will have a taste of intellectual excitement that will lead them on and on. The teacher who has helped a pupil to use his mind on this level deserves the thrill that comes to him.

The authors of MATH WORKSHOP are dedicated to helping elementary school teachers rescue the subject of mathematics from emphasis on rote memorization and present it as basically a problem solving activity. Further, we have tried to present many opportunities that will encourage some pupils to pose questions to themselves and to solve them.

B. Pose Challenging Questions

1. *"What Can You Say?"*

The Report of the Cambridge Conference on School Mathematics said:

> "We hope that many problems can be found (we know a few) that read, 'Here is a situation—think about it—what can you say?' "

The authors of Math Workshop share that hope and have built the pupil pages, the accompanying practice sheets, and notes for teachers around such problems. A mathematical problem is a situation worth thinking and talking about, a situation that suggests exploration and investigation. Consider this situation:

What Can You Say? Situation 1

Make a list of the square numbers up to 100 . . .
$0\times0,\ 1\times1,\ 2\times2,\ 3\times3,\ \ldots,\ 10\times10$
0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100

Making the list is simply computation . . . it is not a problem. But this list provides situations we can think and talk about. For the numbers in this list:

Is there a pattern of even and odd numbers? (Even and odd numbers alternate.)

By which square number (other than 1) can you divide evenly each even square number? (4)

What happens if we divide the odd square numbers by 4? (All odd square numbers give a remainder of 1 when divided by 4.)

Can we find a pair of numbers in the list whose sum is also in the list? (Yes. Two such pairs are: $9+16=25$ and $36+64=100$.)

Are there any more? Could we find any more if the list were extended to include 121, 144, 169, etc.?

Can we find a number in the list that is twice as large as another in the list? Three times as large as another? (After a lot of work, providing meaningful practice, we find that the answer to both questions is "no!")

Is there a number in the list that is four times as large as another in the list? (Surprise!)

What can be observed about the differences between successive terms?

0 1 4 9 16 ----

1 3 5 7 ----

Will this pattern continue?

Each of the questions asked about this situation is simple and direct. If this is a pupil's first encounter with this situation, the answers have not been memorized. Yet each answer is well within the reach of any pupil who could have

created the list in the first place. He need only be ready to find out for himself by putting to work what he already knows.

How does one invent situations worth this kind of consideration? Unexpected sources for such situations are old arithmetic text books and current standardized tests . . . with a *little* modification. Both sources are, for different reasons, limited in their concern to simple, direct answers, so their questions do not lead to *situations worth considering*. They both suffer from a need or compulsion to give too much information. Here is an example: "Adult tickets are 85¢ each and children's are 45¢ each. *How much will 2 children's and 3 adult's tickets cost?*"

As is, there is nothing to think about . . . it only requires a bit of computation. The answer is \$3.45 . . . and we go on to the next example.

This might be called the "customer" approach to such situations—one purchase, and that's the end of it. The clerk, or box office ticket seller, will have a different view of the matter. Here is his situation:

What Can You Say? Situation 2

Adult's tickets—85¢ each Children's tickets—45¢ each

He might make a pair of charts to help him solve the many problems he will face as different people make different purchases:

No. adult's tickets	1	2	3	4	5	6
Price	85¢	\$1.70	\$2.55	\$3.40	\$4.25	\$5.10

No. children's tickets	1	2	3	4	5	6
Price	45¢	90¢	\$1.35	\$1.80	\$2.25	\$2.70

If his charts are long enough, he is now ready to compute each sale by looking up not more than two entries and adding them, perhaps in his head. If a customer wanted 3 adult and 2 children tickets, he would look at his charts

Adult's	3
	\$2.55

Children's	2
	90¢

and ask the customer for \$3.45. He might, however, make a chart to avoid even his addition. It would have to be quite a large chart. One corner of it would look like this:

Children 45¢ each

Adults 85¢ each		0	1	2	3	4	5
	0	~~0~~	45¢	90¢	\$1.35	\$1.80	\$2.25
	1	85¢	\$1.30	\$1.75	\$2.20	\$2.65	\$3.10
	2	\$1.70	\$2.15	\$2.60	\$3.05	\$3.50	\$3.95
	3	\$2.55	\$3.00	\$3.45	\$3.90	\$4.35	\$4.80
	4	\$3.40	\$3.85	\$4.30	\$4.75	\$5.20	\$5.65
	5						

One of his other charts is now in the top row of amounts. The other is in the left hand column. Other entries show the sums of the amounts at the top of the column and the left end of the row. If this chart were extended far enough, the ticket seller would have only to know the number of each kind of ticket to be purchased to find the entry giving the total amount to be charged for any purchase.

Notice that all entries in the chart above are different. So, if the amount of the purchase is known, one can find out exactly how many of each kind of ticket was purchased. If a purchase was \$2.20 and the chart was not available, it might require several trials to find that it could have been 3 children's and 1 adult's tickets and probably more trials to be sure that no other combination would fit.

We might ask if it is possible for a purchase to be exactly \$5.00. Without the chart, this would lead to a lot of computation before we found that the closest we can come to spending exactly \$5.00 is to spend \$4.95 for 11 children's tickets.

What is the smallest amount that can be spent in more than one way? (\$7.65 . . . 17 @ 45¢ or 9 @ 85¢.)

Remember that this investigation started from part of a typical "story problem." It was converted from a customer problem to an investigation of the box office salesman's point of view.

Here are several additional *What Can You Say?* situations, some taken out of mathematics itself, others taken from applications of mathematics to the real world. You can use each as a model for creating many others like it. Each suggests many similar situations that would bear independent investigation.

What Can You Say? Situation 3

Here are three paper cups:

Helen had four pieces of chalk. She put them in the cups.

Here is a situation. Think about it. What can you say? This and somewhat more pointed questions will elicit such observations as:

> If two cups are empty, the other has 4 pieces in it.
>
> There cannot be the same number of pieces of chalk in each cup.
>
> There might be a different number of pieces of chalk in each cup—3 in one, 1 in another, and 0 in the third.
>
> At least one cup has 2 or more pieces of chalk in it.

What can you say if Helen put only one piece of chalk in the cups?

> Two cups are empty.
>
> There cannot be the same number of pieces of chalk in each cup.

What can you say if she put 2 pieces in the cups?

> There cannot be the same number of pieces of chalk in each cup.
>
> They cannot all be different.
>
> Two cups must have the same number of pieces in them—either two cups are empty or two cups have exactly one piece of chalk in each.

What can you say if Helen puts 3 pieces in the cups?

> They can all contain the same number of pieces of chalk.
>
> Each can have a different number of pieces of chalk —one empty, one with a single piece, and one with two pieces.
>
> If two cups have the same number and the third has a different number, then two cups are empty.

What can you say if Helen puts 7 pieces of chalk in the cups?

> They can't all contain the same number.
>
> They can all have different numbers.
>
> In at least one cup, there are at least 3 pieces of chalk.
>
> If no cup has as many as 4 pieces in it, then one has 3 pieces and the others have 2 in each.
>
> It could be that each cup that is to the right of another has twice as many pieces as the cup to its left. (1, 2, 4)

What Can You Say? Situation 4

> Bill asked his father if he had change for a quarter. "Yes. What combination of coins do you want?"

Here is a situation. What can you say?

How many of each kind of coin would Bill's father need in order to make change in any way Bill might ask? We might make a list of all possible combinations, following some plan that will help us know when we are done:

Dimes	2	2	1	1	1	1	0	0	0	0	0	0
Nickels	1	0	3	2	1	0	5	4	3	2	1	0
Pennies	0	5	0	5	10	15	0	5	10	15	20	25
Coins	(3	7	4	8	12	16	5	9	13	17	21	25)

Bill's father would need 32 coins: 2 dimes, 5 nickels, and 25 pennies; with these 32 coins, he could make change in any of the above ways.

There are 12 different ways to make change for a quarter.

(Do you agree with the authors that many of these explorations would make excellent independent activities, homework activities, or activities for exploration by teams of pupils?)

If we make a record (like the one above) of the number of coins in each combination, we notice that:

Since no two combinations involve the same number of coins, we only have to know how many coins were used in order to tell which combination was used.

There is no way to make change for 25¢ using combinations with exactly 2, 6, 10, 11, 14, 15, 18, 19, 20, 22, 23, or 24 coins.

How does the problem change if we investigate ways to make change for a half-dollar?

1. We already know that there would be 12 combinations using just one quarter (one quarter plus each of the 12 combinations listed above) and we know that there is only 1 combination with 2 quarters . . . a total of 13 thus far.
2. Without using quarters, we might list combinations using a specific number of dimes:

5 dimes . . . . 1 combination
4 dimes . . . . 3 combinations
3 dimes . . . . 5 combinations
2 dimes . . . . 7 combinations
1 dime . . . . 9 combinations
0 dimes . . . $\underline{11}$ combinations
36

3. This gives 1+12+36, or 49, combinations.
4. All amounts 30¢ and more can be made by at least 2 combinations that have the same number of coins.

30¢ . . . 5 pennies and 1 quarter (6 coins)
6 nickels (6 coins)
40¢ . . . 1 quarter and 3 nickels (4 coins)
4 dimes (4 coins)
1 quarter and 15 pennies (16 coins)
6 nickels and 10 pennies (16 coins)

Perhaps someone would like to investigate the ways to make change for \$1.00 or \$5.00. Or someone might notice that 90¢ can be made with at least 4 different combinations of 18 coins.

What Can You Say? Situation 5

SALE:	Pencils . . .	5¢ each
	Pens . . .	9¢ each
	Notebooks . . .	25¢ each
Limit: 4 of each to a customer		

What can you say about purchases that can be made under the terms of this sale?

What would be the largest purchase? (4 of each. Since 1 of each would be 39¢, 4 of each must be \$1.56.)

Could a 30¢ purchase be made? (Yes. One way would be to buy a pencil and a notebook.) In what other way? (None)

How might the amounts indicated below have been spent? The record below shows one combination for each amount.

Amount of Purchase	23¢	53¢	76¢	59¢	81¢	89¢	\$1.37
No. of Pencils	1	2	3	0	4	1	2
No. of Pens	2	2	4	1	4	1	3
No. of Notebooks	0	1	1	2	1	3	4

What other combinations of items, other than those indicated above, give purchases of 23¢, 53¢, 76¢, 59¢, 81¢, 89¢, or \$1.37? (None)

Is it possible to spend \$1.25 under the terms of this sale? (No) How many amounts of money can be spent in two or more different ways under the terms of this sale? (Surprisingly enough, every one of the

124 different combinations of items possible under the limit set has a different total price. Thus, if the amount of the purchase is known, the exact combination of items can be found.)

What amounts of money more than 10¢ and less than a dollar cannot be spent under the terms of this sale? (11¢, 12¢, 16¢, 17¢, 21¢, 22¢, 26¢, and 31¢. All other amounts up to $1.25 can be spent.)

Below is a summary of purchases of all combinations of pencils and pens, with no notebooks and with a single notebook:

No Notebooks

Pens at 9¢ \ Pencils at 5¢	0	1	2	3	4
0	~~0¢~~	5¢	10¢	15¢	20¢
1	9¢	14¢	19¢	24¢	29¢
2	18¢	23¢	28¢	33¢	38¢
3	27¢	32¢	37¢	42¢	47¢
4	36¢	41¢	46¢	51¢	56¢

One Notebook and

Pens at 9¢ \ Pencils at 5¢	0	1	2	3	4
0	25¢	30¢	35¢	40¢	45¢
1	34¢	39¢	44¢	49¢	54¢
2	43¢	48¢	53¢	58¢	63¢
3	52¢	57¢	62¢	67¢	72¢
4	61¢	66¢	71¢	76¢	81¢

Additional charts can be made, noting the combinations including 2 notebooks, 3 notebooks, and 4 notebooks. No amount will occur twice—all combinations of items have a unique total price; thus, if we know the amount spent and can find it in the chart, we can easily determine the items purchased.

Notice the five entries circled in the charts above: 36¢, 37¢, 38¢, 39¢, and 40¢—an over-two-and-up-one pattern (the Knight's move in chess). Also notice that 41¢, 42¢, 43¢, 44¢, and 45¢ follow the same over-two-and-up-one pattern. This may help suggest that no amount will be repeated.

If the prices are changed to

Pencils	3¢
Pens	4¢
Notebooks	16¢

and a limit set of 3 each to a customer, the items purchased by a given price are unique, as in the situation discussed above. The smallest purchase is 3¢. The largest purchase is 69¢. All amounts of money between 3¢ and 69¢ can be spent in one and exactly one way except 5¢, 64¢, 67¢, and 68¢—which cannot be spent for any combination of items.

What Can You Say? Situation 6

> Suppose each of 6 pupils was asked to contribute his "favorite" number to a list and these numbers were recorded:
>
> 15, 4, 26, 1, 44, 9

/hat can you say about that list?

> Three of them are odd and three are even.
>
> None is prime.
>
> Their sum is 99.
>
> There is no combination of these numbers whose sum is 32, 33, or 34 (assuming that no number is included more than once in each combination).
>
> | | |
> |---|---|
> | $27=26+1$ | $32=$ ________ |
> | $28=15+9+4$ | $33=$ ________ |
> | $29=15+9+4+1$ | $34=$ ________ |
> | $30=26+4$ | $35=26+9$ |
> | $31=26+4+1$ | $36=26+9+1$ |
>
> How many different numbers can be expressed as the sum of a combination of two or more of those listed? (57)
>
> Can any member of the list be expressed as the sum of some combination of others from the list? (The answer is no. We can come close: $15+26+4=45$, but not 44; $1+9+15=25$, but not 26.)
>
> Can any number be expressed as the sum of a combination of two or more numbers from the list in more than one way? (No. The sum of every combination is different from the sum of every other combination.)

Can you find another list of six numbers such that the answers to the last two questions are No? (The simplest is 1, 2, 4, 8, 16, 32.) There are an unlimited number of such lists but they are hard to find unless the numbers are separated by large gaps, for example: 1, 17, 101, 900, 2000, 7587. In fact, any list in which each new number is more than the sum of all preceding numbers will work. Here are a few that are not at all obvious:

4, 6, 7, 8, 20, 40
1, 4, 7, 9, 18, 36
1, 4, 6, 8, 16, 32

A newcomer to this problem will hurry through hundreds of examples of addition without noticing the arithmetic practice. His goal is the solution of a problem—the practice is incidental.

It's easy to ask a question so easy that almost anyone can answer it. And it's easy to dig up trick questions—there are books full of them. But challenging questions, like those in the *What Can You Say?* situations just presented, are those that encourage independent research and lead to discovery. They must be neither too hard nor too easy, and must not rely on

clever tricks. Rather, challenging questions are less than obvious but will yield answers if the pupil thoughtfully applies what he already knows.

Such questions are hard to come by. They are worth their weight in gold. Often they are hidden in very simple situations. In this section and throughout the MATH WORKSHOP materials, we have presented the largest collection of such problems that have ever appeared in elementary school textbooks.

2. *"What's My Rule?"*

In first grade, Miss Atkins tells the pupils that she has a secret rule (a function) and that she will help them read her mind. "Call out a number and I will answer with a number."

"Five," a pupil calls out.

"Four," Miss Atkins answers. "Someone call out another number."

"One," a pupil calls out.

"Zero," the teacher replies.

"Seven," another pupil calls out.

"Six," the teacher answers. "Have you guessed my rule?"

"Yes," Bill is quick to reply.

"Don't give my secret away. Just call out a number and tell me what you think I will reply."

"If I say 9, you are going to say 8."

"That's right. I *think* that you know my secret." Thus the pupils have learned a way to demonstrate their discovery.

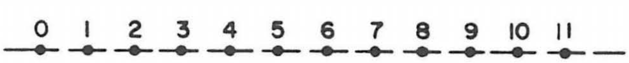
Many pupils are not as quick as Bill. So Miss Atkins may put on the board a record of the development thus far, first listing the number called out and then the reply:

$$5 \to 4$$
$$1 \to 0$$
$$7 \to 6$$
$$9 \to 8$$

The last item was Bill's contribution. Others might be:

$$6 \to 5$$
$$2 \to 1$$
$$4 \to 3$$

When the game is new or the rule difficult, a number line on the board will help:



"Let's act out the game we have played. When someone called out 5, I answered 4. Point to 5 on the number line. Keep your finger on that point. I answered by saying 4. Point to 4 on the line."

Each step is similary acted out.

"Now, if you call out 8, put your finger at the point on the number line."

8→___

"What do you think I will reply?"

"Seven, of course!"

In some such way the idea can be made clear to even the slowest pupils. Care was taken not to talk about the teacher's rule itself—only about the results of applying it. Thus, every pupil must make the discovery for himself—some with minimum clues, others only after several different kinds of clues. There is a bit of artistry in giving clues only when needed and finding ways to reserve at least a little for even the slowest pupil to find out for himself.

Usually some pupil will note a curious fact about this rule.

"Suppose I call out zero. What will you answer?" a pupil may ask.

"Point to zero on the number line," the teacher suggests. "You can see that I would have to ask you to pick some other number." (Unless the teacher would like to talk about positive and negative numbers and zero.)

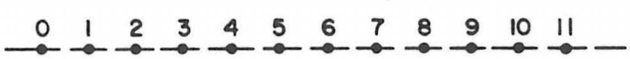
"I have a new rule. Call out a pair of numbers and I will reply by saying a number." Here is a record of a simple rule:

2, 3→5
4, 0→4
3, 1→4
1, 2→3
1, 1→2
3, 1→__

A more difficult rule is behind the following record:

2, 4→3
0, 2→1
4, 0→2
5, 9→7

This last rule may puzzle the pupils. Perhaps a number line can help shed light on the rule:



"If someone called out 2 and 4, I would reply with 3. Point to the numbers called out (2 and 4). I would reply with 3.

"If someone called out 4 and 0, I would reply with 2. Point to 4 and 0 on the number line. I would call out 2.

"Point to 5 and 9—and remember that I would answer 7. What would I answer if someone called out 1 and 7?"

1, 7→___

"Four," Bill might answer, and quickly ask: "But what would you do if I called out 5 and 6 or if I called out 3 and 0?"

"I would have to ask you to pick another pair of numbers." (Unless the teacher is ready to discuss fractions.)

An unlimited number of rules can be devised on any level of sophistication. Further, there are many formats to be used in presenting rules, such as:

a. 1, 4, 7, 10, 13, , , . . .

b.

3	1	4	10	0	9	6
7	3	9	21	1		

c.

7	4	3	1
9	9	0	9
11	6	5	1
	8	6	
6			2

Mathematics is essentially a study of patterns. It has its own fascinating patterns and it is often employed to help reveal patterns in the physical world. When a scientist or any observer is searching for a better understanding of what he notices, he usually tries to set down a description of his observations. Whenever possible he will try to write down numbers that in some way help him discover relationships. He might record the following observations:

	1st	2nd	3rd	4th	5th
minutes	1	3	4	5	10
miles	1	9	16		

In his first three observations he notes time and distance. A pattern is strongly suggested. He may check his guess by making further checks for 5 minutes and 10 minutes. If his results are 25 miles and 100 miles, respectively, he will feel that he can formulate a rule or law that will suggest a pattern in certain real events ($1 \times 1 = 1$, $3 \times 3 = 9$, $4 \times 4 = 16$ . . .).

In mathematics we study patterns because they suggest basic principles of the subject itself and because mathematics helps record and explore patterns of the world we live in. The discovery of *one* basic principle can remove the necessity for learning many separate facts. Learning to discover patterns is a big step toward self-education. The recognition of the mathematical relationship among quantities in a scientific experiment allows further costly experimentation to be replaced by inexpensive computation.

Thus, this exercise or game of *What's My Rule?* has three important dimensions: (1) it can be used to motivate drill in the basic combinations of arithmetic; (2) it poses problems that pupils can think through to a solution; (3) it uses a method fundamental to the experimental sciences.

C. Provide for Individual Differences

As teachers, we strive to reach every pupil. We attempt to *intensify* the instructional process so that learning for all pupils, from the slowest to the

fastest, can proceed at an optimum level. Our strategy for this intensification begins with our concern for each pupil's *level*, *rate*, and *interest*. We further our aim as we choose activities and materials which, by their design, provide for the great variation of ability we find in our classes.

MATH WORKSHOP shares this goal with you—the intensification of the learning process for every pupil. In our concern about *level*, we have built *multi-level* materials into the program. To provide for differences in the learning rates of pupils, we have pioneered in developing problems in categories like *What Can You Say?* and *What's My Rule?*, which, by their nature, make it possible for the slowest pupil to progress while the most able pupil is inspired to investigate the questions more deeply. Our tests and suggestions for informal appraisal will aid you in making provision for the various levels and rates of your pupils. Further, we have developed problems and activities which may excite some pupils so much that they will ask whether they have to go to recess, or lunch, or some other coveted activity.

We have built MATH WORKSHOP with these ideas in mind. We have tested them on pupils for more than ten years. This present edition represents a third revision of the original material. The form of the present books, then, is a result of field testing with tens of thousands of pupils by the authors, their colleagues, and thousands of teachers from all parts of the country and in every conceivable type of school.

There are five specific ways in which we developed and organized MATH WORKSHOP so that its materials and suggestions would contribute to our common goal:

1. Through the Multilevel Structure of the Seven Basic Pupil Books

These Pupil Books were given the following designations: Beginner's Book, Level A, Level B, Level C, Level D, Level E, and Level F—so that at any grade a Pupil Book appropriate to the levels and rates of the pupils could be used. In general, the Beginner's Book would be used for Kindergarten or first grade, Level A for first grade, Level B for second grade, etc. However, a slow class or slow group in grade 3 might use Level A or B rather than Level C. Further, a very bright group may go on to Level D. Suggestions are made in the early pages of the Teacher's Guide to help you make this decision.

2. Through the Multilevel Structure of the Pupil Pages

Pages in the Pupil Book are designed with major consideration given to posing challenging questions. In the previous section we were able to demonstrate through examples how this approach to teaching provides for success for even the slowest pupil. At the same time, such a multi-dimensional technique opens the door to higher level responses and further investigations by the more able pupil.

3. Through the Multilevel Suggestions in the Teacher's Guide

Many suggestions are made throughout each Teacher's Guide for adjustments and extensions of activities to accommodate the various levels of achievement. Further, we should like to describe at this point four of the general ways in which individual pupil pages can be taught:

a. Use the page to conclude a development.
b. Use the page as a basis for discussion.
c. Let the pupils work in teams.
d. Let pupils work independently.

a. In those areas in which the teacher has many ideas of his own, there is no substitute for direct teaching—with liberal use of manipulative materials, the chalkboard, and class discussion. The pupil page is then used late in the development to tie the idea down or simply to record the progress that has been made toward deeper understanding.

There are limitless ways to initiate lively activities around important and useful concepts. MATH WORKSHOP provides many such activities for you to choose from.

After handling manipulative materials, playing number games, and other such preliminary work, pupils are ready to move through the pupil page in the final few minutes of the arithmetic period.

b. In contrast, some pupil pages might be gone through very carefully with an entire class. Such a page or sequence of pages offers many opportunities to dig deeply into "What's going on here?"

c. The team approach involves two or more pupils working on their own. It is very much like the fourth approach except for the rustle of activity as pupils speculate on theories as how to proceed. Teams may include pupils of many different levels of ability. This allows pupils to learn from one another. A contest based on the rate and accuracy of the team's effort is often a good motivational device.

d. A fourth approach might be called *Do-it-all-by-yourself*. Begin it by saying: Open your book to (name the page). Look it over. Think about it. See if you can decide for yourself what is to be done. Wait a while before you ask for help. And let's make today a "quiet day."

Often, the first question a pupil faces when he begins work on a pupil page is: What am I supposed to do? There are almost always sufficient clues provided so that with a bit of study pupils can discover the intent of the exercises. Traditional materials have provided too little incentive for pupils to figure out for themselves the nature of the business at hand.

Now you feign reluctance to help but are, of course, watching carefully for those who haven't sufficient self-confidence to get started on their own. If several pupils are unable to get started, suggest that they work in pairs, helping each other quietly. Enlarge these helper combinations if necessary. Grouping like this will usually get the activity under way for everyone. Let

time run out on the session if you wish, suggesting that those who haven't finished may wish to take the work home with them to finish.

4. Through the Tests and Suggestions for Informal Appraisal in the Pupil Book and Teacher's Guide

The first six practice sheets in Level F are designed to serve as an Informal Arithmetic Inventory that will aid you in identifying the levels of pupils and that will suggest where further stress will be needed in teaching. The Teacher's Guide offers many suggestions for interpreting these tests.

5. Through an Index to Remedial Activities in Earlier Levels of MATH WORKSHOP

An index is provided in this guide indicating the pages in Levels A, B, C, D, and E that may be used for remedial work.

6. Through Supplementary Materials Developed by the Authors of MATH WORKSHOP

Described in Section IV below are a number of MATH WORKSHOP supplementary multilevel teaching aids which are available from Encyclopaedia Britannica Press. These include: (a) a teacher training text, *Discovery in Elementary School Mathematics;* (b) Practice Sheets; (c) MATH WORKSHOP Supplementary Activities; and (d) Games and Enrichment Activities.

Nowhere today has one program collected, invented, and brought together into a comprehensive form such a great variety of material aimed at the intensification of the learning process.

D. Use the Discovery Method

As we have noted repeatedly, each page in MATH WORKSHOP, the activities suggested in the Teacher's Guide, the practice sheets, and other aids are designed so that there are several levels on which pupils can participate—including opportunities for discovery and success for even the slowest learners.

The *What Can You Say?* type of activities we present are truly open-ended discovery exercises with challenge for even the most advanced and able pupils in the class. While some questions can be easily settled, more will have been left open. Some pupils will wish to explore these open questions on their own. Such pupils ought to learn to work independently, sparing the teacher's time for those who need more of it. Actually, what the more able pupils need most is encouragement and the opportunity to develop a taste for independent study. The highest level of learning has been reached when a pupil begins to pose problems of his own and to work toward their solution.

In MATH WORKSHOP there is extensive provision for introducing or practicing the basic facts of arithmetic and for solving problems. In most cases such practice is organized so that it leads to the emergence of a pattern or the solution of a problem.

Discovering such patterns is one of the most rewarding experiences in mathematics. It is called the "Aha!" or the "Gee-Whiz!" experience—a gasp, the light dawns! "I see it!" These are events for which teachers work long and hard. One of the most disheartening facts teachers must face is that some pupils simply "don't see it." Yet this is also a challenge—it calls for exercising skill in teaching. Slow learners need more clues—more obvious clues; but the finest teacher will reserve a wee bit for each pupil to "see" for himself. If the teacher's best efforts fail, he may have to give away the secret and try again next time. This is an art that does not come easily, but the eventual results are deeply rewarding.

When the results of work in arithmetic exhibit an unusual regularity, the inquiring mind looks for an explanation. One problem may be solved but others are suggested. This is an exciting characteristic of mathematics.

When pupils discover a relationship, they are excited and eager to stay with that discovery. They will readily put this new idea to work, showing more examples of *it*. They come alive, often with an "Oo-o-o! I've got it," or with a burst of industry using a newly-discovered relationship to reel off answers with speed and confidence.

What is the *it* they have glimpsed? Often, if we distract their attention with a request for them to *tell about it* or *put it in words*, we have selected a method that could not be better calculated to turn pupils against discovery.

That a pupil can move ahead by putting *it* to work successfully is certainly all the evidence we need in order to know that he *has it*. Our only justification for veering off in the direction of a word-pulling exercise is a feeling that *it* is a generalization that ought to be verbalized in order to be usable. But research does not support this. Transfer to future experiences is not necessarily increased by attempting to *verbalize a generalization*. Subvocal or nonverbal awareness, the new-found ability to move ahead more speedily, more directly, more confidently, is our main goal.

When pupils experience an insight—when they leap ahead because they have discovered a pattern—we ought to be very careful about letting words into the act too soon. The pupils probably do not have the language needed to express their discovery in words. If they try, we must, of course, help them. But too often such help and explanation degenerate into hair-splitting, an activity far removed from the act of discovery.

Gertrude Hendrix of the University of Illinois Committee on School Mathematics, who has done much research on discovery in the teaching of mathematics, warns that imprecise verbalization may *mutilate* a discovery. On the other hand, when some pupils (and teachers) make discoveries, they wish others to know about it. But herein lies a danger. Whenever a person reveals his secret to one who has not been as fortunate, the latter is deprived of the delight of discovering it for himself.

In MATH WORKSHOP we invent ways in which pupils can share their discovery without revealing the secret. We have them whisper their new idea in the teacher's ear; we encourage them to make up additional examples which will provide further clues for those who haven't found a pattern that underlies several examples. But we urge children to keep the secret—for a while at least.

Once in a while we find pupils who have not invented shortcuts for solving certain kinds of problems. For example, suppose that pupils use counting by ones in addition problems which involve fairly large sums. Hopefully, such pupils will become discouraged and begin to look for easier ways of determining the sum, such as counting by fives. If such easier ways are discovered, the pupils should be highly praised and urged to demonstrate these techniques to the entire class. If these examples are not sufficient to shake all pupils loose from counting by ones, then you must step in and urge the conservative ones to drop the slower technique. Please temper the discovery method with common sense.

E. Use Representational Materials

When mathematics loses contact with the world of things and with the events out of which it came, the subject becomes mysterious to most pupils. If they do not understand that in mathematics we are discussing the way things happen as we manipulate objects, pupils must resort to rote memory. MATH WORKSHOP activities stress the fact that arithmetic provides a means of recording or reporting what happens in the physical world.

Many kinds of manipulative or representational materials are suggested and discussed in this Teacher's Guide. These include various types of counting materials, geometric shapes, number lines, etc. Perhaps the most accessible source of representational materials is the pupil pages themselves. We believe that all of elementary mathematics can be abstracted out of experiences in the physical world. The very word *workshop* in the title of our program indicates our belief in the value of learning mathematics by working with physical objects. In some cases, the pupil pages are the workshop—pupils work with the two-dimensional pictorial representations of physical objects and apparatus and use the forms provided to keep records. Frequently the teacher will decide that the pupils need to work with three-dimensional equipment. In those cases, the teacher may provide equipment like that pictured on the pupil pages or like that suggested in the Teacher's Guide.

F. Teach for Mastery of Basic Facts and Algorithms

Mathematics provides a language to help report events that happen in the world of things. As we have noted, the core of elementary school mathematics

is counting. Because this process becomes tedious when many things must be counted, shortcuts to counting have been invented and improved. Thus, elementary school mathematics is essentially concerned with counting and with shortcuts to counting.

To some who have not studied the statements that precede this section, many MATH WORKSHOP activities appear to be highly motivated exercises to provide practice in the basic facts, combinations, and processes of arithmetic. In the vernacular, they wear the garb of "sugar-coated drill." Actually, they are valid educational devices, because they help establish that automatic recall of fundamentals without which pupils are mathematical stutterers.

Every teacher knows that many pupils are mathematical stutterers because they can't easily recall the basic facts, and stumble over the algorithms. They hesitate and flounder; the discussion bogs down; they lose the argument; they miss the feeling of movement toward a solution; they cannot enjoy mathematics.

This area of trouble is so universally known that it is surprising so many are only casually concerned! There is much hue and cry to bring new and unusual content into the elementary school curriculum—numeration using bases other than ten, sets, properties of numbers, and other concepts—yet there seems to be silence about such a crucial matter as helping pupils master the fundamental combinations. Without this mastery their progress toward understanding mathematical ideas is very painful.

While we have emphasized the scope of our program in terms of nine unifying ideas, *the basic facts and algorithms of arithmetic are taught and retaught throughout the program in all basic and supplementary materials.*

IV. MATH WORKSHOP TEACHING AIDS

A. Teacher's Guide

The introduction presents a point of view and a rationale for our approach to teaching elementary school mathematics. You will find this useful in interpreting many of the things we have included in the pupil pages. You will also find it useful in explaining the program to parents.

Each pupil page is reproduced in the Teacher's Guide in full size and in full color with answers printed in red.

Each pupil page is accompanied by a facing guide page that is divided into sections. Let's consider these sections:

1. *Tasks*. This section contains a very brief description of what the pupil will be doing as he works on the pupil page. We suggest that you study the pupil page in a pupil's book and work on it before you read our description and the answers we give.

2. *Purpose.* Here we describe what a pupil is intended to learn as he works on the pupil page or as he carries out some of the activities described in The Lesson. (Naturally, our description contains a conservative estimate of what might happen. Mathematics is full of surprises and so are pupils.) If you are aware of our purpose, you will find it easier to help pupils achieve it.

3. *Unifying Ideas.* In this section we list those of the nine unifying ideas mentioned on pages 12-13 to which the exercises on the pupil page and recommendations in the notes on the page in the Teacher's Guide can be expected to contribute. Please regard the list as a minimum statement, and be prepared for pupils who may wish to engage in activities that go beyond what we have described for the page.

4. *The Lesson.* In this section we describe one or more of the ways we might build a lesson around the pupil page. No teacher could possibly find time to use all our suggestions in any school year. Choose the ones which best suit your style of teaching, and vary your choices from year to year.

B. Practice Sheets

On each of the first 96 pages of the Level F Teacher's Guide, a practice sheet is presented (in reduced size) and discussed in the notes when the purpose and use of the sheet are not self-evident. These pages may be copied or adapted by the teacher. (They are also available in full size from Encyclopaedia Britannica Press.)

Every practice sheet contains the beginning or continuation of the investigation of a problem, the exploration of a mathematical situation. Generally, the activity is a different approach to the development found on the corresponding pupil page, an extension of it, or a self-contained activity that provides extensive practice. No page is just random drill.

C. MATH WORKSHOP Supplementary Activities

A package of supplementary activity worksheets is available for every level, A through F, from Encyclopaedia Britannica Press. Each package includes 32 copies of 64 different activity pages. These can be used independently by pupils for seatwork and homework. Able pupils might be given supplementary activity materials from advanced levels. Slow pupils might work on these materials from lower levels. Each set of supplementary activities includes a Teacher's Manual with brief suggestions for each page and an answer key.

D. Discovery in Elementary School Mathematics

This is a professional development textbook that suggests to teachers many ideas for teaching mathematics at every grade level. This book is divided into two major sections—Multilevel Research Activities and Multilevel Worksheet Activities.

Multilevel Research Activities present 27 exploratory experiences which can be introduced on the chalkboard and with simple teacher-made manipulative materials at any grade level. Each activity starts simply and spirals upward with extensions and elaborations that will appeal to the more mature and sophisticated mind. Each activity focuses on mathematical structure in a context of discovery learning.

The Multilevel Worksheet Activities section contains 24 units, each of which explores some of the unifying ideas of mathematics. Each unit begins with a brief introductory statement regarding mathematical structure and discovery. Each unit includes six activity worksheets, scaled in difficulty so that a teacher can present the ideas discussed in the unit to pupils at any level. These pages may be copied or modified by teachers for classroom use.

E. Games and Enrichment Activities

We have prepared a manual on the use of representational materials, such as Cuisenaire®* rods and Geo-boards. These materials can be used as the basis for games and other enrichment activities, many of which are mentioned or introduced in this Teacher's Guide. They contain many unusual applications of the concepts considered in the pupil pages.

F. Informal Arithmetic Inventory

The first six practice sheets in Level F are designed to serve as an informal inventory. Teachers have used such informal tests in reading and spelling to provide each pupil with material at his instructional level. You might wish to do the same thing in arithmetic.

The psychological principle of success in learning is the basic notion behind such an inventory or placement test. We all know that the ideal learning environment provides for heaping measures of success and challenge, and eliminates or minimizes frustration. Unfortunately, no matter how carefully textbooks are designed—including MATH WORKSHOP, which provides as much help as possible for pupils at different levels of ability—some pupils are frustrated unless they are able to work in easier books in the series. It follows too that some few should work in harder books if they are to be sufficiently challenged.

You may wish to initiate this year's program by having pupils complete the first six practice sheets first. In this case keep a record of their percentage of accuracy on each page for the problems you have assigned. You will then be able to determine whether Level F is the appropriate text for each pupil or if a lower or higher level in the series is called for.

You will notice that we are measuring more than computational skill in these six pages. Consistent with our promise, we place computational problems most often in a context which requires more than automatic recall. You

*Cuisenaire–Registered trademark of Cuisenaire Company of America, Inc.

will be able to learn what each pupil does when confronted by a situation with which he is not familiar. In this way you will have a basis for knowing how pupils go about solving new problems.

We offer the following as a guide to the placement of your pupils as you observe them working on these six pages: (1) if a pupil scores 0-50% on the series, he might be placed in the Level D pupil book; (2) if a pupil scores 51-70%, he might be placed in the Level E pupil book; (3) if a pupil scores 71-100%, he can begin in the Level F pupil book.

As with any evaluation technique for young pupils, care must be taken not to put more faith in tests than they deserve. Your own judgment of a pupil is probably more reliable than scores on tests.

This brings up the general problem of marking or grouping pupils as they work on all of the pupil pages. The pupil pages should be considered opportunities for investigation and exploration on the part of the pupil rather than as pages to mark for a grade. There are many pages on which pupils are not expected to get 100% on the first try. Nothing would be more damaging to their regard for mathematical inquiry than to be pummeled with red marks and low grades. We suggest that you *avoid* marking pages with a *minus* score. The number right or the percentage of accuracy is a more positive approach. By all means, keep a light touch rather than a heavy hand when checking and marking these pages.

V. PARENT EDUCATION AND INVOLVEMENT

Ill-advised and misleading publicity in newspapers and magazines has frightened parents into believing they can no longer help their children at home. The practice sheets may help considerably to soothe these parents. No new mathematical language is used on any of these pages. Most require extensive mental or scratch-paper computations in addition to the results finally recorded on the sheets. None of these computations is beyond the experience of most parents.

A common cause of misunderstanding between parents and children is concern for doing a computation the "right way." The child will often argue that the book's way or the teacher's way is the "right way"—and the parent will prefer his way. Of course, there is no one right way. There are only right answers or right results or valid conclusions. The way one chooses is his own business, a matter of personal taste—so long as it does not lead him to nonsense. MATH WORKSHOP suggests many approaches to problems and urges teachers to place a high premium on the discovery of many ways to find a right answer. We have argued elsewhere that if a child does not have at least one alternate method of finding a right answer, then he does not understand the method he is using.

Parents can do much to help children move ahead in mathematics. If they work together with their children toward solutions of problems, avoiding questions of right and wrong methods, mathematics can become a common

interest, contributing to a more constructive home environment. Parents may even come to realize that the fact that they "can't even keep the checkbook straight" does not really indicate a lack of mathematical talent or prevent them from participating with enjoyment in mathematical investigations. In fact, in these activities, parents are likely to discover that mathematics is a part of their culture that was denied them by an early contact with a crude facsimile of the subject.

Practice sheet activities provide opportunities for children and parents to work together and to think together about problems—a much more productive experience than completing assigned tasks in a prescribed way. Suppose a question is brought home: How many ways are there to make change for a quarter? Out come the coins from father's pocket or mother's purse. Someone must keep a record. How can you be sure that you have all the combinations? What kind of record will help answer that last question?

"Let's try a dollar next!"

And later, in the car, "I'm thinking of a combination of 9 coins that add up to 29¢. What coins am I thinking about?"

Practice sheets were designed with "home consumption" in mind. They may help bridge the gap that misinformation and rigid assignments create. Of course, the Supplementary Activities can also be used in the same way.

VI. PUNCTUATION AND ORDER OF OPERATIONS

In Level F pupils will sometimes face expressions such as "$8-2+3$," which appear to be ambiguous. Does this expression stand for 9 or for 3? That is, should one subtract 2 from 8 and then add 3 to the result, or should one add 3 to 2 and subtract the result from 8? One way to remove the ambiguity is to punctuate the expression with a pair of parentheses. Thus,

$$(8-2)+3=6+3=9$$

and

$$8-(2+3)=8-5=3.$$

A second way to remove the ambiguity is to establish a convention or agreement. The convention we follow in MATH WORKSHOP is that when an expression involves addition and subtraction, one should proceed from left to right in carrying out the indicated operations, unless parentheses are used to indicate a contrary order. Thus,

$$8-2+3=6+3=9$$

and

$$9+4-1=13-1=12$$

and

$$7-2-1-3=5-1-3=4-3=1.$$

When an expression involves multiplication and addition or subtraction, the convention we follow says that, unless parentheses indicate a contrary order,

one should perform the multiplications first, and then the additions or subtractions from left to right. Thus,

$$2\times 3-1=6-1=5,$$
$$4+3\times 2=4+6=10,$$
$$3\times 5+7\times 5=15+35=50,$$
$$\begin{aligned}4\times 2+5\times 3-6\times 2&=8+15-12\\&=23-12\\&=11.\end{aligned}$$

VII. A FINAL WORD

Makers of I.Q. tests search for measures of the pupil's ability to think. They try to rule out questions that merely assess achievement in subject-matter areas. Therefore, they would not ask pupils to complete the following familiar examples from arithmetic. These questions would more likely appear in an achievement test.

$$\begin{array}{r}3\\+7\\\hline\end{array}\qquad \begin{array}{r}41\\-16\\\hline\end{array}\qquad \begin{array}{r}346\\\times\ 27\\\hline\end{array}\qquad 5\overline{)135}\qquad 8\times 3=$$

$$\frac{1}{2}+\frac{2}{3}=\qquad\qquad 1.19=17\%\times\_\_$$

For measurement of I.Q. the questions from mathematics would be more like these:

I. Extend each of the following sequences:

a. 2, 5, 8, 11, 14, __, __, __ .....
b. 1, 2, 4, 8, 16, __, __, __ .....
c. 3, 4, 6, 9, 13, __, __, __ .....
d. 1, 1, 2, 3, 5, __, __, __ .....

II. Fill the blanks in the following examples:

a.

$$\begin{array}{cccccc}\boxed{6}&\boxed{3}&\boxed{1}&\boxed{4}&\boxed{2}&\boxed{}\\10&7&5&\bigcirc&\bigcirc&11\end{array}$$

b.

$$\begin{array}{cccccc}2&4&3&1&5&\_\_\\\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\4&16&9&\_\_&\_\_&0\end{array}$$

c.

$$\begin{array}{lcl}1,3\rightarrow 8&\qquad&3,2\rightarrow 10\\5,0\rightarrow 10&&6,3\rightarrow \_\_\\2,5\rightarrow 14&&\_\_,4\rightarrow 12\end{array}$$

III. Circle the number that seems most out of place in each row:

a. 6 , 10 , 2 , 7 , 8 , 4
b. 25 , 70 , 43 , 16 , 31 , 52
c. 13 , 15 , 19 , 3 , 11 , 7

Such problems, requiring a familiarity with arithmetic, are held to be more appropriate as measures of the ability to think. They are designed to test intelligence rather than achievement.

Mathematics is a fertile source of I.Q. test questions because mathematics is fundamentally a study of patterns and relationships. Many of the questions selected are similar to the examples given above. Behind that choice is the test-maker's estimate that pupils have not already explored and investigated such questions.

What an indictment of the elementary school curriculum in mathematics!

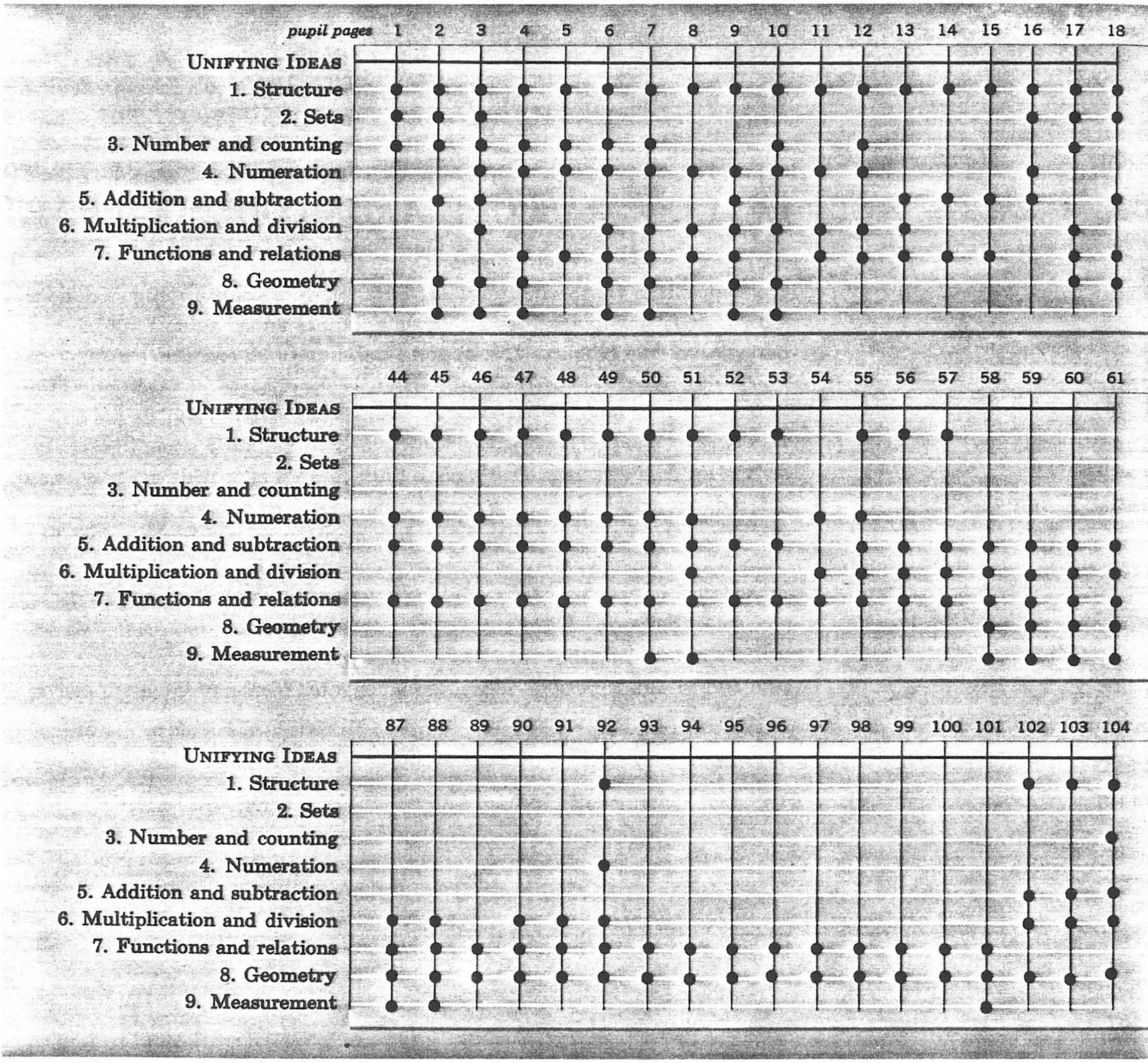
If arithmetic is limited to memory of number combinations and rules for manipulation, then it is a dead, dull, dreary task against which an active, eager, creative mind must surely rebel. However, if we broaden our thinking and look everywhere for patterns and relationships, the subject stimulates the mind to inquire and search out, and discover!

Every page in MATH WORKSHOP contains problems that probe beneath the basic facts and processes. We are always exploring the question: Why does arithmetic work? Each activity asks: What can you say? Each activity invites the pupil to ask: What's going on here?—and urges him to find out.

Robert W. Wirtz
Morton Botel
Max Beberman
W. W. Sawyer

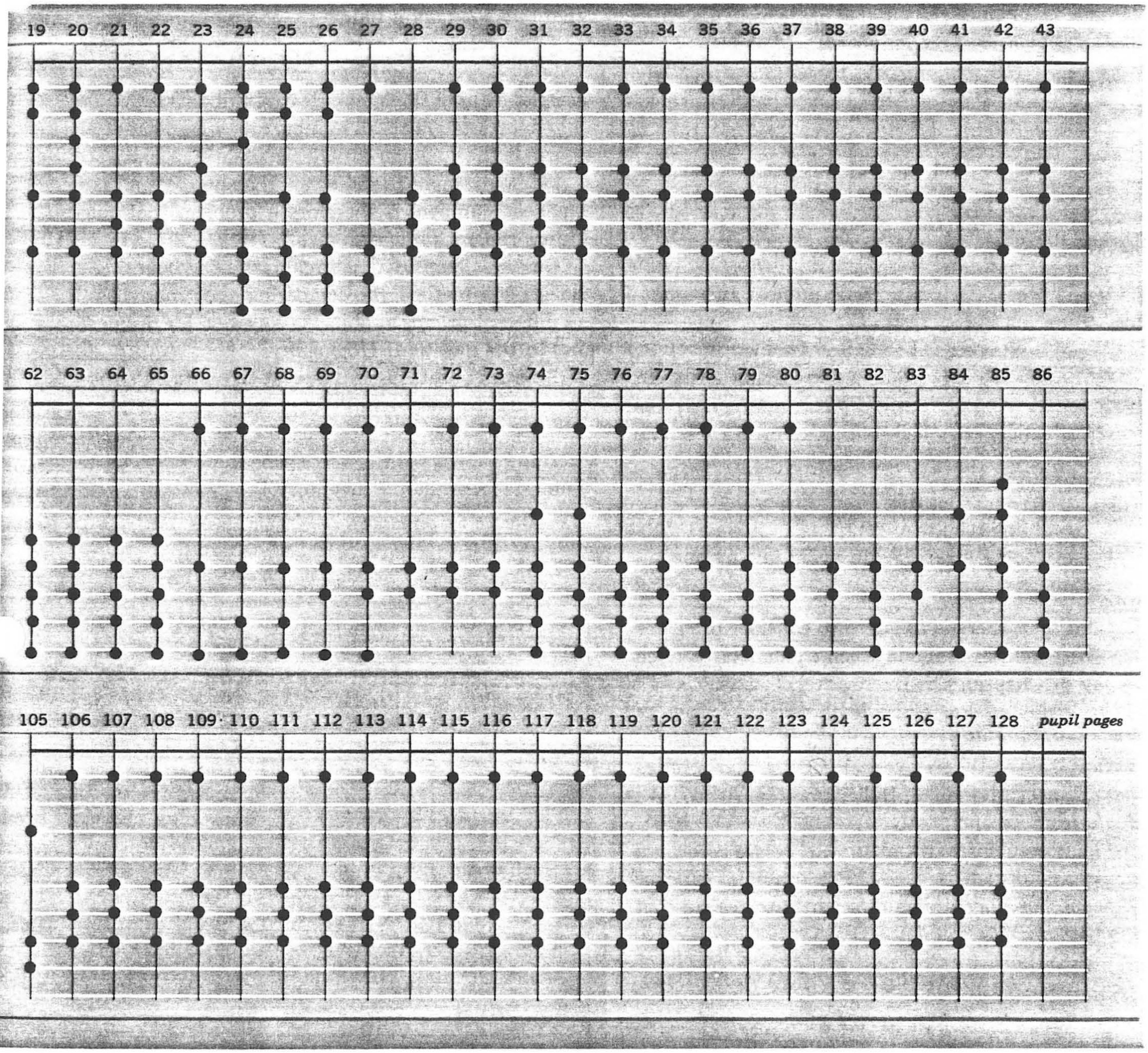
SCOPE AND SEQUENCE CHART

Level F Revised



Consider each unifying idea . . . on how many pages will you find it?

1. Structure	97 pages
2. Sets	15 pages
3. Number and counting	13 pages
4. Numeration	43 pages
5. Addition and subtraction	78 pages
6. Multiplication and division	83 pages
7. Functions and relations	116 pages
8. Geometry	51 pages
9. Measurement	41 pages

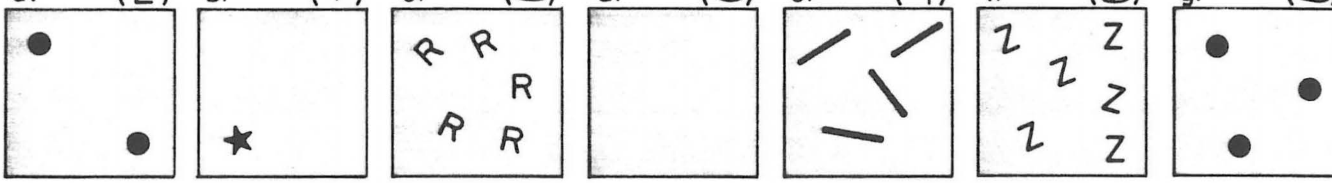


Consider each page . . . how many unifying ideas will you find on it?

11 pages contain two unifying ideas,
15 pages contain three unifying ideas,
62 pages contain four unifying ideas,
25 pages contain five unifying ideas,
11 pages contain six unifying ideas,
3 pages contain seven unifying ideas,
1 page contains eight unifying ideas.

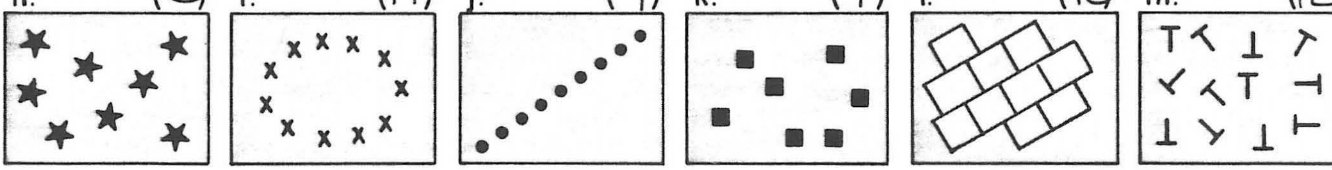
Take a quick glance at one of the sketches below. Now answer the question "How many?" without counting. Glance at the others. "How many?" In how many cases did you need to count?

a. (2) b. (1) c. (5) d. (0) e. (4) f. (6) g. (3)



Try the same experiment with the next group of sketches. Without counting, answer: "How many?" "An odd or even number?" "Which sketch shows the largest number of items?" "Which shows the smallest number of items?"

h. (8) i. (11) j. (9) k. (7) l. (10) m. (12)

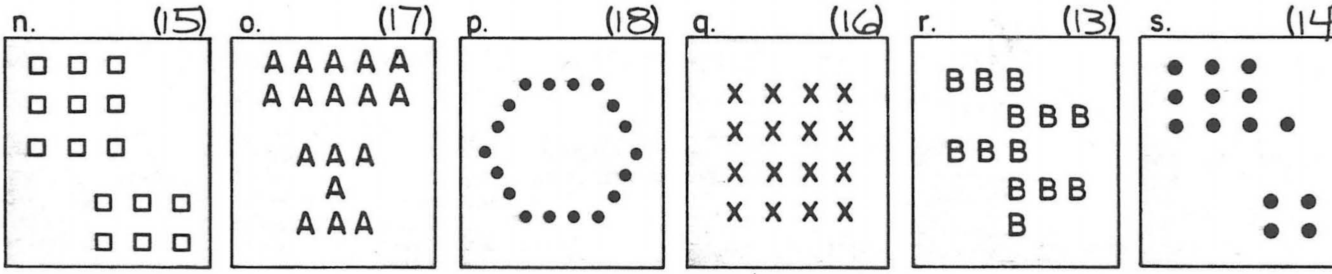


You failed! But don't feel bad. Everyone does unless he has a most unusual kind of vision and memory.

A few persons may have noticed that picture *k* has 7 little squares, but beyond that number of objects you probably needed to count.

Try the same experiment with the sketches below.

n. (15) o. (17) p. (18) q. (16) r. (13) s. (14)



Again, a quick glance is not enough. But the special arrangements certainly help! And you probably depended heavily on your experience.

As quickly as possible:

1. Of the 19 sketches above, list those that represent even numbers: d, a, e, f, h, l, m, p, q, & s.

2. Of the 19 sketches above, list those that represent odd numbers: b, c, g, i, j, k, n, o, & r.

3. Go back to each sketch, count the number of items in it, and write the result in the parentheses ( ) above each example.

(Those numbers you just wrote will help you with the following.)

4. List all of them that are multiples of 3.
0, 3, 6, 9, 12, 15, AND 18.

5. List all of them that are multiples of 4.
0, 4, 8, 12, AND 16.

Because vision fails us so quickly, we need to be able to count so that we can answer "How many?" And we need to be able to record the results of our counting.

Tasks: Without counting, students attempt to determine at a glance the number of items in a picture and to tell whether the number is odd or even. They then count the items in the pictures and indicate those for which the numbers are even, odd, multiples of 3, and multiples of 4.

Purpose: To remind students that vision fails us and that we need a system for comparing the "manyness" of different groups of objects and for recording the results of such a system.

Unifying Ideas: Structure; sets; number and counting.

The Lesson: We hope to help correct any mistaken notion that arithmetic is a difficult and mysterious subject. We sincerely believe that elementary school arithmetic has three simple aspects:

1. One-by-one counting
2. Recording results of counting
3. Developing strategies to avoid one-by-one counting

Beyond that, there is the need to consider problems involving counting (about which we shall have much to say in Chapter Two).

We highly recommend a book written by one of our group of authors: *Vision in Elementary Mathematics* by W. W. Sawyer (Penguin Books, Inc., 3300 Clipper Mill Road, Baltimore, Maryland). We quote from its opening paragraphs:*

"Some results in arithmetic can be grouped by a single act of mental vision. For example, suppose someone has played dice sufficiently to realize that the pattern:

o o o
o o o

represents 6. It is immediately apparent that 6 consists of 2 rows of 3. In this sense the result $6 = 2 \times 3$ is one that we can see directly.

"Unfortunately, this kind of vision leaves off almost as soon as it has begun. In the lines above, the words *immediately*, *unfortunately*, *arithmetic* have occurred. How many letters are there in each of these words? Few people can answer without counting or breaking these words into smaller groups. We have a clear picture of the smallest numbers. Beyond that, we see only a blur.

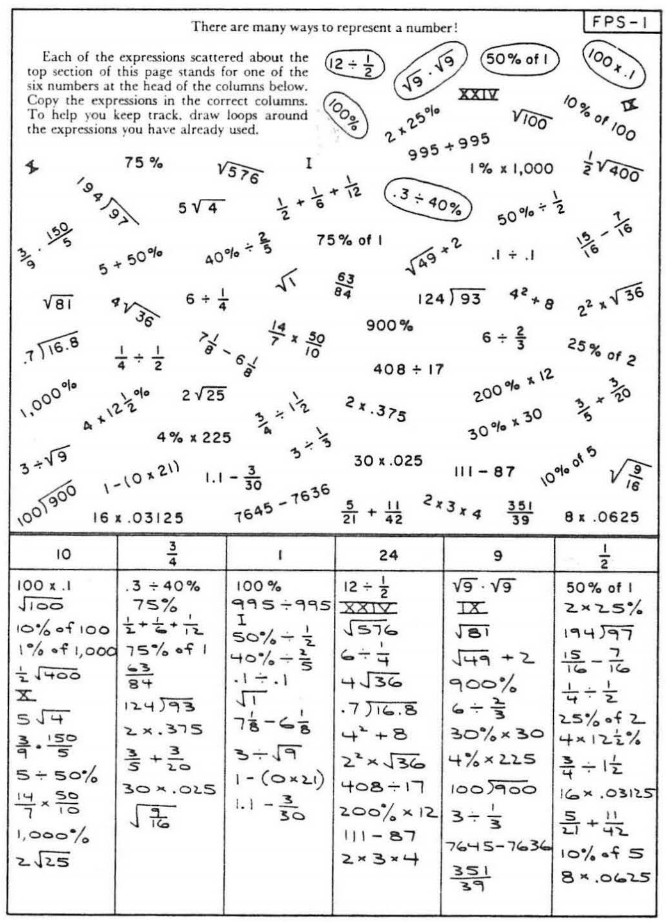
"The average man is often too modest. He sees only a blur. He may blame himself for this; cleverer people, he may think, see clearly. But this is not so. The blur into which all numbers dissolve soon after 4 or 5 is the common experience of us all.

"How to organize the chaos that lies beyond the smallest numbers is therefore a problem that confronts the entire human race."

Our first step in that organization is counting—devising a method for comparing the "manyness" of different groups of objects—a system for responding to the question: *How many?*

Practice sheet 1 is the first of six practice sheets designed to help assess the individual student's fluency in arithmetic. (See page 44 of the Introduction for a discussion of the use of these pages as an Informal Arithmetic Inventory.)

*Quoted by permission of Penguin Books, Inc.



FPS-1

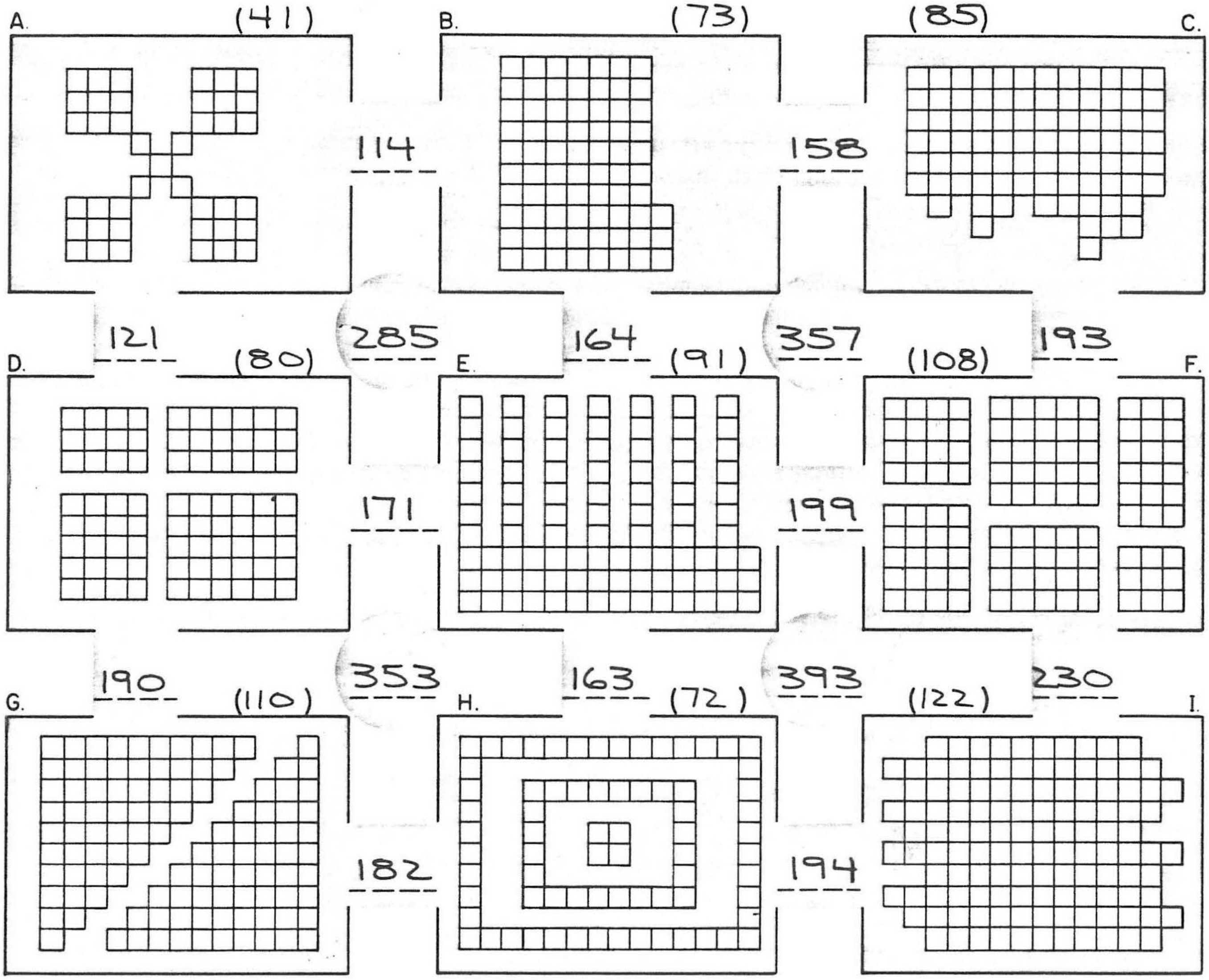
There are many ways to represent a number!

Each of the expressions scattered about the top section of this page stands for one of the six numbers at the head of the columns below. Copy the expressions in the correct columns. To help you keep track, draw loops around the expressions you have already used.

$12 \div \frac{1}{2}$ $\sqrt{9}\cdot\sqrt{9}$ 50% of 1 $100 \times .1$ 100% XXIV $2 \times 25\%$ $\sqrt{100}$ 10% of 100 IX $995 \div 995$ X 75% $\sqrt{576}$ I $1\% \times 1{,}000$ $\frac{1}{2}\sqrt{400}$ $194\overline{)97}$ $5\sqrt{4}$ $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}$ $.3 \div 40\%$ $50\% \div \frac{1}{2}$ $\frac{3}{9}\cdot\frac{150}{5}$ $5 \div 50\%$ $40\% \div \frac{2}{5}$ 75% of 1 $\sqrt{49}+2$ $.1 \div .1$ $\frac{15}{16}-\frac{7}{16}$ $\sqrt{81}$ $4\sqrt{36}$ $6 \div \frac{1}{4}$ $\sqrt{1}$ $\frac{63}{84}$ $124\overline{)93}$ 4^2+8 $2^2 \times \sqrt{36}$ 900% $.7\overline{)16.8}$ $\frac{1}{4} \div \frac{1}{2}$ $7\frac{1}{8}-6\frac{1}{8}$ $\frac{14}{7} \times \frac{50}{10}$ $6 \div \frac{2}{3}$ 25% of 2 $408 \div 17$ 1,000% $4 \times 12\frac{1}{2}\%$ $2\sqrt{25}$ $\frac{3}{4} \div 1\frac{1}{2}$ $2 \times .375$ $200\% \times 12$ $\frac{3}{5}+\frac{3}{20}$ $30\% \times 30$ $4\% \times 225$ $3 \div \frac{1}{3}$ $3 \div \sqrt{9}$ $1-(0 \times 21)$ $1.1-\frac{3}{30}$ $30 \times .025$ $111-87$ 10% of 5 $\sqrt{\frac{9}{16}}$ $100\overline{)900}$ $16 \times .03125$ $7645-7636$ $\frac{5}{21}+\frac{11}{42}$ $2 \times 3 \times 4$ $\frac{351}{39}$ $8 \times .0625$

10	$\frac{3}{4}$	1	24	9	$\frac{1}{2}$
$100 \times .1$	$.3 \div 40\%$	100%	$12 \div \frac{1}{2}$	$\sqrt{9}\cdot\sqrt{9}$	50% of 1
$\sqrt{100}$	75%	$995 \div 995$	XXIV	IX	$2 \times 25\%$
10% of 100	$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}$	I	$\sqrt{576}$	$\sqrt{81}$	$194\overline{)97}$
1% of 1,000	75% of 1	$50\% \div \frac{1}{2}$	$6 \div \frac{1}{4}$	$\sqrt{49}+2$	$\frac{15}{16}-\frac{7}{16}$
$\frac{1}{2}\sqrt{400}$	$\frac{63}{84}$	$40\% \div \frac{2}{5}$	$4\sqrt{36}$	900%	$\frac{1}{4} \div \frac{1}{2}$
X	$124\overline{)93}$	$.1 \div .1$	$.7\overline{)16.8}$	$6 \div \frac{2}{3}$	25% of 2
$5\sqrt{4}$	$2 \times .375$	$\sqrt{1}$	4^2+8	$30\% \times 30$	$4 \times 12\frac{1}{2}\%$
$\frac{3}{9}\cdot\frac{150}{5}$	$\frac{3}{5}+\frac{3}{20}$	$7\frac{1}{8}-6\frac{1}{8}$	$2^2 \times \sqrt{36}$	$4\% \times 225$	$\frac{3}{4} \div 1\frac{1}{2}$
$5 \div 50\%$	$30 \times .025$	$3 \div \sqrt{9}$	$408 \div 17$	$100\overline{)900}$	$16 \times .03125$
$\frac{14}{7} \times \frac{50}{10}$	$\sqrt{\frac{9}{16}}$	$1-(0 \times 21)$	$200\% \times 12$	$3 \div \frac{1}{3}$	$\frac{5}{21}+\frac{11}{42}$
1,000%		$1.1-\frac{3}{30}$	$111-87$	$7645-7636$	10% of 5
$2\sqrt{25}$			$2 \times 3 \times 4$	$\frac{351}{39}$	$8 \times .0625$

TASK ONE: Please find out how many little squares there are in each sketch, and record your results in the parentheses above each sketch. Look for shortcuts.



TASK TWO: Please find the total number of little squares in each pair of adjacent sketches. Record the results in the blue blocks.

TASK THREE: Please find the total number of little squares in the 4 sketches that touch each of the green tinted discs, and record the number in those discs.

The authors sincerely hope you found many shortcuts so that you didn't have to count all those little squares. We are sure you were glad you kept records of your results. In that way, you didn't have to count the squares in each sketch over and over again.

However, you could have counted the squares, one-by-one, to find correct answers to record.

So, arithmetic also includes:

. . . developing strategies or shortcuts to avoid one-by-one counting.

Arithmetic deals with:
(a) one-by-one counting,
(b) recording results,
(c) shortcuts to avoid one-by-one counting.

Tasks: *Students use whatever counting shortcuts they know to determine the number of squares in each sketch and in combinations of the sketches.*

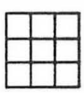
Purpose: To remind students that arithmetic is concerned with (1) developing reliable strategies or shortcuts to avoid one-by-one counting, and (2) recording the results of a count to avoid having to repeat it.

Unifying Ideas: Structure; sets; number and counting; addition and subtraction; geometry; measurement.

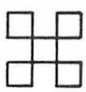
The Lesson: Pupil page 2 is designed to dramatize the need to develop strategies to avoid one-by-one counting and the advisability of recording the results of a count to avoid having to count the same group of things more than once.

What is an appropriate strategy? Any method that saves time and is reliable! Usually, several will be employed. Students ought to be encouraged to develop their own techniques of "being lazy" when they are asked to find out *How many?*

Consider the grouping in sketch *A*. One student might see an arrangement that is familiar to him:

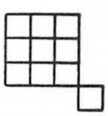


and notice that there are four such pieces—and $4 \times 9 = 36$. Those remaining are in a pattern familiar from dominoes:



and $36 + 5 = 41$—the answer to *How many?*

Another might see:



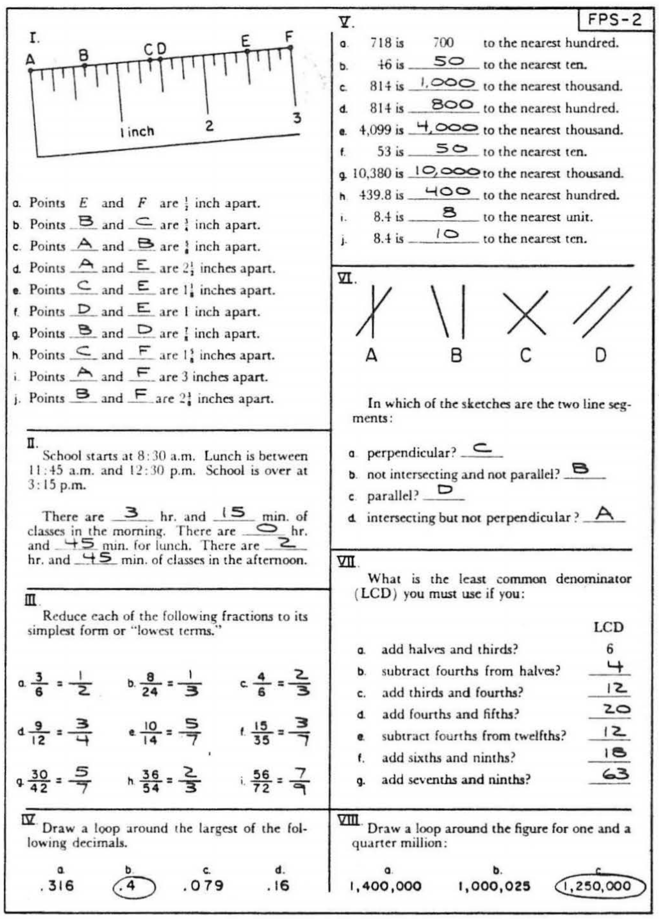
and at a glance realize that this group represents 10. There are four such groups, with a single square left over: $4 \times 10 + 1 = 41$.

Both strategies use the symmetry of the geometric figure in the sketch. Both utilize familiar number facts. The second approach brings in yet another strategy—grouping in 10's. This is a useful method for sketch *B*, where there are 10 squares in each of 7 columns, with 3 squares left over: $7 \times 10 + 3 = 73$.

Sketch *C* contains 6 rows of 12 squares each, with 13 squares left over: $12 \times 6 + 13 = 72 + 13 = 85$. Here again we use known number facts. Thus, use of *memorized facts* is a very efficient strategy to avoid one-by-one counting—the fastest one of all—but it need not be overtaxed. We have developed techniques that require memorizing only a few basic addition and multiplication facts. These are the standard techniques we call *algorithms*.

Students may like to argue about the author's opinion that of all school subjects, arithmetic requires the least memorizing! Some unfortunate people even try to memorize their way through arithmetic, but the fact is that it is much easier to think your way through arithmetic than to memorize your way through arithmetic.

Practice sheet 2 is the second in a series of six practice sheets designed to check individual fluency in arithmetic.



FPS-2

I.

a. Points *E* and *F* are $\frac{1}{2}$ inch apart.
b. Points B and C are $\frac{1}{4}$ inch apart.
c. Points A and B are $\frac{5}{8}$ inch apart.
d. Points A and E are $2\frac{1}{2}$ inches apart.
e. Points C and E are $1\frac{1}{8}$ inches apart.
f. Points D and E are 1 inch apart.
g. Points B and D are $\frac{7}{8}$ inch apart.
h. Points C and F are $1\frac{5}{8}$ inches apart.
i. Points A and F are 3 inches apart.
j. Points B and F are $2\frac{3}{8}$ inches apart.

II.

School starts at 8:30 a.m. Lunch is between 11:45 a.m. and 12:30 p.m. School is over at 3:15 p.m.

There are 3 hr. and 15 min. of classes in the morning. There are 0 hr. and 45 min. for lunch. There are 2 hr. and 45 min. of classes in the afternoon.

III.

Reduce each of the following fractions to its simplest form or "lowest terms."

a. $\frac{3}{6} = \frac{1}{2}$ b. $\frac{8}{24} = \frac{1}{3}$ c. $\frac{4}{6} = \frac{2}{3}$

d. $\frac{9}{12} = \frac{3}{4}$ e. $\frac{10}{14} = \frac{5}{7}$ f. $\frac{15}{35} = \frac{3}{7}$

g. $\frac{30}{42} = \frac{5}{7}$ h. $\frac{36}{54} = \frac{2}{3}$ i. $\frac{56}{72} = \frac{7}{9}$

IV. Draw a loop around the largest of the following decimals.

a. .316 b. .4 c. .079 d. .16

V.

a. 718 is 700 to the nearest hundred.
b. 46 is 50 to the nearest ten.
c. 814 is 1,000 to the nearest thousand.
d. 814 is 800 to the nearest hundred.
e. 4,099 is 4,000 to the nearest thousand.
f. 53 is 50 to the nearest ten.
g. 10,380 is 10,000 to the nearest thousand.
h. 439.8 is 400 to the nearest hundred.
i. 8.4 is 8 to the nearest unit.
j. 8.4 is 10 to the nearest ten.

VI.

In which of the sketches are the two line segments:

a. perpendicular? C
b. not intersecting and not parallel? B
c. parallel? D
d. intersecting but not perpendicular? A

VII.

What is the least common denominator (LCD) you must use if you:

	LCD
a. add halves and thirds?	6
b. subtract fourths from halves?	4
c. add thirds and fourths?	12
d. add fourths and fifths?	20
e. subtract fourths from twelfths?	12
f. add sixths and ninths?	18
g. add sevenths and ninths?	63

VIII. Draw a loop around the figure for one and a quarter million:

a. 1,400,000 b. 1,000,025 c. 1,250,000

And, in the opinion of the authors, that's about all there is to this subject of arithmetic.

Do you agree with us?

If you will look over your work on page 2, you will realize that you seldom depended on one-by-one counting. In each case, you developed your own strategy to save time and to keep from making careless errors that are apt to happen in long counts.

Think About Your Strategies

Patterns

You probably took a glance at each sketch wondering whether it had been drawn in some particular way that would suggest shortcuts. Many patterns include repetitions — and if two or more arrangements are clearly the same size, you need only find out how many there are in one of the arrangements.

Memory

If there is a 3-by-4 chunk, you use your memory to avoid having to count by ones. You remember that there are 12 in such arrangements.

Often, you saw the sketches as several small chunks and used your memory to tell you how many you would find if you counted.

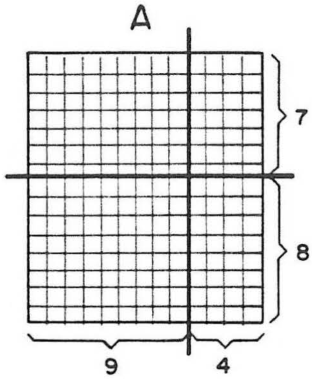
Looking for 10's

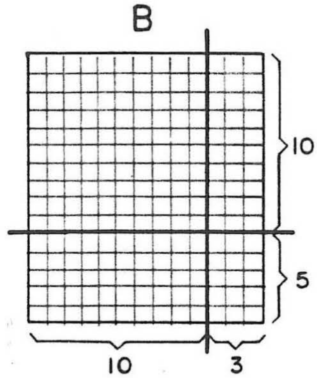
You might have looked for tens and multiples of tens, especially for hundreds. And, if the sketches were larger, you would look for multiples of hundreds, thousands, etc., because this leads us to

this	rather than	this
300 + 500 = 800		237 + 563 = 800
20 x 90 = 1,800		24 x 75 = 1,800
50 x 80 = 4,000		125 x 32 = 4,000
10 x 10 x 21 = 2,100		15 x 35 x 4 = 2,100

15 rows with 13 in a row?

Few of us remember the product of 13 and 15; so we break it up into *smaller* problems. The sketches below suggest two strategies:





A		B	
4 x 8 =	32	3 x 5 =	15
4 x 7 =	28	3 x 10 =	30
9 x 8 =	72	10 x 5 =	50
9 x 7 =	63	10 x 10 =	100
Total	195	Total	195

Which strategy do you prefer?

Both are better than finding the number with one-by-one counting!

Looking for 10ths

Looking for tenths, hundredths, and thousandths, etc., is another very fruitful search. It leads to important shortcuts, to easier bookkeeping as we record the results we would have counted if necessary.

The key to developing good strategies to avoid one-by-one counting is a good understanding of the decimal system, the system based on tens and hundreds and on tenths and hundredths of units.

Your authors believe that developing strategies to avoid one-by-one counting is what arithmetic is all about!

3 Strategies in counting

Tasks: Students consider strategies to avoid one-by-one counting and do some easy computing.

Purpose: To remind students that developing strategies to avoid one-by-one counting (with special emphasis on understanding of the decimal system) is the basis of all arithmetic.

Unifying Ideas: Structure; sets; number and counting; numeration; addition and subtraction; multiplication and division; geometry; measurement.

The Lesson: Just how much is required of the memory in order to master the 390 basic facts of addition, subtraction, multiplication, and division? The class may enjoy a discussion about cutting this job down to a more manageable size. Very little work is needed to master the following set of facts:

a. 0 added to any number leaves the number unchanged;
b. 1 added to any whole number gives the next whole number;
c. 2 added to any whole number gives us the whole number after the next;
d. subtraction facts are addition facts in reverse;
e. 0 times any number is 0;
f. 1 times any number leaves the number unchanged;
g. doubling a number is simply adding that number to itself;
h. all division facts are multiplication facts in reverse;
i. the order in which we add two numbers does not affect the outcome;
j. the order in which we multiply two numbers does not affect the outcome.

So, how much do we have left? We can make two simple tables to summarize the facts that must be memorized:

+	3	4	5	6	7	8	9
3	6	7	8	9	10	11	12
4		8	9	10	11	12	13
5			10	11	12	13	14
6				12	13	14	15
7					14	15	16
8						16	17
9							18

×	3	4	5	6	7	8	9
3	9	12	15	18	21	24	27
4		16	20	24	28	32	36
5			25	30	35	40	45
6				36	42	48	54
7					49	56	63
8						64	72
9							81

These two tables contain fifty-six facts that ought to be memorized so that time-consuming one-by-one counts can be avoided. That's really an amazingly small task.

From that point on, our strategies are essentially based on the technique of breaking problems up into parts, tackling one part at a time, and finally putting partial results together—usually by simple addition or subtraction. The usefulness of this technique depends directly on the individual's understanding of our base-ten place-value system. (See the next chapter.)

Practice sheet 3 is the third in a series of six practice sheets designed to check individual fluency in arithmetic.

FPS-3

Use scratch paper whenever you wish. The "questions" start below the lists of numbers, and the first group of questions refer to those lists.

List A	17	9	26	19	56	4	List B	86	66	274	103	77	239
C	$\frac{1}{2}$	$2\frac{1}{4}$	$1\frac{1}{2}$	$\frac{1}{4}$	$1\frac{3}{4}$	$\frac{3}{4}$	D	.8	6	.3	18	3.7	9.1
E	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{5}{6}$	F	2.57	106	.85	8.7	12.94	.9

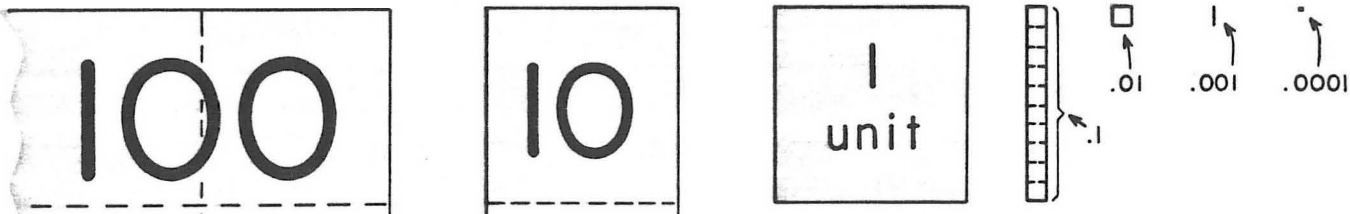
smallest two — middle two — largest two

	List A	List B	C	D	E	F
The difference between the largest and smallest numbers in the list.	52	208	2	17.7	$\frac{2}{3}$	105.15
The sum of the two largest numbers in the list.	82	513	4	27.1	$1\frac{1}{2}$	118.94
The sum of the two smallest numbers in the list.	13	143	$\frac{3}{4}$	1.1	$\frac{5}{12}$	1.75
The difference between the two middle numbers.	2	17	$\frac{3}{4}$	2.3	$\frac{1}{6}$	6.13
The sum of the four numbers listed above in this chart.	149	881	$7\frac{1}{2}$	48.2	$2\frac{3}{4}$	231.97

Which product is largest?
Which product is smallest?

Use scratch paper only if you wish.

A	B	C	D	
7 x 9 = 63	31 x 2 = 62	5 x 13 = 65	8 x 8 = 64	Largest Product: C; Smallest Product: B
297 x 35 = 10,395	297 x 34 = 10,098	296 x 35 = 10,360	296 x 34 = 10,064	Largest Product: A; Smallest Product: D
$\frac{1}{100} \times \frac{1}{100}$ = $\frac{1}{10,000}$	$\frac{1}{10} \times 1$ = $\frac{1}{10}$	$1 \times \frac{1}{1000}$ = $\frac{1}{1000}$	$\frac{1}{10} \times \frac{1}{10}$ = $\frac{1}{100}$	Largest Product: B; Smallest Product: A
$\frac{1}{3} \times \frac{1}{7}$ = $\frac{1}{21}$	$\frac{1}{4} \times \frac{1}{6}$ = $\frac{1}{24}$	$\frac{1}{3} \times \frac{1}{6}$ = $\frac{1}{18}$	$\frac{1}{5} \times \frac{1}{5}$ = $\frac{1}{25}$	Largest Product: C; Smallest Product: D
3.85 x .46 = 1.7710	38.3 x 4.8 = 183.84	3.85 x 4.8 = 18.480	38.3 x 46 = 1761.8	Largest Product: D; Smallest Product: A
.13 x .6 = .078	3 x .6 = 1.8	3 x 6 = 18	.3 x .6 = .18	Largest Product: C; Smallest Product: A
50% x 97 = 48.5	25% x 200 = 50	$33\frac{1}{3}$% x 175 = 58.333	4% x 1000 = 40	Largest Product: C; Smallest Product: D



Chapter ONE: THE DECIMAL SYSTEM — the key to many strategies that help avoid the need for one-by-one counting.

Remember what arithmetic is all about:

COUNTING

RECORDING the results of counting

and

SHORTCUTS to avoid 1-by-1 counting

With only ten digits:

0, 1, 2, 3, 4, 5, 6, 7, 8, and 9

and with the idea of

Place Value

you can write any number you need or wish to write.

It's easy to read large numbers:

$$\underbrace{79}_{\text{Billions}},\underbrace{347}_{\text{Millions}},\underbrace{205}_{\text{Thousands}},870$$

Start reading the number at the left, and use the indicated words as you come to each comma:

"seventy nine *billion,* three hundred forty-seven *million,* two hundred five *thousand,* eight hundred seventy"

. . . unless there are three zeros to the left of a comma. Then, don't say anything when you pass the three zeros and the comma that follows them:

$$2\underset{\text{Billions}}{\hat{,}}120\underset{\text{Millions}}{\hat{,}}\underbrace{000}_{\text{Pass over}},061$$

"two *billion,* one hundred twenty *million,* sixty-one."

Try a few:

48,206	127,138,000
175,175	1,000,001
301,000	100,000,984
800,008	60,001

4-5 Place value and pronouncing decimal notation

Tasks: *Students study representations of the decimal structure and review the reading and writing of large numbers.*

Purpose: To remind students that, using positional or place-value notation, we can write any number as a combination of ten symbols.

Unifying Ideas: Structure; number and counting; numeration; functions and relations; geometry; measurement.

The Lesson: For the next nine pages, we shall range back and forth from very large numbers to very small numbers, from millions to millionths, exploring the tremendous flexibility of our decimal system of numeration.

Students are sometimes surprised to notice that we can write any number whatever as a combination of just ten symbols:

0, 1, 2, 3, 4, 5, 6, 7, 8, and 9

We use different combinations of twenty-six letters to make English words, but we need no more than ten symbols to indicate any whole number we wish. This great economy stems from the use of place-value notation. The uniqueness of the place-value system can be obscured by placing too much stress on the value of various places—units, tens, hundreds, tenths, hundredths, etc. Students should come to realize that the heart of this system is the relation between each place and its neighbor. Suppose that the beginning and ending of a decimal number are covered up:

35779

We don't know how many more digits are to the left or right of those shown, nor do we know whether the decimal point is located to the right or left of the digits we can see. We do not know the specific value of any of the digits showing. However, we do know such things as (a) that the 7 written on the left is "worth" 10 times as much as the 7 on the right, (b) that the value of the place occupied by the digit 3 is 10 times that of the place occupied by the digit 5, and (c) that the value of the place occupied by the 9 is one-tenth that of the place occupied by the first 7 to its left and one-hundredth the value of the place occupied by the second 7 to its left, etc.

Suppose that all we know about the following example of column-addition is that digits occupying places of equal value are written under each other:

$$\begin{array}{rcccccc} & & 1 & 2 & 1 & & \\ & 4 & 2 & 5 & 4 & 9 & 7 \\ + & & & 1 & 0 & 7 & \\ \hline & 4 & 3 & 8 & 6 & 6 & 7 \end{array}$$

Since decimal points are omitted, we do not know the place value of any of the digits. Do we need to know these place values to find the digits in the sum? Certainly not. The important fact is that the value of each place is 10 times that of the place on its right and one-tenth that of the place on its left.

It is when we wish to indicate a specific number that we need to assign a specific value to each place. We do this by indicating the "units place"—either by omitting the decimal point or by placing it to the right of the units place.

Practice sheet 4 is one of six practice sheets designed to check individual fluency in arithmetic.

FPS-4

Based only on the information given in each case,

What Can You Say for Sure?

Mark statements either true T F or false T F

I. I have 3 pockets. In them, I have a total of 4 keys. No two pockets have the same number of keys.

a. All the keys might be in one pocket. T F

b. One pocket might have twice as many as another. T F

c. One pocket has to be empty. T F

d. There must be 3 keys in one of the pockets. T F

e. There must be a single key in one of the pockets. T F

II. In a game, Bill had exactly twice as many points as Harry, and Harry had exactly twice as many points as Al.

f. Altogether, they must have an even number of points. T F

g. Al has exactly half as many points as Harry. T F

h. Bill has exactly three times as many points as Al. T F

i. Altogether, they might have exactly 30 points. T F

j. Together, Al and Harry have fewer points than Bill. T F

k. If altogether they have fewer than 100 points then Al could have 14 points, but no more. T F

l. If Al has an even number of points then altogether the sum of the points must be a multiple of 14. T F

III. There are 12 goats on a hill. Fifty percent of them are black. Each of the others is either white or brown. A third of the goats are eating daisies. Two-thirds of them have bells on.

m. There are 6 black goats. T F

n. Four goats are eating daisies. T F

o. Those not eating daisies have bells on. T F

p. Some black goats have bells on. T F

q. Some white goats have bells on. T F

r. Two-thirds of the black goats are eating daisies. T F

s. There are half as many white goats as black goats. T F

t. If all the goats eating daisies are brown goats then there are fewer white goats than brown goats. T F

IV. Mary is younger than Jane and older than Anne. Louise is not the youngest, but she is younger than Mary.

u. Louise is younger than Jane. T F

v. Anne is older than Jane. T F

w. Jane is not the oldest. T F

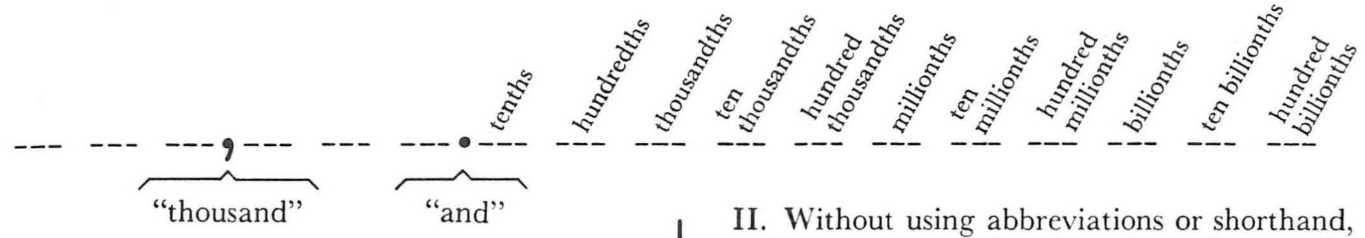
x. Anne is the youngest. T F

y. Louise is older than Mary. T F

V. $1 \times 2 \times 3 \times 5 \times 7 = 210$

210 is divisible by:

A. 8 T F	*F.* 13 T F	*K.* 18 T F
B. 9 T F	*G.* 14 T F	*L.* 19 T F
C. 10 T F	*H.* 15 T F	*M.* 20 T F
D. 11 T F	*I.* 16 T F	*N.* 21 T F
E. 12 T F	*J.* 17 T F	*O.* 22 T F



The *decimal point* is another bit of shorthand. It is read "and" if the number is greater than 1.

3.7 . . . is read "three *and* seven tenths."

108.18 . . . is read "one hundred eight *and* eighteen hundredths."

8,176.04 . . . is read "eight thousand one hundred seventy-six *and* four hundredths."

When there are no nonzero digits to the left of the decimal point, we do not say "and."

.9 . . . is read "nine tenths."

0.12 . . . is read "twelve hundreths."

.084 . . . is read "eighty-four thousandths."

.1072 . . . is read "one thousand seventy-two ten thousandths."

Caution: Confusion can result if "and" is used for any other reason than to indicate the location of the decimal point. A careless person might read the last example above as

"one thousand and seventy-two ten thousandths" . . . which means

1,000.0072.

I. Read the following expressions:

a.	3.08	*h.*	.1078
b.	18.1	*i.*	.010
c.	4.125	*j.*	2.100
d.	.008	*k.*	4.001
e.	.1009	*l.*	.0807
f.	129.018	*m.*	.19473
g.	27.00012	*n.*	.002009

II. Without using abbreviations or shorthand, write out some of the expressions given at the bottom of the last column.

e. ONE THOUSAND NINE TEN THOUSANDTHS

f. ONE HUNDRED TWENTY-NINE AND EIGHTEEN THOUSANDTHS.

g. TWENTY-SEVEN AND TWELVE HUNDRED THOUSANDTHS

i. TEN THOUSANDTHS

l. EIGHT HUNDRED SEVEN TEN THOUSANDTHS

n. TWO THOUSAND NINE MILLIONTHS

III. Could you count by ones to a million? Of course, provided that you had plenty of time and a good reason for doing it.

If a clock makes a "tick" every second, how long does it take for the clock to make a million ticks?

1 in 1 second
60 in 1 minute
3,600 in 1 hour
86,400 in 1 day
604,800 in 1 week
950,400 in 1 week and 4 days
1,036,800 in 1 week and 5 days
31,536,000 in 365 days

So, big numbers really do arise when we wonder about such questions.

What would a stack of a million blocks look like? We've tried to give you some idea on the next page.

Notes on Pupil Page 5

Tasks: *Students read and write expressions for numbers in decimal notation. They get some practical idea of how long it takes a one-tick-per-second clock to make a million ticks.*

Purpose: To help students better understand the decimal system of numeration by focusing on reading and writing numbers containing decimal points and by noting an application of counting large sets.

Unifying Ideas: Structure; number and counting; numeration; functions and relations.

The Lesson: Some people have argued that it would be more sensible to put the decimal point over the units place rather than between the units and tenths place:

$1\dot{2}3$ rather than 12.3

Perhaps your class would like to discuss this idea. Certainly the suggested improvement has the advantage of symmetry with respect to the decimal point:

tens units tenths
$1\dot{2}3$
or
tens units tenths
12.3

Consider another argument. When we multiply by 10 in the decimal system, we move the decimal point one place to the right, and when we divide by 10 we move it one place to the left:

$1\dot{2}3 \times 10 = 12\dot{3}$ or $12.3 \times 10 = 123.$

$1\dot{2}3 \div 10 = \dot{1}23$ or $12.3 \div 10 = 1.23$

In each case we are selecting a new place to designate as the units place. This, it is argued, would be emphasized by putting the decimal point over the digit in the units place rather than on its right.

Of course, we are not actually advocating such changes in the worldwide agreement about the location of the decimal point. However, discussion of possible alternatives may help some students better understand the decimal system.

On pupil page 5 we look at one meaningful situation that gives rise to large numbers—how many times does a clock tick in a year if it ticks once a second? Most clocks tick at a faster rate. Some watches tick several times a second. Students may listen to a watch and estimate the number of ticks per second. They can then ask: "How many ticks per year?" "If the watch lasts ten or twenty-five years, how many ticks?"

How many grains of rice are there in a ten-pound box? How many drops of water in a gallon? How many people are there in the world? How much does the earth weigh? What part of a square inch is the area of the head of a pin? Where else might we encounter very large or very small numbers? Some students will enjoy carrying out investigations of such questions. The whole class might discuss how they would proceed if they were to carry out such investigations.

Practice sheet 5 is one of a series of six practice sheets designed to check individual fluency in arithmetic.

FPS-5

I. In each column, circle a combination of numbers whose sum is given at the bottom of the column. We call this *Punch Card Arithmetic.*

A.	B.	C.	D.	E.	F.	G.	H.	I.	J.	K.	L.
7	7	7	7	7	7	7	7	7	7	7	7
11	11	11	11	11	11	11	11	11	11	11	11
14	14	14	14	14	14	14	14	14	14	14	14
17	17	17	17	17	17	17	17	17	17	17	17
22	22	22	22	22	22	22	22	22	22	22	22
45	45	45	45	45	45	45	45	45	45	45	45
40	31	52	43	73	116	64	85	94	84	109	99

M.	N.	O.	P.	Q.	R.	S.	T.	U.	V.
34	34	34	34	34	34	34	34	34	34
46	46	46	46	46	46	46	46	46	46
49	49	49	49	49	49	49	49	49	49
53	53	53	53	53	53	53	53	53	53
58	58	58	58	58	58	58	58	58	58
67	67	67	67	67	67	67	67	67	67
147	80	104	182	178	307	166	141	240	205

II. In each list, circle a pair of numbers whose difference is given on the left.

9 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96
48 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96
19 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96
21 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96
43 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96
59 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96

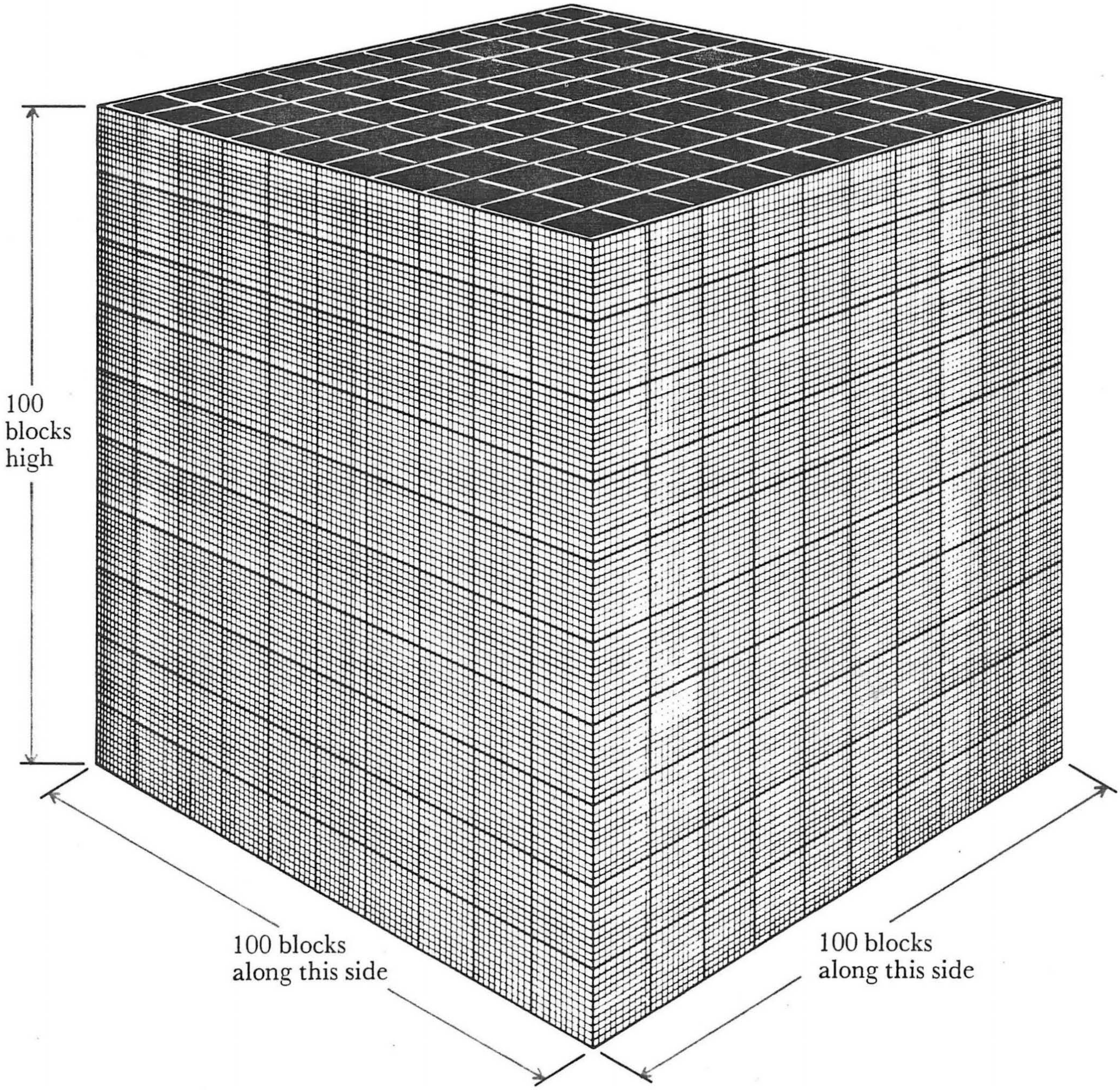
14 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96
29 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96
42 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96
25 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96
28 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96
44 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96
45 : 3, 7, 8, 10, 16, 26, 37, 51, 68, 96

III. What Are My Rules?

A	B	C	D
7 → 15	5 → 24	4, 5 → 18	3, 4 → 6
10 → 21	10 → 99	1, 9 → 20	9, 6 → 27
49 → 99	1 → 0	3, 3 → 12	6, 7 → 21
5 → 11	7 → 48	8, 11 → 38	8, 8 → 32
18 → 37	20 → 399	.4, .6 → 2	3, 14 → 21
86 → 173	38 → 1,443	0, .5 → 1	18, 37 → 333
.4 → 1.8	75 → 5,624	.03, .7 → 1.46	0, 14 → 0
.7 → 2.4	1.2 → .44	1.8, 2 → 7.6	.4, .9 → .18
$\frac{1}{2}$ → 2	$1\frac{1}{2}$ → $1\frac{1}{4}$	$\frac{1}{2}, \frac{1}{3}$ → $\frac{5}{3}$ or $1\frac{2}{3}$	$\frac{1}{2}, \frac{1}{2}$ → $\frac{1}{8}$
$\frac{3}{4}$ → $2\frac{1}{2}$	$1\frac{1}{4}$ → $\frac{9}{16}$	$\frac{2}{5}, \frac{3}{4}$ → $\frac{46}{20}$ or $2\frac{3}{10}$	$\frac{2}{3}, \frac{7}{6}$ → $\frac{7}{18}$
$n \rightarrow 2n + 1$	$t \rightarrow t^2 - 1$	$(a, b) \rightarrow 2 \cdot (a + b)$	$(q, r) \rightarrow (q \cdot r) \div 2$

Here's a big stack of blocks.

You would have to count to a million if you wished to count all the blocks in a stack as big as the one illustrated in this drawing.



The bottom layer of tiny blocks would have 100 in each of 100 rows.

How much is 100 × 100? 10,000

There would be 100 such layers.

How many in 100 layers?

100 × 100 × 100 = 1,000,000

Of course, you see only a few of the blocks in this stack — those on the outside.

. . . one million of them!

Do you think that there are a million grains in a bushel of wheat? No

Believe it or not, there are about 5,000,000 grains in one bushel of wheat.

6-7 From one million blocks to one millionth of a million blocks

Tasks: Students study a representation of a million and determine the count in various layers.

Purpose: To help students gain some feeling for large numbers and the role they play in the real world.

Unifying Ideas: Structure; number and counting; numeration; multiplication and division; functions and relations; geometry; measurement.

The Lesson: Did you ever see a pile of a million of anything? How about a pail of sand, a sky full of stars, blades of grass in a lawn?

A school in Wisconsin once collected (on loan) 1,000,000 pennies and piled them on the floor of the gym. How might one go about finding how much a pile of one million pennies weighs? Writing "one million" or "1,000,000" gives us little feeling for how many pennies that really is, what the pile would look like, what it would weigh, or how much space it would take up.

On pupil page 6, an artist has shown what a stack of a million blocks would look like. There are 100 blocks in every row along each side and 100 blocks in each column. The drawing at the top of pupil page 7 gives us a close-up view of the top part of the stack.

On the pupil page we state that there are approximately 5,000,000 grains of wheat in a bushel. Suppose we wished to check this statement. Would we simply get a bushel of wheat and start counting? Remember, there are about 31,000,000 seconds in a year. Therefore, if we counted a grain a second for four hours every day, we would need just about a year to count 5,000,000 grains! What shortcuts might we use to avoid such forbidding stretches of one-by-one counting? Perhaps we could weigh a bushel of wheat and then weigh one ounce. We could then count the number of grains in that ounce and use multiplication to get a final estimate.

There are many similar problems that students might consider: How many pine needles are there on a particular pine tree? How would you tackle the problem?

How many steps would you take if you were to walk from New York to San Francisco? Since that's about 3,000 miles, and there are about 5,000 feet in a mile, you'd travel about 15,000,000 feet. And, if you covered two feet each step, you'd have to take about 7,500,000 steps. But you wouldn't be able to walk in a straight line, so it would take something like 10,000,000 steps. Do you think you could do it in a year?

Such discussions, using rounded numbers for obtaining very rough approximations, will help suggest the power of arithmetic and help give students some notion of the role large numbers play in thinking about the world we live in.

Practice sheet 6 is the last of the series of practice sheets designed to help assess individual fluency in arithmetic.

FPS-6

I. I have 13 ordinary U.S. coins in my hand. They are worth a total of 29¢. There might be 4 nickels and 9 pennies. Is there a different combination of 13 coins worth exactly 29¢? No (Yes or No.) If yes, what is it? ——

II. In my other hand, I have 13 ordinary U.S. coins worth 37¢. These might be a quarter and 12 pennies. Is there another combination of 13 coins worth 37¢? YES (Yes or No.) If yes, what is it? 6 NICKELS & 7 PENNIES.

III. I'm thinking of a 2-digit even number. It is 1 less than a multiple of 9. It is a multiple of 7. What is the number? 98

IV. Janet and Anne together had exactly 12 books. Anne and Ruth together had 16 books. Janet and Ruth together had 18 books.
Janet had 7 books; Anne had 5 books; and Ruth had 11 books.

V. Prices selected from a newspaper ad:

Turkey Roasts
All white — 2 lbs., 6 oz. $3.59
Light & dark — 2 lbs., 6 oz. $2.99

Canned Ham
5 lb. $5.29
8 lb. $7.79

Coffee
2 lb. can $1.09
3 lb. can $1.63

Shrimp
2 lb. $2.19
10 oz. . . . . . 75¢

Bologna
1 lb. . . . . . 65¢
12 oz. . . . . . 53¢

a. Six pounds of coffee bought in 2-lb. cans would cost $3.27.

b. Six pounds of coffee bought in 3-lb. cans would cost $3.26.

c. In 2-lb. cans, coffee costs $54\frac{1}{2}$¢ per pound, and in 3-lb. cans, coffee costs $54\frac{1}{3}$¢ per pound. The difference is $\frac{1}{6}$¢ per pound.

d. Shrimp is $2.19 for two pounds or 1.09\frac{1}{2}$ per pound. 10-oz. packages of shrimp are 75¢ each, or $1.20 per pound.

e. Bologna is sold at 65¢ a pound. In 12-oz. packages at 53¢ each, that is $70\frac{2}{3}$¢ a pound.

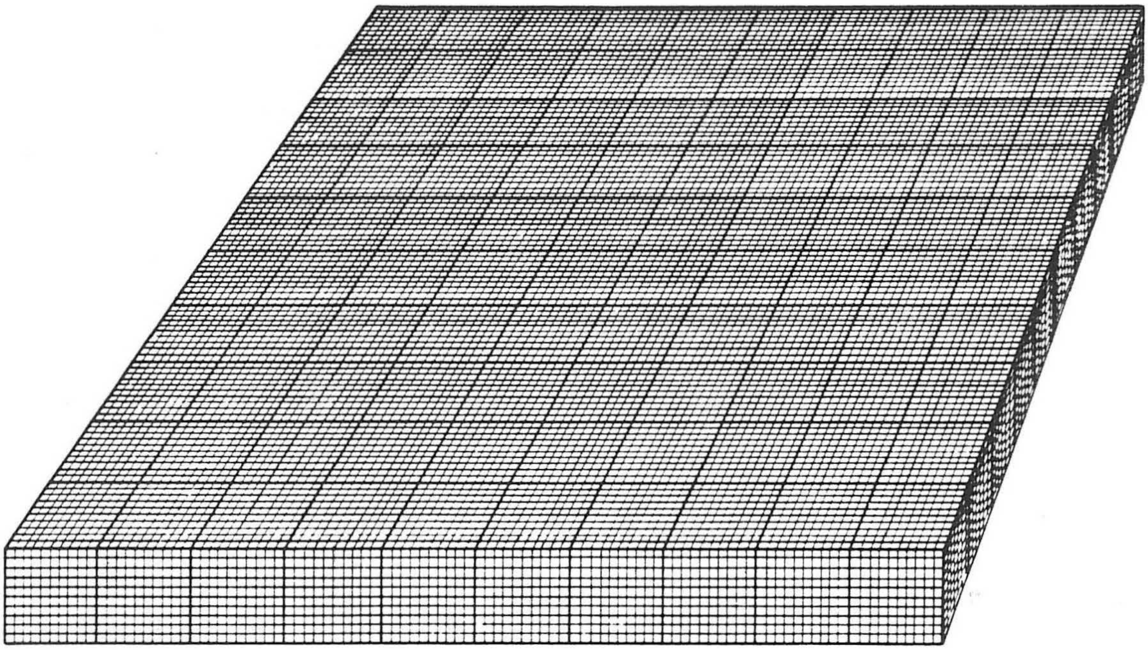
f. In 5-lb. cans, ham costs 1.05\frac{4}{5}$ a pound, and in 8-lb. cans, it costs $.97$\frac{3}{8}$ a pound which is .08$\frac{17}{40}$¢ less per pound.

g. All-white-meat turkey roasts cost $3.59 for 2 lbs. and 6 oz. This is $1.51 per pound (to the nearest cent). The light-and-dark-meat at $2.99 for 2 lbs. and 6 oz. costs $1.26 per pound (to the nearest cent).

h. Mrs. Swanson bought a turkey roast (light and dark), an 8-lb. can of ham, 4 lbs. of shrimp, 1 lb. and 12 oz. of bologna, and 4 lbs. of coffee. The total purchase price of these items is $18.52

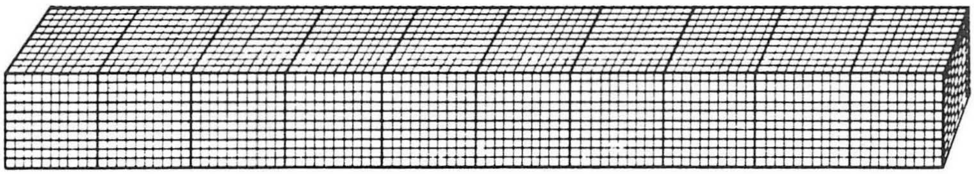
i. Mr. Powell bought eight 10-oz. packages of shrimp. It cost him $6.00. He bought 5 lbs. and 0 oz. of shrimp.

Ten layers of the tiny blocks . . . with 100 × 100 in each layer.



100,000 blocks . . . one hundred thousand of them . . . a tenth of the big stack.

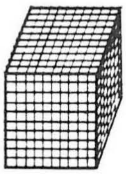
100 × 100 is 10,000. Again, you see only one side of some, two sides of others, and three sides of one. (When you have a little time to spare, you might like to determine how many of these hundred thousand blocks are showing in the picture.)



10,000 blocks . . . ten thousand of them . . . a hundredth of the big stack.

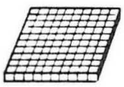
---

a thousandth of the big stack



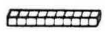
1,000 blocks

a ten-thousandth of the big stack



100 blocks

a hundred-thousandth of the big stack



10 blocks

a millionth of the big stack

1 block

Each sketch illustrates a stack that has 10 times as many blocks as the sketch that follows it . . . or, starting with the single block at the right, each sketch illustrates $\frac{1}{10}$ of the next larger stack of blocks.

Tasks: Students study various layers and sections of the stack of one million blocks shown on pupil page 6.

Purpose: To help students better understand the decimal system of numeration through pictorial representations. To help students develop a sense for making approximations and rounding numbers.

Unifying Ideas: Structure; number and counting; numeration; multiplication and division; functions and relations; geometry; measurement.

The Lesson: The following relationships are pictorially presented on pupil page 7 (with the help of the picture on pupil page 6).

100,000 → .1 of a million
10,000 → .01 of a million
1,000 → .001 of a million
100 → .0001 of a million
10 → .00001 of a million
1 → .000001 of a million

The smallest Cuisenaire® rod*—the white one—is 1 centimeter by 1 centimeter by 1 centimeter. If 100 were put in a row, the row would be 1 meter long. It would take 100 such rows to cover a square meter. It would require 100 such layers to fill a cubic meter, and that would be a total of $100 \times 100 \times 100$, or 1,000,000 blocks. While you will not have 1,000,000 such blocks around, you can stack up a few to indicate how large the pile would have to be to contain a million, a hundred thousand, a thousand, etc.

Every such demonstration and comparison with actual objects or sketches will help make large numbers more meaningful.

Students occasionally see a hundred things and are aware of the number. For example, they may often have counted 100 pennies. But what about 1,000 or 10,000 pennies?

You might obtain several kinds of graph paper and ask the students how many squares are on each sheet. One common variety has lines spaced $\frac{1}{10}$ inch apart, covering an 8-inch by 10-inch region; and $8 \times 10 \times 10 \times 10 = 8{,}000$. Another kind of graph paper has upwards of 10,000 tiny squares in a region the same size.

How many cubic feet in the classroom? (The answer will probably be in the thousands.) How many square inches on a football field? How many bricks in a particular building? How many pennies would be needed to fill a half-gallon milk carton?

In investigating and discussing such questions, students develop skill in making approximations and rounding numbers. Terms such as *about*, *in the neighborhood of*, and *approximately* arise naturally in these discussions. Thus students come to realize that we can ask many questions in arithmetic that are answered by approximation.

Practice sheet 7 introduces *chain addition*—the 2-at-a-time and 3-at-a-time versions. Two or three numbers are sufficient to start it off—and a careless error along the way will produce disastrous results.

The *Pattern Hunt* (of which there are many more to come) is an informal, gradual introduction to some fundamental notions of algebra.

*Cuisenaire—Registered trademark of Cuisenaire Company of America, Inc.

FPS-7

CHAIN ADDITION (2 at a time)

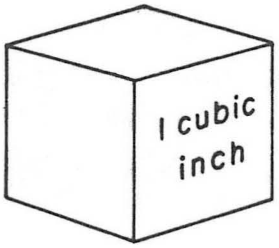
A	B	C	D	E	F	G	H	I
1	2	3	2	3	5	6	10	16
1	1	1	2	2	3	4	6	10
2	3	4	4	5	8	10	16	26
3	4	5	6	7	11	14	22	36
5	7	9	10	12	19	24	38	62
8	11	14	16	19	30	38	60	98
13	18	23	26	31	49	62	98	160
21	29	37	42	50	79	100	158	258
34	47	60	68	81	128	162	256	418
55	76	97	110	131	207	262	414	676
89	123	157	178	212	335	424	670	1094
144	199	254	288	343	542	686	1084	1770

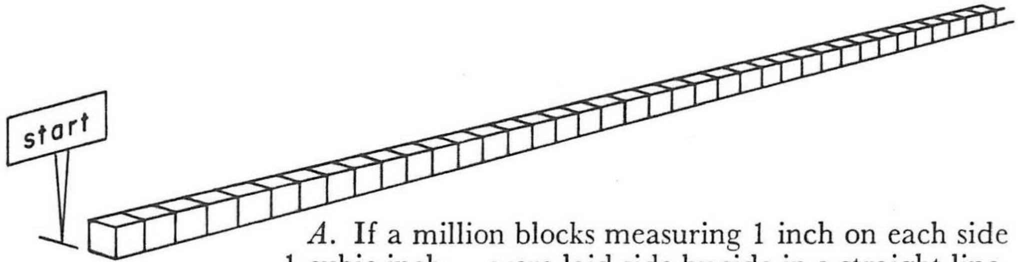
CHAIN ADDITION (3 at a time)

J	K	L	M	N	O	P	Q	R
2	2	5	7	5	7	6	9	14
1	2	3	6	6	5	5	8	26
3	3	4	5	7	6	7	7	53
6	7	12	18	18	18	18	24	93
10	12	19	29	31	29	30	39	172
19	22	35	52	56	53	55	70	318
35	41	66	99	105	100	103	133	583
64	75	120	180	192	182	188	242	1073

A PATTERN HUNT

I		II		III		IV	
3 →	8	7 →	14	1 →	3	2 →	5
0 →	5	23 →	46	2 →	6	5 →	11
10 →	15	0 →	0	10 →	30	7 →	15
5 →	10	1 →	2	5 →	15	50 →	101
7 →	12	19 →	38	8 →	24	17 →	35
19 →	24	178 →	356	19 →	57	29 →	59
87 →	92	1.4 →	2.8	.3 →	.9	0 →	1
346 →	351	.9 →	1.8	.4 →	1.2	.8 →	2.6
$3\frac{5}{7}$ →	$8\frac{5}{7}$	$\frac{1}{7}$ →	$\frac{2}{7}$	$\frac{1}{3}$ →	$\frac{3}{3}$ OR 1	$1\frac{1}{3}$ →	$3\frac{2}{3}$
$\frac{3}{8}$ →	$5\frac{3}{8}$	$\frac{1}{4}$ →	$\frac{2}{4}$ OR $\frac{1}{2}$	$\frac{2}{5}$ →	$\frac{6}{5}$ OR $1\frac{1}{5}$	$\frac{2}{5}$ →	$1\frac{4}{5}$
n →	n + 5	a →	2a	s →	3s	g →	2g + 1





A. If a million blocks measuring 1 inch on each side — 1 cubic inch — were laid side by side in a straight line, how far would they extend?

Which of the following would be your first guess as the closest to being correct?

500 feet; 5 miles; 15 miles; 100 miles; 770 miles

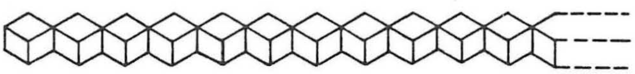
B. There are 5,280 feet in a mile. Can you find out which distance given above is closest to the correct answer? 15 MILES

C. If each one-cubic-inch block of a certain material weighs 1 ounce, how many tons would a million of those blocks weigh? (There are 16 ounces in a pound and there are 2,000 pounds in a ton.) 31^{+} TONS

Which of the weights below is closest to being the correct weight?

1 ton; $19\frac{1}{2}$ tons; $31\frac{1}{4}$ tons; 87 tons

D. If a million one-cubic-inch blocks were laid in the pattern below, how far would they reach? 23^{+} MILES



15 miles; 19 miles; 22 miles; 25 miles; 30 miles

E. How many running feet of 2″ × 4″ blocks can be cut from 1,000,000 cubic inches of wood? 96)1 MILLION = $10{,}416^{+}$ LIN. FT.

F. How many square feet could be covered by 1,000,000 cubes of wood that are 1″ by 1″ by 1″? 144)1 MILLION = $6{,}944^{+}$ SQ. FT.

G.

1,000,000 ÷ 2 = 500,000

1,000,000 ÷ 3 = $333{,}333\frac{1}{3}$

1,000,000 ÷ 4 = 250,000

1,000,000 ÷ 5 = 200,000

1,000,000 ÷ 6 = $166{,}666\frac{2}{3}$

1,000,000 ÷ 7 = $142{,}857\frac{1}{7}$

1,000,000 ÷ 8 = 125,000

1,000,000 ÷ 9 = $111{,}111\frac{1}{9}$

1,000,000 ÷ 10 = 100,000

1,000,000 ÷ 11 = $90{,}909\frac{1}{11}$

1,000,000 ÷ 12 = $83{,}333\frac{1}{3}$

1,000,000 ÷ 13 = $76{,}923\frac{1}{13}$

1,000,000 ÷ 14 = $71{,}428\frac{4}{7}$

1,000,000 ÷ 15 = $66{,}666\frac{2}{3}$

1,000,000 ÷ 20 = 50,000

1,000,000 ÷ 25 = 40,000

1,000,000 ÷ 75 = $13{,}333\frac{1}{3}$

1,000,000 ÷ 100 = 10,000

1,000,000 ÷ 500 = 2,000

1,000,000 ÷ 750 = $1{,}333\frac{1}{3}$

1,000,000 ÷ 1,000 = 1,000

8 Experiments with inch cubes

Tasks: Students plan strategies and answer questions related to weight, distance, and area involving very large numbers. They then plan a strategy for and do division exercises.

Purpose: To help students understand that they can save time in computation by planning a strategy for working with large numbers.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: One can save himself a lot of work if he decides ahead of time how precisely he must work. Question *A* calls for a pure guess; questions *B* and *C* require educated guesses based on approximate calculations; question *D* requires measurement and approximate calculations; questions *E* and *F* call for precise answers.

When making approximate calculations, it is always a good idea to decide ahead of time how precise the final answer should be. In question *B* the choices differ from their neighbors by a factor of 3. Therefore, we can use 5,000 as the approximate number of feet in a mile and obtain:

$$\begin{aligned} 12 \times 5{,}000 &= 60{,}000 \text{ (inches in 1 mile)} \\ 60{,}000 \times 10 &= 600{,}000 \text{ (inches in 10 miles)} \\ 600{,}000 \times 1\tfrac{1}{2} &= 900{,}000 \text{ (inches in 15 miles)} \end{aligned}$$

So, the closest choice must be 15 miles. Certainly not 5 miles nor 100 miles.

In *C* the two choices closest to each other are $19\frac{1}{2}$ tons and $31\frac{1}{4}$ tons—one almost $1\frac{1}{2}$ times the other. Again, the arithmetic can be kept simple:

$$\begin{aligned} 2{,}000 \times 16 &= 32{,}000 \text{ (ounces in 1 ton)} \\ 32{,}000 \times 10 &= 320{,}000 \text{ (ounces in 10 tons)} \\ 320{,}000 \times 3 &= 960{,}000 \text{ (ounces in 30 tons)} \end{aligned}$$

So the best choice is clearly $31\frac{1}{4}$ tons.

Question *D* requires a bit more care since the choices are reasonably close together. We can begin by drawing a square, 1 inch on each side, and measuring the diagonal; the result is between $1\frac{3}{8}$ inches and $1\frac{7}{16}$ inches. If we select the smaller measurement, our results will be on the low side. Now, look back to question *B:* Since $1{,}000{,}000 \times 1$ inch is a little more than 15 miles then $1{,}000{,}000 \times 1\frac{3}{8}$ inches is a little more than $1\frac{3}{8} \times 15$ miles, or a little more than $20\frac{1}{2}$ miles. Remember that 15 miles is only an approximation—and $20\frac{1}{2}$ miles is midway between two of the choices, 19 miles and 22 miles. This indicates a more precise approximation is needed. Have the students discuss what should be done next. Perhaps the $1\frac{3}{8}$-inch measurement should be improved by using a drawing of 10 squares diagonally arranged. Perhaps one ought to find out whether the approximation of 15 miles for 1,000,000 inches is a little high or a little low. Urge different students to try different methods and discuss the results. The discussions are of tremendous value, and correct answers will come out of them.

Questions *E* and *F* call for precise answers.

In the divisions called for in the right-hand column, consider grouping the problems: divide by 2, 4, and 8; then 3 and 6; then 5, 10, 20, 100, 500, and 1,000; 7 and 14; etc. This will suggest helpful shortcuts.

Chain subtraction on practice sheet 8 is again designed in part to focus on accuracy of computation, since a little slipup will grow with each succeeding step. Providing clues at different points in the chain requires educated trial and error—one important technique in solving problems.

FPS-8

A. CHAIN SUBTRACTION

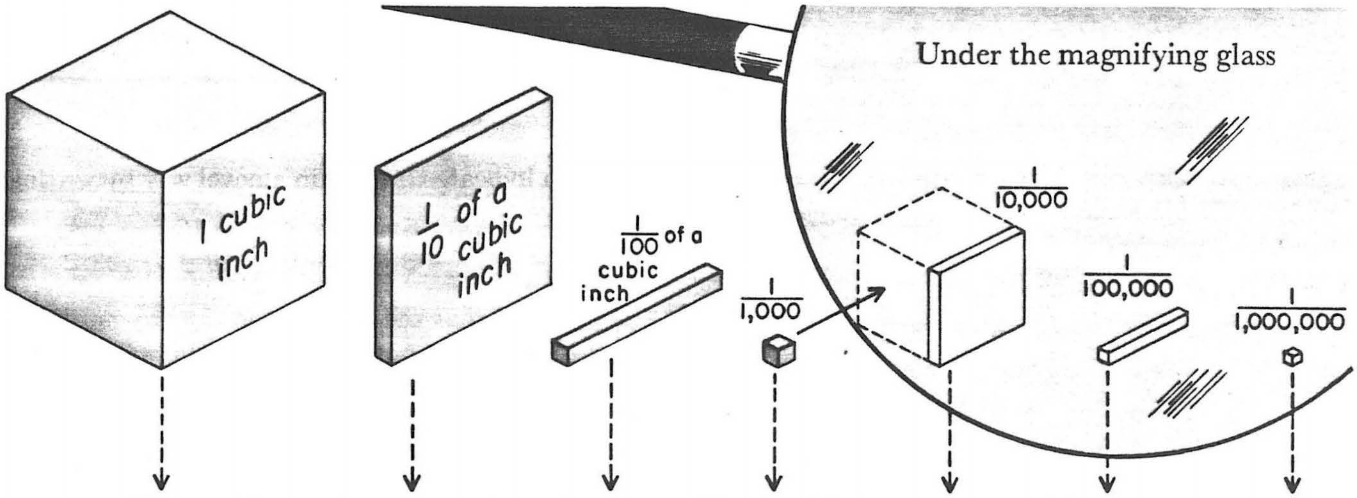
a	b	c	d	e	f	g	h	i
76	144	212	178	343	335	678	7640	8330
47	89	131	110	212	207	419	4722	5145
29	55	81	68	131	128	259	2918	3185
18	34	50	42	81	79	160	1804	1960
11	21	31	26	50	49	99	1114	1225
7	13	19	16	31	30	61	690	735
4	8	12	10	19	19	38	424	490
3	5	7	6	12	11	23	266	245
1	3	5	4	7	8	15	158	245
2	2	2	2	5	3	8	108	0

B. CHAIN SUBTRACTIONS . . . plus a bit of a puzzle

j	k	l	m	n	o	p	q	r	s
71	92	147	229	343	189	246	419	450	645
44	57	91	142	213	117	152	259	279	399
27	35	56	87	130	72	94	160	171	246
17	22	35	55	83	45	58	99	108	153
10	13	21	32	47	27	36	61	63	93
7	9	14	23	36	18	22	38	45	60
3	4	7	9	11	9	14	23	18	33
4	5	7	14	25	9	8	15	27	27

C. Another PATTERN HUNT

V	VI	VII	VIII
3 → 21	1 → 19	7 → 24	7 → 43
10 → 70	20 → 38	19 → 48	1 → 49
5 → 35	100 → 118	25 → 60	20 → 30
8 → 56	39 → 57	12 → 34	38 → 12
0 → 0	18 → 36	46 → 102	50 → 0
1.3 → 9.1	3.2 → 21.2	100 → 210	.1 → 49.9
.25 → 1.75	10.75 → 28.75	.3 → 10.6	1.1 → 48.9
9.1 → 63.7	.8 → 18.8	1.7 → 13.4	.25 → 49.75
19 → 133	.04 → 18.04	.5 → 11	.5 → 49.5
$\frac{1}{3} \to \frac{7}{3}$ or $2\frac{1}{3}$	$2\frac{3}{8} \to 20\frac{3}{8}$	$\frac{3}{4} \to 11\frac{1}{2}$	$7\frac{1}{2} \to 42\frac{1}{2}$
$\frac{2}{5} \to \frac{14}{5}$ or $2\frac{4}{5}$	$\frac{5}{18} \to 18\frac{5}{18}$	$1\frac{2}{3} \to 13\frac{1}{3}$	$18\frac{3}{8} \to 31\frac{5}{8}$
$h \to 7h$	$t \to t + 18$	$x \to 2x + 10$	$y \to 50 - y$



The weight of each piece above in ounces, if a 1-cubic-inch block weighs 1 ounce:

1	.1	.01	.001	.0001	.00001	.000001

Does one-millionth of a cubic inch seem very small? Each side would measure .01 of an inch. Do you think you could see it without a magnifying glass? YES

It might look like a grain of sand. But there are things much smaller. If this tiny piece, one-millionth of a cubic inch, were hollow, it could hold many bacteria — about 15 million of them. Molecules are much smaller than bacteria. Atoms are still smaller. Electrons are smaller still.

I. In arithmetic, *of* and *times* indicate the same operation. We also have several ways to write fractions. Complete the following:

a.	$\frac{1}{10}$ of $1 = \frac{1}{10}$	.1 of 1 = .1
b.	$\frac{1}{10}$ of $\frac{1}{10} = \frac{1}{100}$	.1 of .1 = .01
c.	$\frac{1}{10}$ of $\frac{1}{100} = \frac{1}{1{,}000}$	.1 x .01 = .001
d.	$\frac{1}{100} \times \frac{1}{100} = \frac{1}{10{,}000}$	.01 x .01 = .0001
e.	$\frac{1}{10} \times \frac{1}{1{,}000} = \frac{1}{10{,}000}$	.1 x .001 = .0001
f.	$\frac{2}{10} \times \frac{2}{10} = \frac{4}{100}$	.2 x .2 = .04
g.	$\frac{2}{10} \times \frac{6}{10} = \frac{12}{100}$	.2 x .6 = .12
h.	$\frac{5}{100} \times \frac{5}{100} = \frac{25}{10{,}000}$	.05 x .05 = .0025
i.	$\frac{3}{10} \times \frac{15}{1{,}000} = \frac{45}{10{,}000}$	.3 x .015 = .0045

II. Some ways to express the number 1:

a. .3 + .7	i. .1 x 10	q. .7 ÷ .7
b. 1.4 − .4	j. .19 ÷ .19	r. .92 + .08
c. 0 + 1.0	k. 1.012 − .012	s. .25 x 4
d. .2 x 5	l. .07 + .93	t. 10 x .1
e. 8 − 7	m. 100 x .01	u. 2.3 − 1.3
f. 100 x .01	n. .81 + .19	v. 93 ÷ 93
g. 2.81 − 1.81	o. 2 x .5	w. .16 + .84
h. 40 x .025	p. .007 + .993	x. .02 x 50

III. Some ways to express .1:

a. .2 ÷ 2	d. .04 + .06	g. 1.4 − 1.3
b. 1 x .1	e. .2 x .5	h. 1 ÷ 10
c. 100 x .001	f. .01 x 10	i. .011 + .089

IV. Some ways to express .01:

a. .01 + 0	e. .01 x 1	i. .8 − .79
b. .001 + .009	f. 10 x .001	j. 5 x .002
c. .133 − .123	g. .009 + .001	k. 4 x .0025
d. 2 x .005	h. .0099 + .0001	l. 100 x .0001

Tasks: Students consider the relative size and weight of a pictured row of blocks in which each block is one-tenth the size of its predecessor on the right. They then complete the mathematical sentences involving decimal notation for fractions.

Purpose: To help students understand fractions in decimal notation and to give them practice in adding, subtracting, multiplying, and dividing such fractions.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations; geometry; measurement.

The Lesson: We now turn our attention from very large numbers to very small numbers. The idea of using numbers to express smallness has already crept into the discussion. Each little block on pupil pages 6 and 7 was referred to as "one millionth of the stack." On pupil page 8, an inch was seen to be about a millionth of 15 miles; an ounce, about a millionth of 31¼ tons; etc.

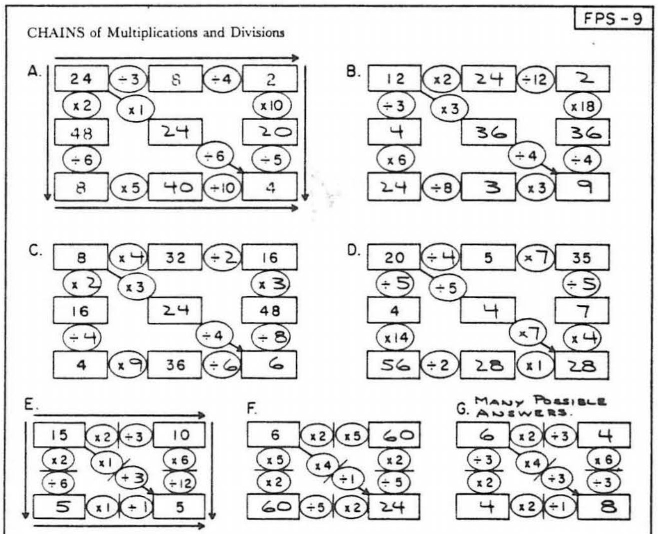
Could we see a grain of sand that is a millionth of a cubic inch? It would be .01 inch by .01 inch by .01 inch. How big is a hundredth of an inch? Some rulers are marked in thirty-seconds of an inch. A third of $\frac{1}{32}$ is $\frac{1}{96}$, and that's about a hundredth of an inch. The paper on which this book is printed is about half of a hundredth of an inch thick. So most people could easily see a grain of sand one millionth of a cubic inch in size.

Of course, there are things much smaller than that—so small that we cannot see them with the most powerful microscopes. An atom is about six sextillionths (decimal point followed by 20 zeroes and then a 6) of the size of a drop of water. Within the atom, an electron has about two thousandths of the mass of a proton or a neutron. The astronomer explores fantastic distances in space, and the physicist studies incredibly small bits of matter; both use appropriate numbers to report their results.

The problems on pupil page 9 will give the students exercise in handling computations involving decimals. They will also reveal any basic weakness and help you discover where reteaching may be required.

Practice sheet 9 introduces chains of multiplications and divisions. This material is reminiscent of a lengthy development in MATH WORKSHOP Level E in which a fraction was considered as a pair of operations—a multiplication and a division. (You may wish to look at that section in Level E: pupil pages 57 to 87.) We do not propose to repeat that discussion here. Rather, we wish to provide practice in multiplication in an activity that embraces certain aspects of problem solving.

For the first time in the *Pattern Hunt*, students are asked to complete the "pattern indicators" at the bottom of each example—another little step toward algebra. In example XII they may write $3 \times x$, $x \times 3$, $3 \cdot x$, or $x \cdot 3$—all of which are adequate. At your discretion, you may point out that mathematicians have decided to denote this product by simply writing $3x$.

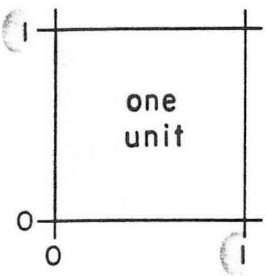


H. Another PATTERN HUNT

IX		X		XI		XII	
4 →	16	5 →	22	6 →	4	6 →	9
5 →	25	10 →	97	30 →	20	10 →	15
3 →	9	2 →	1	9 →	6	30 →	45
1 →	1	4 →	13	21 →	14	8 →	12
0 →	0	12 →	141	51 →	34	100 →	150
9 →	81	2.5 →	3.25	.6 →	.4	62 →	93
.4 →	.16	7.0 →	46	4.2 →	2.8	510 →	765
.3 →	.09	70 →	4,897	0.12 →	.08	3.0 →	4.5
1.2 →	1.44	100 →	9,997	900 →	600	.2 →	.3
$\frac{1}{2}$ →	$\frac{1}{4}$	$2\frac{1}{2}$ →	$3\frac{1}{4}$	$\frac{1}{2}$ →	$\frac{1}{3}$	$1\frac{1}{3}$ →	2
$\frac{3}{5}$ →	$\frac{9}{25}$	$3\frac{1}{3}$ →	$8\frac{1}{9}$	$\frac{1}{3}$ →	$\frac{2}{9}$	$\frac{1}{2}$ →	$\frac{3}{4}$
n → n x n or n^2		b → b^2 − 3		r → 2r ÷ 3		x → 3x ÷ 2	

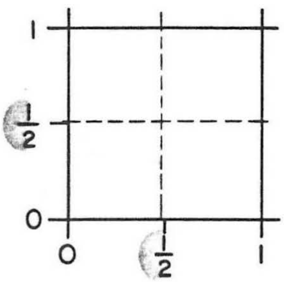
When in doubt, make a sketch. That's often a quick way to decide whether a statement in arithmetic is sensible or not.

A.



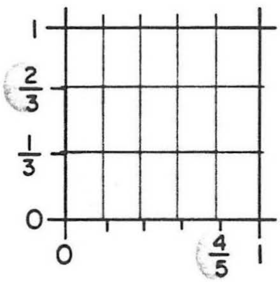
$1 \times 1 = 1$

B.



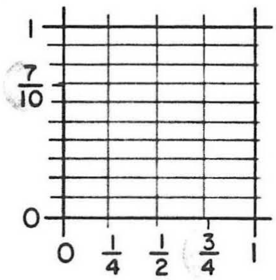
$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

C.



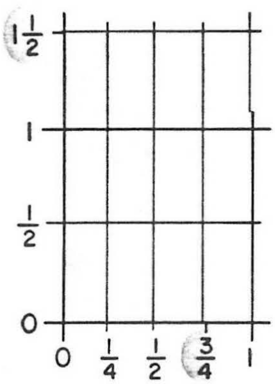
$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$

D.



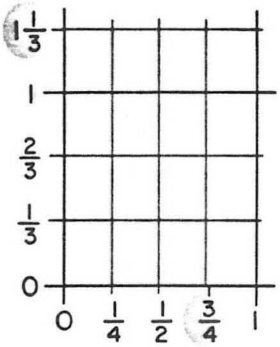
$\frac{7}{10} \times \frac{3}{4} = \frac{21}{40}$

E.



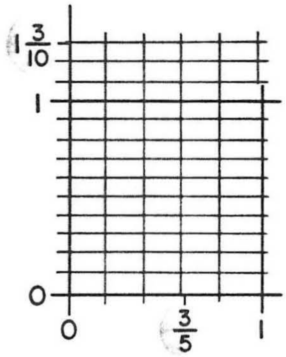
$\frac{3}{2} \times \frac{3}{4} = \frac{9}{8} = 1\frac{1}{8}$

F.



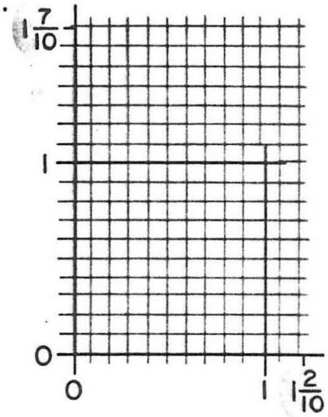
$\frac{4}{3} \times \frac{3}{4} = \frac{12}{12} = 1$

G.



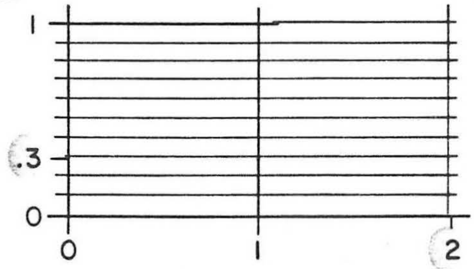
$\frac{13}{10} \times \frac{3}{5} = \frac{39}{50}$

H.



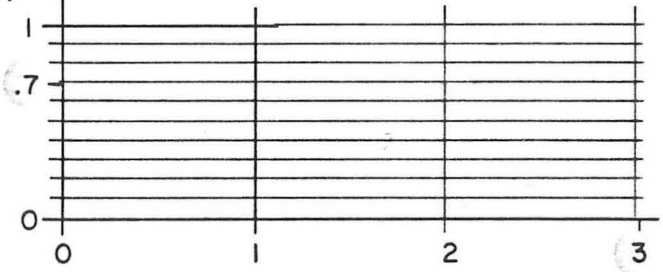
$\frac{17}{10} \times \frac{12}{10} = \frac{204}{100} = 2\frac{4}{100} = 2\frac{1}{25}$

I.



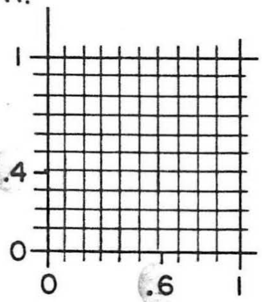
$.3 \times 2 = .6 \quad 2 \times .3 = .6$

J.



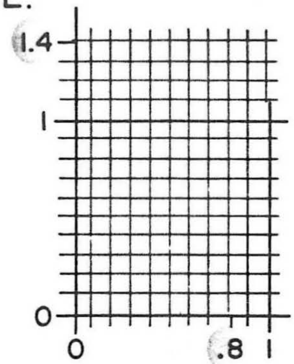
$.7 \times 3 = 2.1 \quad 3 \times .7 = 2.1$

K.



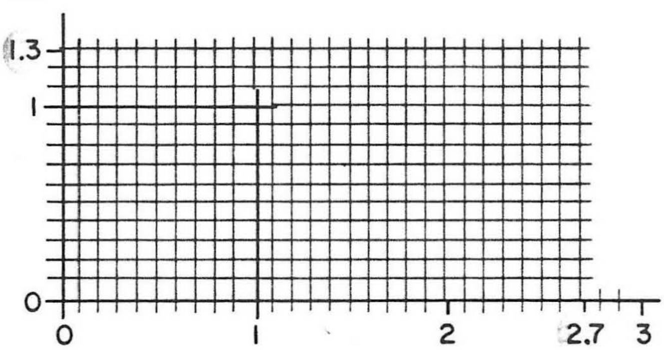
$.4 \times .6 = .24$

L.



$1.4 \times .8 = 1.12$

M.



$1.3 \times 2.7 = 3.51$

Tasks: Students complete mathematical statements involving multiplication of fractions suggested by sketches.

Purpose: To remind students that they can rely on sketches for representation when in doubt about how to proceed in solving problems that involve common and decimal fractions.

Unifying Ideas: Structure; number and counting; numeration; multiplication and division; geometry; measurement.

The Lesson: The lesson of this page is simple: when in doubt, make a sketch. If students have a good understanding of multiplication of common fractions and decimal fractions then this page can be skipped. If their understanding is faulty then you may need to expand on the idea presented here.

The examples given are by no means exhaustive. Our purpose is simply to provide one kind of diagram students can resort to when any doubt arises, a sketch that can be used to investigate multiplication of whole numbers, common fractions, and decimal fractions. Of course, there are other kinds of sketches, and we urge you to use any with which you are familiar.

We also hope that you will encourage students to rely on familiar areas of experience as valuable sources of clarification. Problems involving decimals can often be thought of in terms of money. A tenth of a dollar is a dime; a tenth of a dime is a penny; a hundredth of a dollar is a penny; a hundredth of ten dollars is a dime; three tenths of a dime is three pennies. In shorthand:

$\$1 \times .1 = \$.10$ or $\$.1$

$\$.10 \times .1 = \$.01$

$\$1 \times .01 = \$.01$

$\$10 \times .01 = \$.10$ or $\$.1$

$\$.10 \times .3 = \$.1 \times .3 = \$.03$

Thus students can fall back on sketches and on money when in doubt, rather than being helpless slaves to rules.

In practice sheet 10, more chains of multiplication and division are provided. Notice that in example *A* there are three ways to get from 120 to 60. We can indicate this fact in a novel way by writing:

$$\frac{\times 2}{\div 3} \times \frac{\times 6}{\div 8} = \frac{\times 1}{\div 5} \times \frac{\times 10}{\div 4} = \frac{\times 1}{\div 2}$$

If we agree that the numbers on top are multipliers and those on the bottom are dividers, we have:

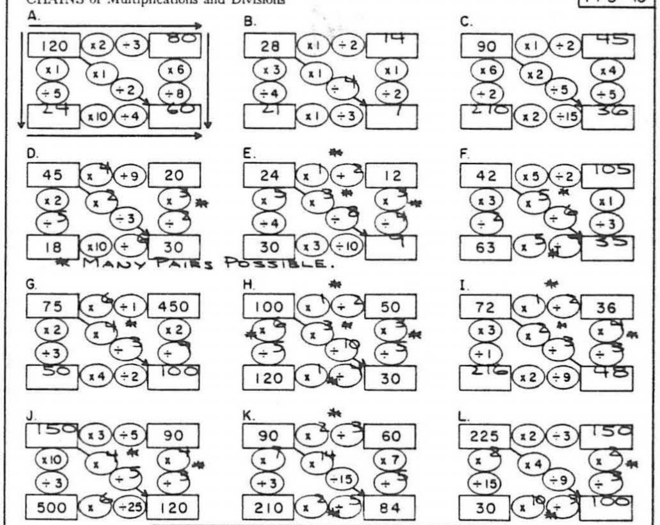
$$\frac{2}{3} \times \frac{6}{8} = \frac{1}{5} \times \frac{10}{4} = \frac{1}{2}$$

which is quite sound arithmetic. (You may wish to refer to Level E for development of this idea.)

Examples *A-C* can be correctly completed in exactly one way, but there are an unlimited number of ways to complete examples *D-L*. For instance, in example *E* we can use the numerator and denominator of any fraction equal to ½ to fill the first and second ovals in the top row.

In the *Pattern Hunt* notice that XIII and XIV show exactly the same pattern—their rules are equivalent. In XVI, a simpler statement of the rule would be $y \rightarrow 2y$, and this is the idea we are pointing to when later we say that $3y - y = 2y$.

CHAINS of Multiplications and Divisions FPS-10



M. Another PATTERN HUNT

XIII		XIV		XV		XVI	
7	→ 24	7	→ 24	10	→ 32	36	→ 72
0	→ 3	11	→ 36	5	→ 52	11	→ 22
5	→ 18	25	→ 78	18	→ 0	0	→ 0
15	→ 48	15	→ 48	1	→ 68	5	→ 10
80	→ 243	100	→ 303	15	→ 12	.5	→ 1
.7	→ 5.1	.12	→ 3.36	.5	→ 70	.05	→ .1
3.2	→ 12.6	.01	→ 3.03	.25	→ 71	75.8	→ 151.6
$\frac{1}{2}$	→ $4\frac{1}{2}$	$\frac{1}{3}$	→ 4	$\frac{1}{4}$	→ 71	$\frac{3}{4}$	→ $1\frac{1}{2}$
$\frac{3}{4}$	→ $5\frac{1}{4}$	$\frac{3}{4}$	→ $5\frac{1}{4}$	$\frac{1}{3}$	→ $70\frac{2}{3}$	$1\frac{2}{3}$	→ $3\frac{1}{3}$
$1\frac{2}{3}$	→ 8	$\frac{1}{100}$	→ $3\frac{3}{100}$	$1\frac{1}{2}$	→ 66	$\frac{5}{6}$	→ $1\frac{2}{3}$
n → 3n + 3		x → (x + 1) x 3		m → 72 − 4m		y → 3y − y	

A. Engineers usually use "decimal equivalents" in noting dimensions that include fractions of inches. They write halves, fourths, and eighths as thousandths. Often they write whole numbers with 3 zeros to the right of the decimal point. Odd 16ths and 32nds of an inch require 4 and 5 places. You can make your own table:

Decimal	Fraction
1.000	
.96875	$\frac{31}{32}$
.9375	$\frac{15}{16}$
.90625	$\frac{29}{32}$
.875	$\frac{7}{8}$
.84375	$\frac{27}{32}$
.8125	$\frac{13}{16}$
.78125	$\frac{25}{32}$
.750	$\frac{3}{4}$
.71875	$\frac{23}{32}$
.6875	$\frac{11}{16}$
.65625	$\frac{21}{32}$
.625	$\frac{5}{8}$
.59375	$\frac{19}{32}$
.5625	$\frac{9}{16}$
.53125	$\frac{17}{32}$
.500	$\frac{1}{2}$
.46875	$\frac{15}{32}$
.4375	$\frac{7}{16}$
.40625	$\frac{13}{32}$
.375	$\frac{3}{8}$
.34375	$\frac{11}{32}$
.3125	$\frac{5}{16}$
.28125	$\frac{9}{32}$
.250	$\frac{1}{4}$
.21875	$\frac{7}{32}$
.1875	$\frac{3}{16}$
.15625	$\frac{5}{32}$
.125	$\frac{1}{8}$
.09375	$\frac{3}{32}$
.0625	$\frac{1}{16}$
.03125	$\frac{1}{32}$

B. You have probably used the shorthand for "squaring" and, perhaps, "cubing" a number.

$7^2 = 7 \times 7 =$ 49 $\quad 7^3 = 7 \times 7 \times 7 =$ 343

$4^2 = 4 \times 4 =$ 16 $\quad 4^3 = 4 \times 4 \times 4 =$ 64

$2^2 = 2 \times 2 =$ 4 $\quad 2^3 = 2 \times 2 \times 2 =$ 8

$9^2 = 9 \times 9 =$ 81 $\quad 9^3 = 9 \times 9 \times 9 =$ 729

$10^2 = 10 \times 10 =$ 100 $\quad 10^3 = 10 \times 10 \times 10 =$ 1,000

$5^4 = 5 \times 5 \times 5 \times 5 =$ 625

$8^4 = 8 \times 8 \times 8 \times 8 =$ 4,096

$6^4 = 6 \times 6 \times 6 \times 6 =$ 1,296

$10^4 = 10 \times 10 \times 10 \times 10 =$ 10,000

$10^5 = 10 \times 10 \times 10 \times 10 \times 10 =$ 100,000

$10^6 = 10 \times 10 \times 10 \times 10 \times 10 \times 10 =$ 1,000,000

$10^7 =$ 10,000,000 $\quad 10^8 =$ 100,000,000

$10^9 =$ 1,000,000,000 $\quad 10^{10} =$ 10,000,000,000

$10^{15} =$ 1,000,000,000,000,000

Scientists use this shorthand because it helps indicate very large numbers in such a short space. They read 10^{15} as "ten to the 15th power" or simply as "ten to the 15th."

They extend this system, usually referred to as "scientific notation," to express numbers that are not powers of 10. Here is the plan:

$700 = 7 \times 100 = 7 \times 10^2$

$19{,}000 = 19 \times$ 1,000 $= 19 \times$ 10^3

$330{,}000 =$ 33 x 10,000 $=$ 33×10^4

$6{,}000 = 6 \times 10^3$ $\quad 2{,}400 =$ 24×10^2

$500{,}000 =$ 5×10^5 $\quad 31{,}000{,}000 =$ 31×10^6

$80{,}000 =$ 8×10^4 $\quad 30{,}000{,}000 =$ 3×10^7

Often the plan is extended in this way:

$1{,}200 = 12 \times 100 = 1.2 \times 1{,}000 = 1.2 \times 10^3$

$87{,}000 = 87 \times 1{,}000 = 8.7 \times 10{,}000 =$ 8.7×10^4

$12{,}500 = 125 \times 100 = 1.25 \times 10{,}000 = 1.25 \times 10^4$

$3{,}870 = 387 \times 10 = 3.87 \times 1{,}000 =$ 3.87×10^3

$15{,}000 = 1.5 \times$ 10^4

$7{,}180{,}000 = 7.18 \times$ 10^6

11 Decimal equivalents and scientific notation

Tasks: Students complete (a) a chart of decimal equivalents, and (b) mathematical statements involving the shorthand of exponents and powers of 10.

Purpose: To help students become familiar with tables of conversion and with exponents, part of the shorthand for writing very large and very small numbers.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations.

The Lesson: Pupil page 11 acquaints students with a conversion chart or list of decimal equivalents used almost universally by engineers who work in terms of English units of measure—feet, inches, etc. (Such conversion charts are, of course, unnecessary for engineers who work in the metric system.)

The remainder of pupil pages 11 and 12 provide a quick look at the scheme of scientific notation which is in use throughout the world—a shorthand for writing large and small numbers.

The system is based on use of exponents and powers of 10. It is most important that students distinguish clearly the difference between, for example,

$$7^2 \text{ and } 7 \times 2$$

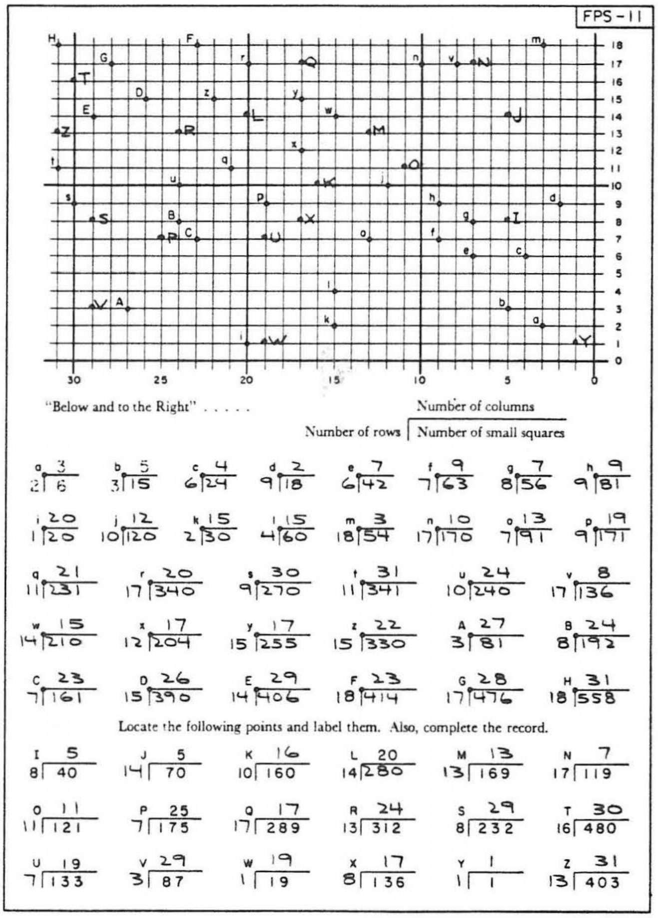
$$10^3 \text{ and } 10 \times 3$$

If the practice provided at the top of the right-hand column is not sufficient, take time out to explore the question more fully before proceeding. A clear understanding of this bit of shorthand is indispensable to the work that follows and will be required often in Level F material.

Newspapers have begun using a somewhat similar system of notation: instead of five billion or 5,000,000,000, they write 5 billion; instead of 3,700,000, they write 3.7 million. Urge students to keep an eye out for such usage, and introduce examples into class discussions.

We do not expect students to master scientific notation at this time; rather, this is an introduction. Scientific notation will arise in a very different context later in Level F.

Practice sheet 11 is a quickie. The arithmetic is purposely quite easy so that the basic notion of the relationship between multiplication and division is brought into focus. It is preliminary to several practice sheets that follow in which we reexamine a long-division algorithm with a rather novel approach.



Astronomers have observed stars that are

$$2{,}500{,}000{,}000{,}000{,}000 \text{ miles}$$

from the earth. We save a lot of time and space by writing this distance as:

$$25 \times 10^{14} \text{ miles, or } 2.5 \times 10^{15}$$

Just a brief note on still other extensions of this system.

Scientists often need to indicate very small numbers — such as a millionth of an inch.

$$.000001 \text{ or } \frac{1}{1{,}000{,}000}$$

They use negative exponents to indicate tenths, hundredths, etc.

$.1 = 10^{-1}$ $.01 = 10^{-2}$ $.001 = 10^{-3}$

$.0001 = 10^{-4}$ $.00001 = 10^{-5}$

$.000001 = 10^{-6}$

Scientists read 10^{-1}, 10^{-3}, etc., as "ten to the negative first, ten to the negative third," etc.

As you would expect, they extend this as follows:

$.7 = 7 \times .1 = 7 \times 10^{-1}$ $.08 = 8 \times .01 = 8 \times 10^{-2}$

$.012 = 12 \times .001 = 1.2 \times 10^{-2}$

$.00091 = 91 \times .00001 = 9.1 \times 10^{-4}$

To complete the pattern, we agree that any number to the zero power is 1.

$1{,}000{,}000 = 10^6$
$100{,}000 = 10^5$
$10{,}000 = 10^4$
$1{,}000 = 10^3$
$100 = 10^2$
$10 = 10^1$
$1 = 10^0$
$.1 = 10^{-1}$
$.01 = 10^{-2}$
$.001 = 10^{-3}$
$.0001 = 10^{-4}$

The earth is about 93,000,000 miles from the sun, or

$$9.3 \times 10^7 \text{ miles from the sun.}$$

The furthest planet from the sun, Pluto, is about 40 times as far.

$$40(9.3 \times 10^7) = 4(9.3 \times 10^8) = 3.72 \times 10^9$$

Saturn is only about 10 times as far from the sun as our earth is — or 10 × 93,000,000 miles.

$$10(9.3 \times 10^7) = 9.3 \times 10^8$$

Before they used radio telescopes, astronomers could see only about 30,000,000,000 stars, or

$$3 \times 10^{10} \text{ stars.}$$

One of the largest stars they could see has a diameter of 2,400,000,000 miles, or

$$2.4 \times 10^9 \text{ miles.}$$

Our moon varies in distance from the earth from between 221,000 to 253,000 miles. When it is 250,000 miles away, we express this distance as:

$$2.5 \times 10^5 \text{ miles.}$$

Uranus, another planet, has four moons. Uranus is 1,800,000,000 miles from our sun, or

$$1.8 \times 10^9 \text{ miles from the sun.}$$

Here is another scientific problem that requires a shorthand for writing large numbers:

How many atoms are there in the universe? Approximately

300,000,000,000,000,000,000,000,000,000,000,
000,000,000,000,000,000,000,000,000,000,000,
000,000,000,000,000,000 — or

$$3 \times 10^{74}$$

12 Astronomical numbers

Tasks: Students complete mathematical statements involving positive and negative exponents.

Purpose: To help students better understand the role that notation plays in arithmetic.

Unifying Ideas: Structure; number and counting; numeration; multiplication and division; functions and relations.

The Lesson: The idea of negative numbers and zero as exponents may seem strange to students, so we need to point out that we are only talking about a shorthand—another way of noting familiar ideas.

Some students who look at "10^4" and say "that's 1 with 4 zeroes after it" may be upset that "10^{-4}" is "1 with 3 zeroes in front of it." They may be happier to consider an intermediate step:

$$10^2 = 10 \times 10 = 100$$

$$10^{-2} = \frac{1}{10^2} = \frac{1}{100} = .01$$

$$10^{-4} = \frac{1}{10^4} = \frac{1}{10000} = .0001$$

You might revive the discussion suggested in the Notes on Pupil Page 5 regarding our placement of the decimal point to the right of the units place. We write:

$$10^2 = 100. \text{ and } 10^{-2} = .01$$

$$10^4 = 10000. \text{ and } 10^{-4} = .0001$$

Notice the symmetry developed when we use the other notation:

$$10^2 = 10\dot{0} \text{ and } 10^{-2} = \dot{0}01$$

$$10^4 = 1000\dot{0} \text{ and } 10^{-5} = \dot{0}0001$$

$$10^1 = 1\dot{0} \text{ and } 10^{-1} = \dot{0}1$$

Such discussions may help some students understand the role that notation plays in arithmetic.

Again, remember that this is a brief introduction of an idea; mastery will come later.

In almost any book on science, students can find extensive use of scientific notation.

Practice sheet 12 presents an activity called *From the Lists* that looks at the idea behind a long-division algorithm from a somewhat different point of view. It has the very distinct advantage of getting the multiplications done and out of the way so that many computational distractions do not arise. Further, as students first look at the problems, they may think only of a trial-and-error approach. As they proceed, they will of necessity develop a strategy very similar to the one embodied in the usual algorithm.

Notice exercise 5. We show how 9×67 and 6×67 might be selected; however, it is better strategy to select 10×67 and 5×67, though both methods will lead to the same result. Encourage variety—it is the sure sign of understanding.

FPS-12

From the Lists

Complete the lists as multiplication tables.

In the examples below, find numbers in the lists whose sum is indicated below the double line.

10	670
7	469
	1,139

List A x	67	List B x	67	List C x	67
0	0	0	0	0	0
100	6,700	10	670	1	67
200	13,400	20	1,340	2	134
300	20,100	30	2,010	3	201
400	26,800	40	2,680	4	268
500	33,500	50	3,350	5	335
600	40,200	60	4,020	6	402
700	46,900	70	4,690	7	469
800	53,600	80	5,360	8	536
900	60,300	90	6,030	9	603

1.

20	1,340
3	201
	1,541 *

2.

40	2,680
7	469
	3,149

3.

200	13,400
50	3,350
	16,750 *

4.

700	46,900
8	536
	47,436

5.

9	603
6	402
	1,005

6.

100	6,700
9	603
	7,303 *

7.

80	5,360
8	536
	5,896

8.

90	6,030
1	67
	6,097 *

9.

100	6,700
20	1,340
3	201
	8,241 *

10.

400	26,800
50	3,350
7	469
	30,619

11.

200	13,400
90	6,030
2	134
	19,564 *

12.

300	20,100
10	670
8	536
	21,306

13.

800	53,600
70	4,690
6	402
	58,692

14.

500	33,500
40	2,680
1	67
	36,247 *

15.

700	46,900
50	3,350
5	335
	50,585

16.

900	60,300
80	5,360
7	469
	66,129 *

Please find a shortcut for these examples of long division. (See the starred exercises above.)

a. $\begin{array}{r} 23 \\ 67\overline{)1{,}541} \end{array}$ b. $\begin{array}{r} 250 \\ 67\overline{)16{,}750} \end{array}$ c. $\begin{array}{r} 109 \\ 67\overline{)7{,}303} \end{array}$ d. $\begin{array}{r} 91 \\ 67\overline{)6{,}097} \end{array}$

e. $\begin{array}{r} 123 \\ 67\overline{)8{,}241} \end{array}$ f. $\begin{array}{r} 292 \\ 67\overline{)19{,}564} \end{array}$ g. $\begin{array}{r} 541 \\ 67\overline{)36{,}247} \end{array}$ h. $\begin{array}{r} 987 \\ 67\overline{)66{,}129} \end{array}$

Since the word "problem" can be used in many ways, the authors would like to explain what they mean when they use that word.

There are several ways in arithmetic of asking questions in shorthand.

$$\begin{array}{r} 33 \\ -18 \\ \hline 15 \end{array} \quad \begin{array}{r} 84 \\ +98 \\ \hline 182 \end{array} \quad \begin{array}{r} 78 \\ \times 5 \\ \hline 390 \end{array} \quad 13\overline{)234}^{\,18}$$

$35 + 7 = \underline{42}$ $31 + \underline{19} = 50$

$17 \times 9 = \underline{153}$ $\underline{896} \div 8 = 112$

$15 \times \underline{16} = 240$

Perhaps these questions or examples were, once upon a time, problems for you — and it would take lots of time to find correct answers. Now you can probably finish them in a few minutes — most of them without even using scratch paper. Try them.

But we consider those as nothing more than questions you need to be able to answer quickly . . . and when you're finished, you don't feel that you've learned a thing as a result of the effort.

Problems are not like that.

As you answered the first question — "33 − 18 = ?" — did you notice that both 33 and 18 are divisible by 3, and that the answer is, too.

Is the difference between any two multiples of 3 itself a multiple of 3?

That's more like the beginning of what we shall call a problem.

Let's try a few examples of finding the difference between multiples of 3:

$$\begin{array}{r} 30 \\ -12 \\ \hline 18 \end{array} \quad \begin{array}{r} 42 \\ -27 \\ \hline 15 \end{array} \quad \begin{array}{r} 75 \\ -36 \\ \hline 39 \end{array} \quad \begin{array}{r} 132 \\ -51 \\ \hline 81 \end{array}$$

Try other examples that you make up yourself. Can you say confidently that if a multiple of 3 is subtracted from a multiple of 3, the result is a multiple of 3? YES

Is there anything unusual about multiples of 3? Will multiples of 7 behave in much the same way?

$$\begin{array}{r} 21 \\ -14 \\ \hline 7 \end{array} \quad \begin{array}{r} 70 \\ -42 \\ \hline 28 \end{array} \quad \begin{array}{r} {}^{*}91 \\ -49 \\ \hline 42 \end{array} \quad \begin{array}{r} {}^{*}63 \\ -21 \\ \hline 42 \end{array}$$

* MANY RESPONSES POSSIBLE.

Would experiments with multiples of 5, 9, 12, 100, etc., strengthen your conclusion?

Does this same idea hold as well in addition, multiplication, and division as it does in subtraction? No (Yes or No) NOT IN DIVISION.

Now you have an illustration of what the authors call a problem. Work at it. It's not a very hard problem. You ought to settle it by a few minutes of experimenting. But even this simple problem may have a little surprising twist before you're done.

Bill had 12 marbles, Alex had 29, and Harvey had 17. How many do they have altogether?

That is nothing but a question. The answer is 58 marbles . . . but that's not a problem!

Here is another statement about boys and marbles:

Barney has some marbles. He has exactly 3 times as many as Al and exactly 4 times as many as Henry. They have less than 100 marbles among them. What can you say that isn't obvious?

That's a problem!

By "obvious," we might be referring to the fact that Barney has more marbles than either Al or Henry.

(continued on page 14)

13 The authors' meaning of "problem"

Tasks: Students study the explanation of the difference between examples and problems in arithmetic; then they do some examples and solve some problems.

Purpose: To help students understand that a problem is a situation which is open-ended and involves investigation rather than simple recall of known facts.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: A person's attitude toward mathematics depends on his understanding of what mathematical problems are and how he reacts when he encounters them.

Do problems arise when difficult and lengthy computations are called for? Is a mathematical problem created by long written sentences involving numbers (the typical Farmer Jones type of story problem)?

Anyone who gives an affirmative answer to these questions has a distorted view of mathematics. If such matters are the essence of mathematics then the mind demonstrates its health by rejecting such lack of challenge.

Rather, we have a problem when we can say, "Here is a situation—let us think about it. What can we say that isn't obvious?" In this type of situation, memory is relegated to its proper place as an essential tool, while the creative capacities of the mind are called into play.

On pupil pages 13, 14, and 15, the authors have suggested different kinds of problems. The first grows out of a collection of simple computations: Is the difference between any two multiples of 3 itself a multiple of 3? If the problem is not understood at once, a few examples will clarify it. The answer to the problem does not lie in the memory; the student's mind must call up example after example until it has enough evidence to warrant making a conclusion. And the joy of having succeeded under one's own power is an adequate reward for the effort required.

In the course of solving such a problem, other problems often suggest themselves, especially after one has acquired a little experience. Does the same result hold for multiples of other numbers? Does it hold for addition? For multiplication? For division? The discovery that it does not hold for division leads one to wondering why it falters at this point.

The best problems are usually the simplest to state, and lead naturally to an extension of the investigation.

Practice sheet 13 extends the activity called *From the Lists.* Since lists *A* and *B* are 100 times and 10 times the entries in list *C*, they are not shown in full. Urge students to discuss the strategies they have developed to avoid trial-and-error in finding entries whose sum is indicated. Someone will undoubtedly point to the similarity with the usual long-division algorithm.

FPS-13

From the Lists

We didn't have enough room for lists A and B. You can use a piece of scratch paper — if you need to.

A

x	23
0	0
100	2,300
200	4,600

B

x	23
0	0
10	230
20	460

C

x	23
0	0
1	23
2	46
3	69
4	92
5	115
6	138
7	161
8	184
9	207

1.

20	460
7	161
	621

2.

100	2,300
5	115
	2,415

3.

300	6,900
40	920
	7,820

4.

60	1,380
8	184
	1,564

5.

90	2,070
1	23
	2,093

6.

200	4,600
6	138
	4,738

7.

100	2,300
20	460
3	69
	2,829

8.

500	11,500
0	0
9	207
	11,707

9.

300	6,900
70	1,610
5	115
	8,625

10.

100	2,300
80	1,840
8	184
	4,324

11.

0	0
90	2,070
7	161
	2,231

12.

400	9,200
60	1,380
2	46
	10,626

13.

900	20,700
50	1,150
3	69
	21,919

14.

700	16,100
80	1,840
9	207
	18,147

A wholesale dealer had sent out a notice of a special offer:

Belts..........	23¢ each
Ties..........	46¢ each
Hats..........	69¢ each
Shirts..........	92¢ each

The orders came in fast. As shipments were made, a list was kept showing the customer's name and the total amount for any of the items he bought. The shipping clerk forgot to note the numbers of items. Can you help complete the record? (Notice that the price of each item is a multiple of 23¢.)

Brown:	27	Belts	$ 6.21
	103	Ties	$ 47.38
	85	Shirts	$ 78.20
Smith:	41	Hats	$ 28.29
	68	Belts	$ 15.64
Moore:	94	Ties	$ 43.24
	47	Shirts	$ 43.24
Powell:	97	Belts	$ 22.31
	231	Ties	$106.26

It's a little less obvious that Al has more than Henry does, and that Barney has more than Al and Henry put together.

But let's look more deeply into the situation. Could Barney have 7 marbles? . . . 11 marbles? . . . 18 marbles? What is the smallest number of marbles Barney could have? 12. In that case, how many would the boys have altogether? 19

What is the largest number of marbles Barney could have? 60

So, Barney must have 12 or 24 or 36 or 48 or 60 marbles.

Together the three boys must have 19 or 38 or 57 or 76 or 95 marbles.

That was not at all obvious at the outset. We have shed considerable light on the problem.

A Very Famous Problem

Can every even number 4 or larger be obtained by adding two prime numbers?

To understand this problem, we need to know what we mean by *every even number 4 or larger*. We can begin the list:

4, 6, 8, 10, 12, 14, 16, 18, etc.

and we can extend the list as far as we would care to.

We also need to know what *prime numbers* are. Each is larger than 1 and has exactly two different factors — itself and 1. Let's list all that are less than 100, in order:

2, 3, 5, 7, 11, 13, 17, 19, 23
29, 31, 37, 41, 43, 47, 53, 59, 61
67, 71, 73, 79, 83, 89 and 97

Next we set out to obtain numbers in the first list by adding two prime numbers (alike or different) from the second list:

4 =	2 + 2	28 =	5 + 23
6 =	3 + 3	30 =	13 + 17
8 =	3 + 5	32 =	3 + 29
10 =	3 + 7	34 =	3 + 31
12 =	5 + 7	36 =	5 + 31
14 =	3 + 11	38 =	7 + 31
16 =	5 + 11	40 =	3 + 37
18 =	7 + 11	42 =	5 + 37
20 =	7 + 13	44 =	3 + 41
22 =	5 + 17	46 =	3 + 43
24 =	7 + 17	48 =	5 + 43
26 =	7 + 19	50 =	3 + 47

* OTHER POSSIBLE PAIRS etc.

You may wish to extend the investigation further — to 100 — to 500 — as far as you like.

But, remember — we said it was a famous problem, and it's famous because no one knows the answer. No one has ever found an even number 4 or larger that couldn't be obtained by adding two primes. But not even the finest mathematicians have been able to prove that such an even number does not exist.

If you join the search, you will need lots of scratch paper.

One Problem Leads to Another

Such a good problem almost always suggests other lines of investigation.

As you were working, you probably noticed that some even numbers can be obtained by adding two prime numbers in only one way, and some in more than one way.

(8 = 3 + 5 but 20 = 3 + 17 = 7 + 13)

For each even number, we wish to find all possible pairs whose sum is the even number.

Let's start at 12.

14-15 A very famous problem - Can every even number 4 or larger be obtained by adding two prime numbers?

Tasks: Students investigate two problems, keeping records and completing lists for each problem.

Purpose: To help students develop strategies for tackling more difficult problems.

Unifying Ideas: Structure; addition and subtraction; functions and relations.

The Lesson: That a problem may be simple yet profound is dramatized on this page. To understand the problems on this page, one need only know what *even numbers* and *prime numbers* are. (You may wish to use practice sheet 14 to refresh the students' memory about prime numbers.)

Problem: *Can every even number 4 or larger be obtained by adding two prime numbers?*

Certainly this is a simple problem to state. (Most questions that might arise will be answered by the first few examples shown in the right-hand column.) Clearly you may repeat the same prime as with 4 (=2+2) and 6 (=3+3), but you may not need to repeat the prime as with 8 (=5+3). We explore all the even numbers from 4 through 50. Each one can be obtained by adding two primes.

But what about beyond 50? Are you now willing to answer yes to the main question?

Christian Goldbach (1690-1764) made the conjecture that the answer is yes, but no mathematician has been able to prove it. Of course, no one has found an exception (a counterexample), but no one has searched *all* the even numbers.

This is a fascinating problem to explore, and it involves a great deal of practice in computation. How much better it is than routine drill and isolated computational tasks that lead nowhere! Much more and much better practice is provided by investigations of significant mathematical problems.

An enormous amount of computation is required if we extend the problem to finding how many different combinations of primes can be used to obtain an even number. One must perform at least fourteen examples of additions (or subtractions) to discover the nine different ways in which 90 can be so obtained. (See pupil page 15.)

In making these explorations, students should be urged to limit the statements of their results to what they have demonstrated. If we have investigated all even numbers up to 100, we can say that all even numbers 100 or less but greater than 2 can be obtained by adding two primes; but this tells us nothing about even numbers greater than 100 in this respect.

Another conjecture by Goldbach was that every odd number larger than 9 can be expressed as the sum of three odd primes. (See the bottom half of practice sheet 14.) Mathematicians have had a little better success with this. In 1937, I. Vinogradoff, a Russian mathematician, proved this for odd numbers that were "sufficiently large." How large the numbers have to be, however, he could not decide.

FPS-14

Eratosthenes' Sieve

Cross out every multiple of 2, 3, 5, or 7
(except those already shown with a loop drawn around them).

1	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~
101	~~102~~	103	~~104~~	~~105~~	~~106~~	107	~~108~~	109	~~110~~
~~111~~	~~112~~	113	~~114~~	~~115~~	~~116~~	~~117~~	~~118~~	~~119~~	~~120~~

Draw a loop around each of the numbers that are not crossed out.

Numbers greater than 1 which are divisible only by themselves and by 1 are called *prime numbers.*

Can you be sure that all numbers greater than 1 and crossed out are non-prime numbers — *composite numbers?*

Can you be sure that all the numbers circled are prime numbers?

List in increasing order the prime numbers shown in the chart.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61,
67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, ...

Can you find the next three after 113?

Problem: Can all odd numbers greater than 5 and less than 102 be obtained by adding three prime numbers?

OTHER COMBINATIONS POSSIBLE EXCEPT *

* 7 = 2 + 2 + 3	39 = 3 + 7 + 29	71 = 5 + 7 + 59
9 = 2 + 2 + 5	41 = 5 + 7 + 29	73 = 7 + 7 + 59
11 = 2 + 2 + 7	43 = 3 + 3 + 37	75 = 3 + 5 + 67
13 = 3 + 3 + 7	45 = 3 + 5 + 37	77 = 3 + 7 + 67
15 = 2 + 2 + 11	47 = 3 + 7 + 37	79 = 5 + 7 + 67
17 = 2 + 2 + 13	49 = 3 + 3 + 43	81 = 7 + 7 + 67
19 = 3 + 5 + 11	51 = 3 + 5 + 43	83 = 3 + 7 + 73
21 = 2 + 2 + 17	53 = 3 + 7 + 43	85 = 5 + 7 + 73
23 = 2 + 2 + 19	55 = 5 + 7 + 43	87 = 7 + 7 + 73
25 = 3 + 3 + 19	57 = 7 + 7 + 43	89 = 3 + 3 + 83
27 = 3 + 3 + 21	59 = 3 + 3 + 53	91 = 3 + 5 + 83
29 = 3 + 3 + 23	61 = 3 + 5 + 53	93 = 3 + 7 + 83
31 = 3 + 5 + 23	63 = 3 + 7 + 53	95 = 5 + 7 + 83
33 = 3 + 7 + 23	65 = 5 + 7 + 53	97 = 7 + 7 + 83
35 = 2 + 2 + 31	67 = 7 + 7 + 53	99 = 3 + 7 + 89
37 = 3 + 3 + 31	69 = 3 + 7 + 59	101 = 5 + 7 + 89

The answer to the problem is YES. Are you a little curious about what happens with odd numbers greater than 102?

All pairs of primes:

12 = 5 + 7

14 = 7 + 7 = 3 + 11

16 = 5 + 11 = 3 + 13

18 = 5 + 13 = 7 + 11

20 = 3 + 17 = 7 + 13

22 = 5 + 17 = 3 + 19 = 11 + 11

24 = 5 + 19 = 7 + 17 = 11 + 13

26 = 3 + 23 = 7 + 19 = 13 + 13

You may think you see a pattern developing, but:

28 = 5 + 23 = 11 + 17

30 = 7 + 23 = 11 + 19 = 13 + 17

32 = 3 + 29 = 13 + 19

34 = 3 + 31 = 11 + 23 = 17 + 17 = 5 + 29

We skip a bit, and find only two ways to obtain 68:

68 = 7 + 61 = 31 + 37

and nine ways to obtain 90:

90 = 7 + 83 = 11 + 79 = 17 + 73
= 19 + 71 = 23 + 67 = 29 + 61
= 31 + 59 = 37 + 53 = 43 + 47

Perhaps you might be interested in exploring the number of ways other even numbers less than 100 can be obtained by adding two primes. Do you think the nine ways for 90 is a record? It is, but 84 and 86 are close seconds. There are eight ways to obtain each.

Another Problem

The authors have worked out all the pairs whose sums are even numbers up to and including 100.

We can report that all even numbers larger than 12 and through 100 can be obtained by adding two primes in at least 2 different ways.

We wonder: Is there an even number greater than 12 that can be obtained by adding two primes in only 1 way?

All we know is that if there is such a number, it must be larger than 100.

Our records also show that all even numbers 70 or larger can be obtained in at least 4 different ways by adding two prime numbers. The numbers 80, 88, and 98 just barely make the grade, so you might turn one up soon after 100.

What is the smallest even number larger than 100 that cannot be obtained by adding two primes in at least 4 different ways?

The chances of finding one seem encouraging when you realize that 68 turned up suddenly as a number for which there were only 2 ways of adding two primes.

Further, as we reported, our study shows that 90 can be obtained by adding two primes in 9 different ways. That's the record for even numbers 100 or less.

What is the first even number larger than 100 that ties the record, and what is the first even number that beats that record?

We hope you now know what the authors mean when they speak of problems. A good problem requires a lot of thinking and hard work. We hope you will work through some problems with us — and we hope you will tackle some on your own.

In the next chapter, we shall outline many problems. We hope you will pick the one you like best — and begin working at it. You may wish to expand the one you select, or take on others that interest you.

Tasks: Students investigate four problems related to the problems introduced on pupil page 14.

Purpose: To help students develop strategies for tackling difficult problems.

Unifying Ideas: Structure; addition and subtraction; functions and relations.

The Lesson: Pupil page 15 continues the discussion of Goldbach's conjecture and some of the problems it suggests.

Before the investigation is left, questions are asked about even numbers larger than 100. As the numbers become larger, it becomes important to devise a plan. Suppose that we begin a list of prime numbers larger than 50. (In order for the sum of two prime numbers to be more than 100, at least one of them must be larger than 50.)

	102	104	106	108		120
53	~~49~~	~~51~~	53	~~55~~		67
59	43	~~45~~	47	~~49~~		61
61	41	43	~~45~~	47		59
67	~~35~~	37	~~39~~	41		53
71	31	~~33~~	~~35~~	37		~~49~~
73	29	31	~~33~~	~~35~~		47
79	23	~~25~~	~~27~~	29		41
83	19	~~21~~	23	~~25~~		~~39~~
89	13	~~15~~	17	19		31
97	5	7	~~9~~	11		23
101	X	3	5	7		19
103		X	3	5		17
107				X		13
109						11
113						7
Different ways	(8)	(5)	(6)	(8)		(13)

In the chart we list the difference between the even number being studied and each prime number smaller than the even number and greater than 50. We then cross out the nonprimes in each column of differences and count how many we have left. The results are recorded at the bottom of each column. (We would find 7 ways to obtain 112; 10 ways for 114; 6 ways for 116; and 5 ways for 118.)

Now we are prepared to say that all even numbers from 100 to 120 can be obtained in at least 5 different ways; that 114 breaks 90's 9-way record; and that 120 tops them all with 13 different ways to obtain it by adding two prime numbers.

Notice that devising the chart leads to a valuable shortcut: each row turns out to be a sequence of consecutive odd numbers. For example, the first row is 49, 51, 53, 55, etc. So, after computing the left-hand column inside the chart, the work is much easier. Working out good ways to record results helps reveal patterns one might not otherwise see.

Practice sheet 15 reintroduces *From the Lists,* this time with larger numbers.

FPS-15

From the Lists

Please complete the three lists of multiples of 283.

In each of the examples below, select 3 numbers from the lists so that their sum in each case is the number indicated under the double line.

List A x	283	List B x	283	List C x	283
0	0	0	0	0	0
100	28,300	10	2,830	1	283
200	56,600	20	5,660	2	566
300	84,900	30	8,490	3	849
400	113,200	40	11,320	4	1,132
500	141,500	50	14,150	5	1,415
600	169,800	60	16,980	6	1,698
700	198,100	70	19,810	7	1,981
800	226,400	80	22,640	8	2,264
900	254,700	90	25,470	9	2,547

1.	2.	3.	4.
28,300	56,600	226,400	141,500
8,490	16,980	0	25,470
1,415	2,264	1,981	566
38,205	75,844 *	228,381	167,536

5.	6.	7.	8.
113,200	169,800	0	254,700
8,490	22,640	19,810	25,470
2,264	0	1,981	2,547
123,954 *	192,440	21,791	282,717 *

In the following, select one number from each list as indicated.

	9.	10.	11.	12.
A	0	84,900	0	28,300
B	5,660	11,320	19,810	0
C	283	0	2,547	1,981
	5,943	96,220	22,357	30,281

	13.	14.	15.	16.
A	113,200	198,100	28,300	226,400
B	2,830	5,660	22,640	5,660
C	1,698	566	1,132	849
	117,728	204,326	52,072	232,909

Please find a shortcut for each of these long division examples. (Notice the starred examples above.)

17. $283 \overline{)75{,}844}$ = 268

18. $283 \overline{)123{,}954}$ = 438

19. $283 \overline{)282{,}717}$ = 999

We have a suggestion for you as you start this chapter.

We have outlined many problems. It is our hope that you will find some that interest you particularly.

You will probably have time to work at only a few of these problems, so select carefully.

Read through the chapter as rapidly as you can. Don't try any of the examples. Such a skimming will give you an idea about its contents.

Then read through it again with pencil and paper in hand. Spend a few minutes working at any problems that interest you at all. Try your hand at all of them — just to get a little feeling about each of the problems.

Finally, select your first problem, and begin working at it.

Problem I

Bill said he was looking for combinations of 50 common U.S. coins with a total value of \$5.00 — without using dimes — using only silver dollars, halves, quarters, nickels, and pennies. He found one:

6 halves	\$3.00
6 quarters	1.50
3 nickels	.15
35 pennies	.35
50 coins	\$5.00

Bill said that was the only combination he could find.

Can you find other combinations? If so, how many others can you find? If you can't find others, can you give a reason to believe that there are no others?

Problem II*a*

Can you arrange the 10 digits — 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 — to form whole numbers whose sum is exactly 100, using each digit once and only once?

Here is an attempt that's 1 short:

$$39 + 8 + 27 + 10 + 4 + 6 + 5 = 99$$

Here's another:

$$0 + 8 + 16 + 25 + 7 + 31 + 4 + 9 = 100$$

but the digit 1 was used twice.

Keep records of your attempts. Which numbers are easy to obtain as sums?

As you begin, remember that every digit must be used *once and only once.*

And if you get the feeling it is impossible to obtain 100 then try to find a reasonable argument to convince yourself and others you are right.

Problem II*b*

Perhaps you would like to try an easier version of the problem. Use every digit once and only once, but you may use subtraction as well as addition and you must keep the digits in order.

Here are two successful attempts:

$$98 + 7 + 6 - 5 - 4 - 3 + 2 - 1 + 0 = 100$$
$$0 + 1 + 2 + 34 - 5 + 67 - 8 + 9 = 100$$

How many more can you find?

Problem II*c*

Or, if you are having a hard time, remove the requirement that the order be kept:

$$68 - 27 + 59 + 0 - 1 - 3 + 4 = 100$$

Can you find at least 5 different ways?

Tasks: Students try their hand at each problem and choose one or more that interest them for further study.

Purpose: To help students think about and develop strategies for the solution of problems.

Unifying Ideas: Structure; sets; numeration; addition and subtraction.

The Lesson: The authors' purpose in Chapter Three is to help students consider a variety of mathematical problems. We are eager for students to work on the problems that interest them most. Hopefully, these problems will suggest related questions or lead to the invention of new problems that they will wish to investigate on their own.

None of these problems is based on tricks! None is easy. Each will require considerable time, work, and study. Many are open-ended, leading on as far as time and interest allow.

Students need to realize at the outset that they are not expected to solve all of these problems. It is better for a student to do a thorough job on one problem than slipshod work on several.

You may decide to have groups of students cooperating on some of the work.

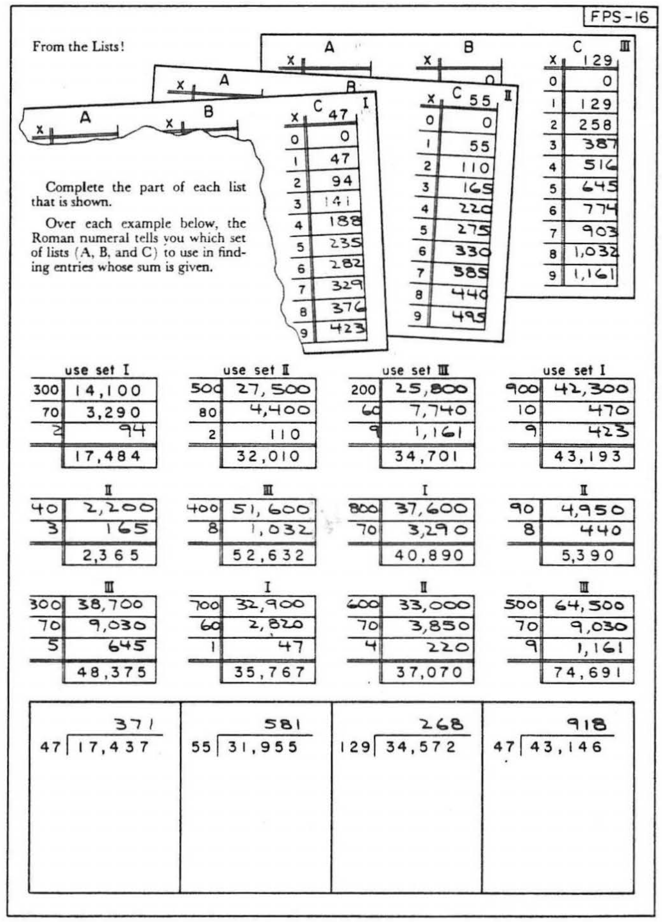
Some students might look toward a result that could be entered in a Science Fair or similar competition. In any event, the work ought to lead to some kind of report that will be of interest to others.

How much should the teacher know? Many of the problems outlined in this chapter will be unfamiliar to teachers. For this reason, students may well come up with methods and results that you do not anticipate. In such cases, allow them to explain their findings—and to defend their conclusions—to the class.

Before individual problems are selected, we hope sufficient time will be allowed for a brief discussion of each.

In this chapter, we will suggest some approaches to some of the problems.

Practice sheet 16 is a further extension of *From the Lists*. Notice that, for each example in the top row, the sum of the numbers in the left-hand column is 1 more than the quotient of the corresponding division example at the bottom.


FPS-16

From the Lists!

Complete the part of each list that is shown.

Over each example below, the Roman numeral tells you which set of lists (A, B, and C) to use in finding entries whose sum is given.

x	C 47 (I)	C 55 (II)	C 129 (III)
0	0	0	0
1	47	55	129
2	94	110	258
3	141	165	387
4	188	220	516
5	235	275	645
6	282	330	774
7	329	385	903
8	376	440	1,032
9	423	495	1,161

use set I	use set II	use set III	use set I
300 14,100	500 27,500	200 25,800	900 42,300
70 3,290	80 4,400	60 7,740	10 470
2 94	2 110	9 1,161	9 423
17,484	32,010	34,701	43,193

II	III	I	II
40 2,200	400 51,600	800 37,600	90 4,950
3 165	8 1,032	70 3,290	8 440
2,365	52,632	40,890	5,390

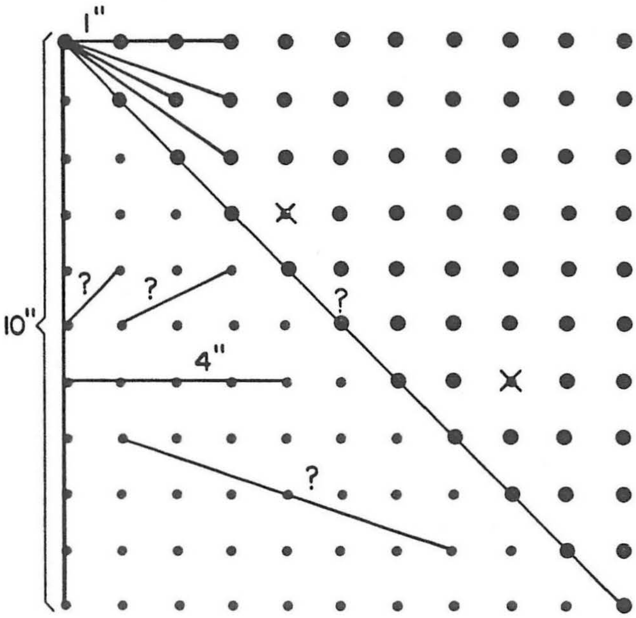
III	I	II	III
300 38,700	700 32,900	600 33,000	500 64,500
70 9,030	60 2,820	70 3,850	70 9,030
5 645	1 47	4 220	9 1,161
48,375	35,767	37,070	74,691

371	581	268	918
47 ⟌ 17,437	55 ⟌ 31,955	129 ⟌ 34,572	47 ⟌ 43,146

Problem III

On a piece of graph paper, locate points one inch apart in an 11-by-11 array.

Measuring with a ruler, how many different lengths of line segments can you draw that connect two points?



The shortest line segment will be 1″ long. The longest line segment will be a diagonal of the array — a little more than $14\frac{1}{8}$ inches.

You will need to plan your work ahead. How will you know when you've got all of the different lengths? Perhaps it will help to have several 11-by-11 arrays so you won't have too many segments on each drawing.

Would you like to guess before you begin about how many can be found?

Problem IV

Try to find a number which when multiplied by itself gives 2.

We start by knowing that no whole number will work.

1 x 1 = 1 (too small)
2 x 2 = 4 (too large)

So, our number is some place between 1 and 2. We guess at 1.3 and go to work.

1.3 x 1.3 = 1.69 (too small)
1.4 x 1.4 = 1.96 (too small)
1.5 x 1.5 = 2.25 (too LARGE)

Now we know that the number is some place between 1.4 and 1.5. Keep closing in from both sides, step by step, decimal place by decimal place. How far must you go? Do you think it is possible?

Next, try to find a number which when multiplied by itself gives 3. Then try 4, 5, 6, 7, 8, 9, and 10. What are your conclusions?

Problem V

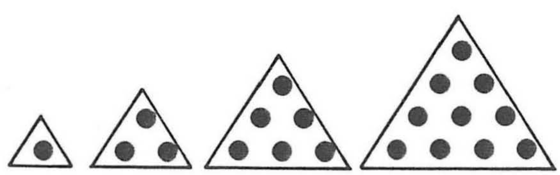
Make a chart showing which letters of the alphabet are used most often and which least often.

Consider noting the frequency of all letters in a newspaper story, the Gettysburg Address, the Pledge of Allegiance to the Flag, a page in a history book — any reading matter available.

If you enjoy this kind of problem, you may be interested in reading *Goldbug* by Edgar Allan Poe. The code for a map of buried treasure is broken by using information you will have uncovered.

Problem VI

You may have already met a sequence of numbers called *triangular numbers*.



etc.

1 3 6 ___

You might go on drawing sketches to extend

(continued on page 18)

Tasks: Students try their hand at each problem and choose one or more that interest them for further study.

Purpose: To help students think about and pursue strategies for the solution of open-ended problems.

Unifying Ideas: Structure; sets; number and counting; multiplication and division; functions and relations; geometry.

The Lesson: Here are a few suggestions that can serve as thought-starters with some of the problems:

Problem I. As you think about this problem (pupil page 16), you realize that if any pennies get into a combination, the number of pennies must be a multiple of 5. If there are no pennies, the number of nickels must be a multiple of 5. There must be some nickels. They cannot all be nickels, so there must be no more than 45 nickels. Since 35 nickels ($1.75) leaves $3.25 to be accounted for with 15 coins each a quarter or larger, there cannot be as few as 35 nickels. Here is the way we might record an investigation along these lines:

D	H	Q	N	P	after nickels & pennies, the problem is
2	0	3	45	0	$2.75 in 5 coins
0	2	8	40	0	$3.00 in 10 coins

And that finishes the problem if pennies are excluded. The greatest number of pennies that can be used is 40. If 45 were used, that would leave $4.55 to be made up with 5 coins. That can't be done. So the chart might resume:

D	H	Q	N	P	after pennies, this problem is
3	1	4	2	40	$4.60 in 10 coins
2	4	2	2	40	
				35	$4.65 in 15 coins

Problem II. Considerable light will be shed on these problems (pupil page 16) if a record is kept of all unsuccessful attempts. This problem provides an unusual insight into the place-value principle used in the decimal system. (See *Discovery in Elementary School Mathematics* by Wirtz, Botel, and Nunley, published by Encyclopaedia Britannica Press; pages 16-20.)

Problem III. If one selects a corner of the grid and draws segments to each of the circled dots (as indicated on the answer key), he will have found every possible length. Notice the two points with cross marks on them; they give exactly the same length as the 5-unit and 10-unit segments, respectively. (A little acquaintance with the Pythagorean Theorem would help considerably here. Each of the segments except those drawn vertically or horizontally can be seen as the hypotenuse of a right triangle, each of whose other sides is a discernible number of units of length.)

Practice sheet 17 introduces *Fun with Factors.* Many shortcuts of a general nature may be found that students can add to their arsenal of time-savers.

FPS-17

Fun with Factors

(You had better keep looking for shortcuts. If you do, you won't need scratch paper.)

2 x 3 = 6 2 x 3 x 5 = 30
2 x 3 x 5 x 7 = 210
4 x 3 x 5 x 7 = 420
4 x 6 x 5 x 7 = 840
4 x 6 x 10 x 7 = 1,680
4 x 6 x 10 x 14 = 3,360
4 x 2 x 10 x 14 = 1,120
2 x 2 x 2 x 14 = 112

1 x 2 x 3 = 6
2 x 2 x 3 = 12
2 x 2 x 4 = 16
2 x 2 x 5 = 20
3 x 2 x 5 = 30
4 x 2 x 5 = 40
4 x 3 x 5 = 60
4 x 4 x 5 = 80
4 x 5 x 5 = 100
4 x 4 x 6 = 96
8 x 4 x 3 = 96
8 x 2 x 6 = 96
9 x 2 x 6 = 108
9 x 4 x 6 = 216
7 x 5 x 6 = 210
8 x 5 x 6 = 240
8 x 5 x 12 = 480
8 x 4 x 12 = 384
8 x 3 x 12 = 288

2 x 3 x 3 = 18
4 x 3 x 3 = 36
4 x 3 x 6 = 72
4 x 6 x 6 = 144
4 x 6 x 7 = 168
4 x 3 x 7 = 84
2 x 3 x 7 = 42
3 x 3 x 7 = 63
6 x 3 x 7 = 126
6 x 6 x 7 = 252
5 x 7 x 7 = 245
5 x 6 x 8 = 240
5 x 6 x 4 = 120
5 x 3 x 4 = 60
5 x 3 x 2 = 30
5 x 9 x 6 = 270
5 x 11 x 5 = 275
5 x 8 x 7 = 280
6 x 8 x 7 = 336

2 x 3 x 5 x 7 x 11 = 2,310
2 x 2 x 3 x 5 x 5 x 7 x 11 = 23,100
3 x 7 x 11 = 231
6 x 7 x 11 = 462
12 x 7 x 11 = 924
13 x 7 x 11 = 1,001
7 x 11 x 13 x 17 = 17,017
3 x 7 x 11 x 13 x 17 = 51,051
2 x 3 x 5 x 7 x 11 x 13 x 17 = 510,510
2 x 3 x 5 x 7 x 11 x 13 = 30,030
3 x 77 = 231
7 x 33 = 231
11 x 21 = 231
2 x 5 x 11 x 21 = 2,310
2 x 5 x 11 x 42 = 4,620

List the prime numbers in order to 101: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101

Are the results of the following examples prime (P) or not (N)?

(a) 2 x 3 + 1 = 7 P N
(b) 2 x 3 x 5 + 1 = 31 P N
(c) 2 x 3 x 5 x 7 + 1 = 211 P N
(d) 2 x 3 x 5 x 7 x 11 + 1 = 2,311 P N
(e) 2 x 3 x 5 x 7 x 11 x 13 + 1 = 30,031 P N

The results in examples *a* and *b* are in the list of primes written above. But is the result in example *c* prime?

Since it is smaller than 15 × 15, one of its factors must be smaller than 15. Can you explain why? If it were divided by 2, 3, 5, or 7, it would have a remainder of 1, so none of these divides it exactly. How about 4, 6, 8, 9, 10, 12, and 14? If none is a factor, then the number must be prime. (None can be. Why?)

To test the results in examples *d* and *e* you might have to try dividing by prime numbers up to 48 and 179, respectively. It might take a very long time.

this list — or you can extend the following:

$$1 = 1$$
$$3 = 1 + 2$$
$$6 = 1 + 2 + 3$$
$$10 = 1 + 2 + 3 + 4$$

Or, extend this sequence, noting the differences between each two terms:

1, 3, 6, 10, 15, 21, 28, 36 etc.

2 3 4 5 6 7 8

Mary claims that she can obtain every whole number through 50 by adding not more than three of these triangular numbers. (A number may be repeated as one of the three or fewer triangular numbers.)

Do you agree with Mary?

She starts her list in this way:

1 = 1	7 = 6 + 1
2 = 1 + 1	8 = 6 + 1 + 1
3 = 3	9 = 6 + 3
4 = 3 + 1	10 = 10
5 = 3 + 1 + 1	11 = 10 + 1
6 = 6	12 = 6 + 6

Do you believe Mary can keep this up all the way to 50?

How far beyond 50 can you go with this list?

Can some numbers be obtained in more than one way by adding three or fewer triangular numbers?

Yes. For example:

30 = 10 + 10 + 10 = 15 + 15 = 21 + 6 + 3 = 28 + 1 + 1

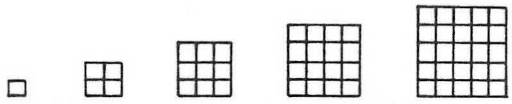
Which numbers can be obtained in only 1 way? Which can be obtained in 2, 3, 4, or more ways?

Investigate these or your own questions. Keep notes of interesting information you uncover.

Problem VII

Mary had another claim.

You are familiar with *square numbers*. They are:



1 4 9 16 25

(diff.) 3 5 7 9

They continue:

25, 36, 49, 64, 81, 100 etc.

11 13 15 17 19

Mary's second claim is that she can obtain every whole number less than 50 by adding not more than four square numbers.

She tried to do it with three square numbers, but stumbled on 7:

7 = 4 + 1 + 1 + 1

and again on 15:

15 = 9 + 4 + 1 + 1

At 28, she noted that she needed four again, but found 3 different ways to do it:

28 = 25 + 1 + 1 + 1 = 16 + 4 + 4 + 4 = 9 + 9 + 9 + 1

When asked why she didn't carry her study beyond 50, she said that she was more interested in another problem involving square numbers.

What interesting developments can you find in checking Mary's claim and extending it to 100?

What numbers more than 20 and less than 100 can be obtained in only one way as the sum of four or fewer square numbers? 32, 56, 96

18 Problems VI-VII - triangular and square numbers

Tasks: Students try their hand at each problem and choose one or more that interest them for further study.

Purpose: To help students think about and pursue strategies for the solution of open-ended problems involving the sums of triangular and square numbers.

Unifying Ideas: Structure; sets; addition and subtraction; functions and relations; geometry.

The Lesson:

Problem IV. Students try to find a decimal fraction whose square is 2. No matter how far the search is carried (pupil page 17), it will fall short of success. Failure is also in store when we try to find a decimal fraction whose square is exactly 3, 5, 6, 7, 8, or 10. The numbers 1, 4, and 9 are the only exceptions among the whole numbers less than 11.

Problem V. Suggest that someone look in a reference book under *typesetting machines* for a picture or diagram of a keyboard. A good look at one should help explain why misprints in newspapers often show such sequences of letters as *etaoin* and *shrdlu*.

Problems VI and VII. Mathematicians have proved that every whole number can be obtained by adding three or four triangular numbers and by adding four or fewer square numbers.

We do not attempt such generality here, but consider only whole numbers up to 50 (Problem VI) and up to 100 (Problem VII). Some students may wish to extend their investigation to larger whole numbers.

In each problem, the question is raised as to how many ways a number can be obtained in the manner specified. Surprises such as the following turn up:

> 90 can be obtained as the sum of four or fewer squares in 9 different ways;
> 96 can be obtained as the sum of four or fewer squares in only 1 way: $64+16+16$.

(See *Discovery in Elementary School Mathematics*, pages 25-28.)

How does one know when he has found all the different ways to obtain a number in a specified way? This problem will lead to a time-consuming trial-and-error process, but many students will invent ways of tackling the problem systematically.

Encourage interested students to invent similar problems. Can every number 2 or larger be obtained by adding two or fewer numbers, both of which are different triangular numbers, both different square numbers, or one of each?

$2 = 1+1\ (\square,\triangle)$	$7 = 4+3\ (\square,\triangle)$
$3 = 3\ (\triangle)$	$8 = ?$
$4 = 4\ (\square)$	$9 = 9\ (\square)$
$5 = 4+1\ (\square,\triangle$ or $\square,\square)$	$10 = 10\ (\triangle)$
$6 = 6\ (\triangle)$	$11 = 10+1\ (\triangle,\square$ or $\triangle,\triangle)$

What is the first exception after 8?

Practice sheet 18 extends the *Fun with Factors* activity. What is the greatest number of factors (other than 1) one can use to obtain a given number? All the factors will turn out to be prime numbers. (Every number except 1 is either itself a prime number or else can be expressed in only one way as the product of prime numbers. This result is the *fundamental theorem of arithmetic*.)

FPS-18

Fun with Factors

Once upon a time, a mathematician stated that there is only one way in which a number can be obtained as a product of the greatest number of factors other than 1 — if the order in which the factors are used is not considered.

For example:

$12 = 4 \times 3 = 2 \times 6 = \underline{2 \times 2 \times 3}$ and

there is no other set of three factors (other than 1) whose product is 12.

$30 = 5 \times \underline{6} = 15 \times \underline{2} = \underline{5} \times \underline{3} \times \underline{2}$

$16 = 4 \times \underline{4} = \underline{2} \times 8 = \underline{2} \times \underline{2} \times \underline{2} \times \underline{2}$

Is there another way to obtain 16 as a product of four factors (1 excluded)? No

Obtain each of the following as the product of the largest number of factors other than 1.

a. $8 = 2 \times 2 \times 2$
b. $24 = 2 \times 2 \times 2 \times 3$
c. $27 = 3 \times 3 \times 3$
d. $36 = 2 \times 2 \times 3 \times 3$
e. $40 = 2 \times 2 \times 2 \times 5$
f. $21 = 3 \times 7$
g. $18 = 2 \times 3 \times 3$
h. $60 = 2 \times 2 \times 3 \times 5$
i. $78 = 2 \times 3 \times 13$
j. $22 = 2 \times 11$
k. $48 = 2 \times 2 \times 2 \times 2 \times 3$
l. $54 = 2 \times 3 \times 3 \times 3$
m. $68 = 2 \times 2 \times 17$
n. $95 = 5 \times 19$
o. $100 = 2 \times 2 \times 5 \times 5$

List each of the factors used in the preceding examples in order from the smallest to the largest.

2, 3, 5, 7, 11, 13, 17, 19

None of these factors can be obtained as the product of two numbers if 1 is excluded. They are called *prime numbers* for this reason. If a factor is a prime number, it is called a *prime factor*.

Obtain each of the following as the product of its prime factors.

p. $10 = 2 \times 5$
q. $32 = 2 \times 2 \times 2 \times 2 \times 2$
r. $120 = 2 \times 2 \times 2 \times 3 \times 5$
s. $360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5$
t. $64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$
u. $90 = 2 \times 3 \times 3 \times 5$
v. $20 = 2 \times 2 \times 5$
w. $25 = 5 \times 5$
x. $50 = 2 \times 5 \times 5$
y. $1{,}000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5$
z. $128 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$

In any of the examples on this page, can you find another set of prime factors for any of the numbers indicated? No

Do you believe the mathematician was correct in his statement? YES

What is the product of all the prime factors that are less than 10? (No factor is to be repeated.)

$2 \times 3 \times 5 \times 7 = 210$

Problem VIII

Mary explained her new problem in this way:

"I'm looking for a square number that is exactly twice as large as another square number.

"I couldn't find any square number less than 100 exactly twice as large as another less than 100 — but I came close:

<u>49</u> is almost twice <u>25</u>

"So, I'm working on square numbers larger than 100. I've found 144 (12×12) and 289 (17×17). That's close. So, I'm still working!"

Maybe you would like to join in the search.

Problem IX

But Mary hadn't finished.

"I'm also looking for a square number that is 3 times as large as another. Again, it's easy to come close, but hard to be exact.

<u>49</u> is just 1 more than 3 x <u>16</u>

"I'm carrying on these two searches at once."

Perhaps you would like to join the search. You might even broaden it — searching the list of square numbers for couples in which one is 2, 3, 4, 5, 6, 7, 8, 9, or 10 times the other.

In thinking about any problem or working at it, keep your eyes open for interesting questions of your own that you would like to explore. Very often such investigations are by far the most interesting.

Problem X

A snack bar had this menu.

At first glance, there seems to be little of interest in those prices.

However, they must have been carefully selected!

What is unusual about those particular prices?

Let's suppose that no customer bought more than one of a given item at a time, but that he could buy one or all five or any combination of items.

A group of 6 boys bought snacks. Each paid his own check. Here are the amounts they paid. (There was no tax.)

Al – 27¢	Chet – 34¢	Ed – 52¢
Bill – 42¢	Don – 32¢	Fred – 54¢

Think about that information and study the menu.

List different combinations of items each of the 6 boys might have had.

Al might have had coffee and a hot dog — $9¢ + 18¢ = 27¢$. What other combination adds to 27¢?

What about the others?

What combination of items could amount to 30¢, 35¢, 44¢, 50¢?

How many different combinations of items are there?

What is unusual about the prices on this menu?

Could you work out another menu that is unusual in the same way? Start with 3 items; then 4 items; then 5 items. You may wish to try for a much longer list that is unusual.

Tasks: *Students try their hand at each problem and choose one or more that interest them for further study.*

Purpose: To help students think about and pursue strategies for the solution of open-ended problems.

Unifying Ideas: Structure; sets; addition and subtraction; functions and relations.

The Lesson:

Problem VIII. One might begin working at this problem by writing down the squares of all numbers up to 100:

1, 4, 9, 16, 25, 36, 49, 64, etc.

Now the inspection begins. Is any number in this list half of another?

$$4 + 4 = 9 - 1 \qquad (2^2 + 2^2 = 3^2 - 1)$$
$$25 + 25 = 49 + 1 \qquad (5^2 + 5^2 = 7^2 + 1)$$
$$144 + 144 = 289 - 1 \qquad (12^2 + 12^2 = 17^2 - 1)$$

Toward the end of the inspection we find that

$$4900 + 4900 = 9801 - 1 \qquad (70^2 + 70^2 = 99^2 - 1)$$

Several close calls, but no direct hits. So perhaps the list of the first 100 square numbers ought to be extended.

Problem IX. Next we try to find a number in the list that is a third of another. No better luck—none can be found.

Try finding an item in the list that is a fourth of another:

$$1 + 1 + 1 + 1 = 4 \qquad 4 \times 1^2 = 2^2$$
$$4 + 4 + 4 + 4 = 16 \qquad 4 \times 2^2 = 4^2$$
$$9 + 9 + 9 + 9 = 36 \qquad 4 \times 3^2 = 6^2$$

Here there are several successes; indeed, for every square number, there is another that is exactly four times as large.

Problem X. Eventually a student working at this problem notices that there is no combination of those particular prices (if none can be repeated in any one combination) with a sum equal to the sum for another combination.

How is such a list constructed? A simple way is to make each entry larger than the sum of all previous entries, such as:

1, 2, 4, 8, 16, 32, etc., or 2, 3, 6, 12, 24, etc., or 3, 5, 9, 37, 96, etc.

However, if we start with 9, 11, 14, and 18 (as on pupil page 19), how do we determine the next larger number that won't give trouble? We already have 9, 11, 14, 18, and the following which are all the combinations of two or more in the list: 20 (9 + 11), 23 (9 + 14), 25 (11 + 14), 27 (9 + 18), 32 (14 + 18), 34 (9 + 11 + 14), 38 (9 + 11 + 18), 41 (9 + 14 + 18), 43 (11 + 14 + 18), and 52 (9 + 11 + 14 + 18). Since 19 is not in our extended list, it works as well as 22. But once 22 is selected for the fifth item, choosing as small a number as possible for the sixth is not at all easy.

Practice sheet 19 is an exercise in searching for "hidden" 10's and 100's, which explores another source of valuable shortcuts. You might like to put the following series of examples on the board:

7	3.5	17.5	87.5	4.375
x 3	x 6	x 1.2	x .24	x 4.8

All have the same product! Why?

FPS-19

In Search of Hidden 10's and 100's

Rearranging factors sometimes leads to unusual shortcuts. Study the next series of examples, filling in the blanks.

1. 2 x 5 = 10
2. 2 x 2 x 5 x 5 = 2 x 5 x 2 x 5 = 100
3. 25 x 12 = 25 x 4 x 3 = 300
4. 12 x 75 = 3 x 4 x 25 x 3 = 900
5. 15 x 18 = 3 x 5 x 2 x 9 = 270
6. 65 x 14 = 13 x 5 x 2 x 7 = 910
7. 18 x 15 = 9 x 2 x 5 x 3 = 270
8. 16 x 35 = 8 x 2 x 5 x 7 = 560
9. 12 x 25 = 3 x 4 x 25 = 300
10. 16 x 25 = 4 x 4 x 25 = 400
11. 16 x 75 = 4 x 4 x 25 x 3 = 1,200
12. 48 x 75 = 12 x 4 x 25 x 3 = 3600
13. 5 x 28 = 5 x 2 x 14 = 140
14. 4 x 75 = 4 x 25 x 3 = 300
15. 18 x 45 = 9 x 2 x 5 x 9 = 810
16. 35 x 24 = 7 x 5 x 2 x 12 = 840
17. 75 x 32 = 3 x 25 x 4 x 8 = 2,400
18. 120 x 45 = 6 x 10 x 2 x 5 x 9 = 5,400
19. 36 x 125 = 9 x 4 x 25 x 5 = 4,500
20. 48 x 125 = 12 x 4 x 25 x 5 = 6,000
21. 72 x 125 = 9 x 8 x 125 = 9,000
22. 72 x 375 = 9 x 8 x 125 x 3 = 27,000
23. 180 x 15 = 9 x 10 x 2 x 5 x 3 = 2,700
24. 180 x 150 = 9 x 10 x 2 x 5 x 3 x 10 = 27,000
25. 350 x 16 = 14 x 25 x 4 x 4 = 5,600
26. 24 x 125 = 3 x 2 x 2 x 2 x 5 x 5 x 5 = 3 x 2 x 5 x 2 x 5 x 2 x 5 = 3,000
27. 56 x 375 = 7 x 2 x 4 x 3 x 5 x 25 = 7 x 2 x 5 x 3 x 4 x 25 = 21,000

Whenever a multiple of 5 and an even number are multiplied together, look for 10's and 100's and even 1000's.

Another way to make the search and keep track is suggested below.

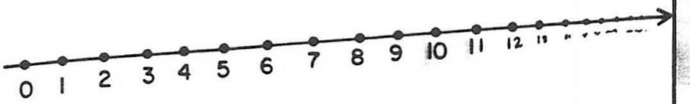
28.		29.		30.		31.	
26	2 x 13	95	5 x 19	75	25 x 3	175	25 x 7
x 35	5 x 7	x 14	2 x 7	x 28	4 x 7	x 56	4 x 14
910	10 x 91	1,330	10 x 133	2,100	100 x 21	9,800	100 x 98

32.		33.		34.		35.	
65	5 x 13	95	5 x 19	375	25 x 15	425	25 x 17
x 24	2 x 12	x 46	2 x 23	x 52	4 x 13	x 164	4 x 41
1,560	26	4,370	57	19,500	45	69,700	17
	13		38		15		68
	10 x 156		10 x 437		100 x 195		100 x 697

Can you complete these "in your head"?

36.	37.	38.	39.	40.	41.	42.
14	18	45	75	175	180	650
x 35	x 55	x 22	x 28	x 36	x 45	x 120
490	990	990	2,100	6,300	8,100	78,000

Problem XI



Imagine that you walked out along a number line labeled with whole numbers and picked up all those made only of the digits 0 or 1. While 0 is itself such a label, let's not include it in the collection.

Thus, the first one you would pick up is 1, the next is 10, the next 11, etc. Let's list the first 15 you would pick up:

1	1st
10	2nd
11	3rd
100	4th
101	5th

110	6th
111	7th
1000	8th
1001	9th
1010	10th

1011	11th
1100	12th
1101	13th
1110	14th
1111	15th

(Make long columns rather than the short ones we made above.)

Find a pattern that will help you know without looking at the list which will be the 50th label, the 71st label, the 99th, the 150th, and any label anyone would ask for.

110,010	50th
1,000,111	71th

1,100,011	99th
10,010,110	150th

As a hint for your search for such a pattern, make a list of these labels:

1st	2nd	4th	8th	16th	32th	etc.
1	10	100	1,000	10,000	100,000	

Problem XII

Try the same basic problem, but pick up all labels after 0 that contain no digits other than 0, 1, 2, 3, and 4.

1	1st
2	2nd
3	3rd

4	4th
10	5th
11	6th

12	7th
13	8th
14	9th

20	10th
21	11th
22	12th

Again, find a pattern that would help you know without checking your list what the 78th, 215th, etc., labels would be.

Make a similar study of the digits 0 through 7; then a study of 0 through 9.

Problem XIII

Palindromic numbers do not change if their digits are reversed. Examples are:

11, 22, 101, 383, 1771, 81618

The number 78 is not palindromic. Reversing digits changes it to 87. If these two numbers are added:

$$\begin{array}{r} 78 \\ +\,87 \\ \hline 165 \end{array}$$

a reversal of digits

the sum is not palindromic because 165 and 561 are different. But let's add them and, if the sum is not palindromic, we'll reverse its digits and add . . . until we reach one of these numbers.

$$\begin{array}{rl} 78 & \\ +\,87 & \text{reverse digits and add} \\ \hline 165 & \\ 561 & \text{reverse digits and add} \\ \hline 726 & \\ 627 & \text{reverse digits and add} \\ \hline 1353 & \\ 3531 & \text{reverse digits and add} \\ \hline 4884 & \leftarrow \text{It's palindromic!} \end{array}$$

Here are a few short ones for you to try:

$$\begin{array}{r} 67 \\ +\,76 \\ \hline 143 \\ 341 \\ \hline 484 \end{array} \qquad \begin{array}{r} 59 \\ +\,95 \\ \hline 154 \\ 451 \\ \hline 605 \\ 506 \\ \hline 1111 \end{array} \qquad \begin{array}{r} 38 \\ +\,83 \\ \hline 121 \end{array} \qquad \begin{array}{r} 27 \\ +\,72 \\ \hline 99 \end{array}$$

Tasks: Students try their hand at each problem and choose one or more that interest them for further study.

Purpose: To help students think about and pursue strategies for the solution of open-ended problems involving the structure of the binary and quinary systems of numeration and palindromic numbers.

Unifying Ideas: Structure; sets; number and counting; numeration; addition and subtraction; functions and relations.

The Lesson:

Problem XI. You may recognize this as a rather unusual way to approach the structure of the binary system of numeration.

Every other label picked up has a 1 in the units place. The pattern in the tens column is 2 ones, 2 zeroes, 2 ones, etc. In the hundreds column, it's 4 ones, 4 zeroes, 4 ones, etc. In the thousands column, it's 8 ones, 8 zeroes, 8 ones, etc. And it would be surprising if the next pattern weren't 16 ones, 16 zeroes, 16 ones, etc.

Any student interested in this problem should be encouraged to look up the *binary system of numeration* in reference material. (He might also look in mathematics reference books for a game called *Nim.)*

Problem XII. After a few changes in the rules for Problem XI, the problem solver is studying the structure of the quinary (base-five) system of numeration.

If he follows the final suggestion and studies the results of picking up labels consisting only of digits 0 through 9, he will, of course, pick up all labels—the 23rd label will read 23, the 517th will read 517, and so on.

Problem XIII. Palindromic words are words that do not change if spelled backwards: *did, noon, madam,* etc. Similarly, palindromic numbers are numbers that do not change when written backwards: 77, 626, 8338, 13531, etc.

The problem on palindromic numbers is almost a diversion—but it provides splendid practice in using the addition algorithm.

This problem is concerned with manufacturing palindromic numbers from nonpalindromic numbers. The procedure is simple: Take a nonpalindromic number, reverse its digits, and add; if the sum is not palindromic, repeat the process. Eventually, a sum will be obtained that is palindromic. Most numbers less than 100 yield palindromic numbers rather rapidly, although some are stubborn. By examining the starting number, can you tell in advance how many additions must be performed before a palindromic number is obtained?

Most students have played the game *What's My Rule?* Practice sheet 20 will help recall this game and provide more experience with pattern indicators or shorthand for rules, all of which help build readiness for algebra.

FPS-20

When Mr. Dean walked into the arithmetic class one day, he saw the chart below on the board. Jenny was calling out numbers.

"Four," she said, and wrote 4 at the top of the third column from the left.

The others wrote numbers opposite their names in the same column.

"I've got the rules," Henry spoke up. "After Mary's name and under the 10, I would write a 17. In the same column, I would write a 20 after Bill's name, a 9 after Alice's name, a 21 after George's name, and a 30 after Al's name."

Numbers Jenny called out	3	1	4	10	9	Rules	Shorthand n
Mary	10	8	11	17	16	Add seven to the number called out.	$n+7$
Bill	6	2	8	20	18	Double the number called out.	$2n$
Alice	2	0	3	9	8	Subtract one from the number called out.	$n-1$
George	7	3	9	21	19	Double the number called out and add one to the result.	$2n+1$
Al	9	3	12	30	27	Multiply the number called out by three.	$3n$

(Complete the record in the fourth and fifth columns.)

Because Henry had discovered the rules in the game above, he took Jenny's place and called out the numbers. He selected ten members of the class to make up secret rules. Can you discover the rules from the records given? Can you find a shorthand expression for each of the rules?

Numbers called out	2	9	25	7	13	Rules	Shorthand a
Joe	4	18	50	14	26	DOUBLE THE NUMBER CALLED OUT.	$2a$
Bob	0	7	23	5	11	SUBTRACT TWO FROM THE NUMBER CALLED OUT.	$a-2$
Catherine	5	19	51	15	27	DOUBLE THE NUMBER CALLED OUT AND ADD ONE TO THE RESULT.	$2a+1$
Sandra	13	20	36	18	24	ADD ELEVEN TO THE NUMBER CALLED OUT.	$a+11$
Sue	3	17	49	13	25	DOUBLE THE NUMBER CALLED OUT SUBTRACT ONE FROM THE RESULT.	$2a-1$
Bill	6	27	75	21	39	TRIPLE THE NUMBER CALLED OUT.	$3a$
Dorothy	7	21	53	17	29	DOUBLE THE NUMBER CALLED OUT AND ADD THREE TO THE RESULT.	$2a+3$
June	9	37	101	29	53	QUADRUPLE THE NUMBER CALLED OUT AND ADD ONE TO THE RESULT.	$4a+1$
Ed	4	25	73	19	37	TRIPLE THE NUMBER CALLED OUT AND SUBTRACT TWO FROM THE RESULT.	$3a-2$
Helen	11	46	126	36	66	MULTIPLY THE NUMBER CALLED OUT BY FIVE AND ADD ONE TO THE RESULT.	$5a+1$

Few numbers less than 100 are interesting when we repeat the digit reversing and adding. However, 97 does not give up easily.

$$\begin{array}{r} 97 \\ +79 \\ \hline 176 \\ +671 \\ \hline 847 \\ +748 \\ \hline 1595 \\ +5951 \\ \hline 7546 \\ +6457 \\ \hline 14003 \\ +30041 \\ \hline 44044 \end{array}$$

6 reversals and 6 additions are required

$\longleftarrow$ It's palindromic.

Can you find any number less than 100 that would require more than 6 reversals and 6 additions?

Try some numbers between 100 and 200. For example, try 168, 157, 175, 188, 198.

Can you find a number that does not become palindromic no matter how many digit reversals and additions you carry out?

Make notes of interesting examples you try.

Problem XIV

What is the largest number that can be obtained as the product of two or more numbers whose sum is given?

For example, suppose 23 is given as the sum. We can begin by finding several combinations of numbers whose sum is 23, and finding the products of those same numbers.

$11 + 12 = 23$ and $11 \times 12 =$ 132

$3 + 20 = 23$ and $3 \times 20 =$ 60

$10 + 10 + 3 = 23$ and $10 \times 10 \times 3 =$ 300

$8 + 8 + 7 = 23$ and $8 \times 8 \times 7 =$ 448

$2 + 6 + 8 + 7 = 23$ and $2 \times 6 \times 8 \times 7 =$ 672

Keep trying combinations of numbers whose sum is 23 until you have the largest product possible. (The largest product is 3,072.)

You will be surprised! Did the fact that we began with 23 as the sum lead to that surprising result?

To find the answer to that, try the same investigation with 17 in place of 23; and with 46 . . . or with any number that comes to mind.

Report your results.

Consider working on the next problem. It is a variation on this one.

Problem XV

This problem is slightly different from the last. The combination of numbers whose sum is a given number must all be different.

What is the largest number that can be obtained as the product of *different* factors whose sum is given?

Again, begin by considering 23 as the sum.

$11 + 12 = 23$ and $11 \times 12 =$ 132

$6 + 8 + 9 = 23$ and $6 \times 8 \times 9 =$ 432

$4 + 5 + 6 + 8 = 23$ and $4 \times 5 \times 6 \times 8 =$ 960

. . . and keep going until you are able to find a combination of different numbers whose sum is 23 and whose product is 1,260.

Try other numbers (in place of 23) as the sum. Begin with small numbers.

$1 + 4 = 5$ and $1 \times 4 = 4$

$2 + 3 = 5$ and $2 \times 3 = 6$

21 Problems XIV-XV - maximizing the product of addends for a given sum

Tasks: Students try their hand at each problem and choose one or more that interest them for further study.

Purpose: To help students think about and pursue strategies for the solution of open-ended problems involving palindromic numbers and the product of two or more factors whose sum is given.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson:

Problems XIV through XVII. These problems explore the idea of factorization (with special attention given to prime numbers as factors) and culminate in a consideration of the unique factorization principle—often called the *fundamental theorem of arithmetic.* This principle states that there is only one set of prime numbers that are factors of any composite number.

Problem XIV. Suppose that we consider different combinations of numbers whose sum is 7, and then find the product of those numbers. Which combination gives the largest product?

$$5+2=7 \text{ and } 5\times 2=10$$

$$3+3+1=7 \text{ and } 3\times 3\times 1=9$$

$$2+2+2+1=7 \text{ and } 2\times 2\times 2\times 1=8$$

$$2+2+3=7 \text{ and } 2\times 2\times 3=12$$

A rather surprising result emerges: For even numbers, the combination will consist of 2's; for odd numbers, one 3 and the rest 2's. This outcome is perhaps unexpected, but it is certainly nice and simple.

Problem XV. This is a variation of Problem XIV and uses the familiar problem-expansion technique of changing the restrictions of the earlier problem. In this version we insist that no number be repeated in any combination—all else remains the same.

Problem XVI. The restrictions (pupil page 22) become a little tighter. No number can be repeated in a combination, and only prime numbers can be used.

Practice sheet 21 presents problems closely related to the *What Are My Rules?* activities. In example I the rules are clear, but in the eighth column there is a problem: Find two numbers whose sum is 7 and whose product is 12. This kind of problem challenges more than the memory.

The last column in example III presents a problem that may be stated algebraically as follows:

$$2(x+y)=26$$

$$xy+15=57$$

$$x = \_\_\_$$

$$y = \_\_\_$$

At this point, such problems may require considerable trial-and-error experimentation.

FPS-21

Complete each column based on the given information.

I.

	1	2	3	4	5	6	7	8	9	10	11	
sum	5	16	60	19	13	15	12	7	10	17	16	
addends	2	9	10	7	8	4	5	4	7	15	13	factors
addends	3	7	50	12	5	11	7	3	3	2	3	factors
	6	63	500	84	40	44	35	12	21	30	39	product

II.

	1	2	3	4	5	6	7	8	9	10	11	
sum + 10 ($s+10$)	20	22	22	21	24	22	24	30	20	27	21	
sum (s)	10	12	12	11	14	12	14	20	10	17	11	
addends	4	9	6	7	8	7	7	10	5	9	6	factors
addends	6	3	6	4	6	5	7	10	5	8	5	factors
	24	27	36	28	48	35	49	100	25	72	30	product (p)
	31	34	43	35	55	42	56	107	32	79	37	product + 7 ($p+7$)

III.

	1	2	3	4	5	6	7	8	9	10	11	
$x+y+9$	15	20	17	$10\frac{1}{2}$	10	$12\frac{3}{4}$	10	$11\frac{1}{2}$	14	$17\frac{1}{2}$	22	
$2(x+y)$	12	22	16	3	2	$7\frac{1}{2}$	2	5	10	17	26	
$x+y$	6	11	8	$1\frac{1}{2}$	1	$3\frac{3}{4}$	1	$2\frac{1}{2}$	5	$8\frac{1}{2}$	13	
x	2	5	4	1	$\frac{1}{2}$	$\frac{3}{4}$	.5	$\frac{1}{2}$	3	7	7	x
y	4	6	4	$\frac{1}{2}$	$\frac{1}{2}$	3	.5	2	2	$1\frac{1}{2}$	6	y
	8	30	16	$\frac{1}{2}$	$\frac{1}{4}$	$2\frac{1}{4}$	.25	1	6	$10\frac{1}{2}$	42	$x \cdot y$ or xy
	23	45	31	$15\frac{1}{2}$	$15\frac{1}{4}$	$17\frac{1}{4}$	15.25	16	21	$25\frac{1}{2}$	57	$xy+15$
	16	60	32	1	$\frac{1}{2}$	$4\frac{1}{2}$	.5	2	12	21	84	$2 \cdot xy$ or $2xy$

IV.

	1	2	3	4	5	6	7	8	9	10	11	12
$m+n$	.5	1	1.5	5.8	10.2	.6	18	1	1.5	9.9	2.4	5
m	.2	.4	.5	5	1.2	.3	9	.5	.8	9	1.2	2.5
n	.3	.6	1	.8	9	.3	9	.5	.7	.9	1.2	2.5
mn	06	.24	.5	4	10.8	.09	81	.25	.56	8.1	1.44	6.25

There is no other combination to try because $1 + 1 + 3$ and $1 + 2 + 2$ break the rule that all must be different.

Try 6, then 7, then 8, etc., until you notice a pattern. Test your pattern with 15 or 27 or some other number.

Can you develop a system so that you can quickly find the best combination as soon as you know the sum?

If you are enjoying the investigation, try the next which is based on another change in the rules.

Problem XVI

What is the largest number that can be obtained as the product of different prime numbers when the sum of the primes is given?

In this variation, we are limited to primes as factors and it is required that they be different.

If, again, we start with 23, we don't have many combinations of primes whose sum is 23. There are a total of only three:

$3 + 7 + 13 = 23$ and $3 \times 7 \times 13 = \underline{273}$

$5 + 7 + 11 = 23$ and $5 \times 7 \times 11 = \underline{385}$

$2 + 3 + 7 + 11 = 23$ and $2 \times 3 \times 7 \times 11 = \underline{462}$

Let's try another number, such as 30. There are three combinations of two primes whose sum is 30.

$7 + 23 = 30$ and $7 \times 23 = \underline{161}$

$11 + 19 = 30$ and $11 \times 19 = \underline{209}$

$13 + 17 = 30$ and $13 \times 17 = \underline{221}$

How many combinations of three primes have 30 as sum? And why must 2 be one of the primes?

Can you show that there is no combination of four primes whose sum is 30?

There is a combination of five primes whose sum is 30 and whose product is 2730. What are the five prime numbers?

$\underline{2} \times \underline{3} \times \underline{5} \times \underline{7} \times \underline{13} = 2730$

Problem XVII

This problem is related to the three previous problems, but *sum* and *product* change places.

What is the smallest sum of factors whose product is given?

Suppose the given product is 36. Since the factors need not all be different, there are quite a few.

$18 \times 2 = 36$ and $18 + 2 = \underline{20}$

$12 \times 3 = 36$ and $12 + 3 = \underline{15}$

$9 \times 4 = 36$ and $9 + 4 = \underline{13}$

$6 \times 6 = 36$ and $6 + 6 \ \underline{12}$

Since we are looking for the smallest sum, we are moving in the right direction. Let's use more factors.

$9 \times 2 \times 2 = 36$ and $9 + 2 + 2 = \underline{13}$

$6 \times 2 \times 3 = 36$ and $6 + 2 + 3 = \underline{11}$

When you have obtained a sum as small as 10, you will have found the smallest sum of factors whose product is 36.

Try other numbers that have several combinations of factors such as 12, 18, 24, 30, 42, 56, etc. Include 64 and 81 as interesting examples. Find other interesting examples.

In each study, what do you notice about all the factors in the combination having the smallest sum?

Can you find an example in which there are two different combinations having the smallest sum — provided that 4 is always written as 2×2?

The number 4 is a little peculiar. It is one of the two numbers which is both the sum and product of a pair of numbers. What is the other number?

$4 = \underline{2} + \underline{2} = \underline{2} \times \underline{2}$

$\underline{0} = \underline{0} + \underline{0} = \underline{0} \times \underline{0}$

Tasks: Students try their hand at each problem and choose one or more that interest them for further study.

Purpose: To help students think about and pursue strategies for solving open-ended problems involving the product of different prime numbers when the sum of the primes is given and involving the smallest sums of the factors when their product is given.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

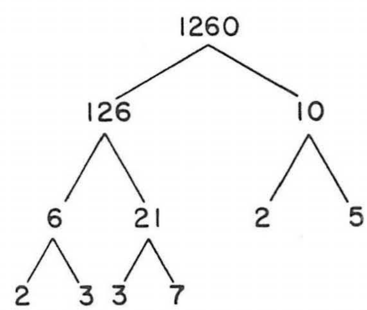
The Lesson:

Problem XVII. In this problem, the roles played by addends and factors in the previous problems are switched.

Here we start with a given number and obtain it in different ways as a product of factors. We are looking for the combination of factors with the smallest sum.

In every case this will lead us to obtaining the number as the product of its prime factors!

With many numbers, this can be done quickly by using the method suggested by the following sketch:



Thus, $1{,}260 = 2 \times 2 \times 3 \times 3 \times 5 \times 7$.

Notice the discussion at the bottom of the right-hand column. Except for 4, any composite number is larger than the sum of its prime factors. Therefore, factoring any composite (except 4) into primes will produce a smaller sum of factors. Factoring 4 into 2×2 does not produce a smaller sum, because $2 \times 2 = 2 + 2$; thus, $7 + 4 = 7 + 2 + 2$. To avoid this difficulty, we agree that 4 will be factored into primes even though this does not produce a smaller sum.

Problems XVIII, XIX, and XX (pupil page 23) lend themselves to countless variations. We hope that students will discover and investigate some of these on their own.

In writing expressions to fulfill the requirements in these problems, questions arise as to what signs of arithmetic are permissible. Should we admit:

$$4^4 = 256 \quad \sqrt{4} = 2 \quad 4! = 24$$

which are read 4 to the 4th power, the square root of 4, and 4 factorial $(4 \times 3 \times 2 \times 1)$? There are no set rules. Let the problem solver set his own limitations and work within them.

Practice sheet 22 illustrates another kind of mathematical problem. The authors feel that this type of problem avoids the pitfalls of the "story problems" in older arithmetic books in which too much information was given and there was scarcely anything to think about.

FPS-22

Sale Artists' Supplies: —

PENCILS .... 7¢ or 3 for 20¢
BOARDS .... 9¢ or 3 for 25¢

(Note: No person bought more than 9 of any one item.)

I.

WHAT CAN YOU SAY!

True or False

1. Five pencils cost 35¢. T F
2. Five pencils and five boards cost a total of 77¢. T F
3. No one can spend exactly 26¢ for pencils and boards. T F
4. There are at least 2 ways to spend exactly 43¢ — 5 pencils and 1 board, or 5 boards and 0 pencils. T F
5. There are at least 2 ways to spend exactly 88¢ at this sale. T F
6. There are at least 3 ways to spend exactly 54¢ at this sale. T F

Any difficulties you had with the previous questions will disappear when you complete the chart in the bottom right-hand corner of the page.

This chart will also help you with the following:

II.

Pencils	Boards	Amount
3	2	38¢
5	2	52¢
2	3	39¢
4	5	70¢
9	9	$1.35
8	4	88¢
9	0	60¢
9	2	78¢
4	7	86¢

Pencils	Boards	Amount
9	5	$1.03
8	6	1.04
7	7	1.06
6	8	1.08
5	9	1.09
9	6	1.10
8	7	1.13
9	7	1.19
8	9	1.29

Is there an amount of money that can be spent exactly in 3 different ways? No

In how many different ways can you spend exactly one dollar? None

III.

Since no one bought more than 9 of any item, we can list all possible combinations in the chart on the right indicating the cost of each. There are 99 combinations if we exclude the combination consisting of no pencils and no boards.

Number of Pencils @ 7¢; 3 for 20¢ (columns); Number of Boards @ 9¢; 3 for 25¢ (rows)

	0	1	2	3	4	5	6	7	8	9
0		7¢	14¢	20¢	27¢	34	40	47	54	60
1	9¢	16¢	23	29	36	43	49	56	63	69
2	18¢	25	32	38	45	52	58	65	72	78
3	25¢	32	39	45	52¢	59	65	72	79	85
4	34	41	48	54	61	68	74	81	88	94
5	43	50	57	63	70	77	83	90	97	103
6	50	57	64	70	77	84	90	97	104	110
7	59	66	73	79	86	93	99	106	113	119
8	68	75	82	88	95	102	108	115	122	128
9	75	82	89	95	102	109	115	122	129	135

Problem XVIII

Can you write each whole number from 0 through 20 with four 4's?

You may use addition, subtraction, multiplication, division, or any combination of these. You may use decimal points and symbols of grouping. But it must be with exactly four 4's — no more, no less.

Suppose that we tackle a similar problem — writing numbers with four 5's. Here are a few:

$0 = 55 - 55$

$1 = 55 \div 55$

$2 = (5 \div 5) + (5 \div 5)$

$3 = (5 + 5 + 5) \div 5$

$4 = [(5 \times 5) - 5] \div 5$

$5 = (5 \times .5) + (5 \times .5)$

$6 = (55 \div 5) - 5$

$7 = 5 + [(5 + 5) \div 5]$

$8 = 5 + .5 + (.5 \times 5)$

$9 = 5 + 5 - .5 - .5$

$10 = 5 + [5 \times (5 \div 5)]$

Of course, someone else might have an entirely different list:

$0 = 55 \times (5 - 5)$

$1 = 5 - 5 + .5 + .5$

$2 = .5 + .5 + .5 + .5$

$3 = 5 - [(5 \times .5) - .5]$

$4 = [(5 - .5) \div .5] - 5$

$5 = [(5 - 5) \times 5] + 5$

Now that you know the rules, you can begin writing as many numbers as you can with four 4's. Try for all numbers 0 through 20 — and as many beyond 20 as you can find.

Next, try four 6's; then on to four 7's. Perhaps you can get a contest underway. Get your parents and friends to join in the search.

Problem XIX

Here is a slight variation of the last problem:

Write each whole number from 0 through 20 using any group of four digits — such as 1, 9, 6, 8 — in place of the four 4's.

$0 = (8 + 1 - 9) \times 6$

$9 = 8 - (6.1 + .9)$

$2 = 8 - [6 \times (.1 + .9)]$

$3 = 18 - 9 - 6$

$4 = 8 + 6 - 9 - 1$

$5 = 19 - 8 - 6$

$6 = 6 \times 1 \times (9 - 8)$

$7 = (6 \times 1) + (9 - 8)$

Problem XX

Another variation of the problem is the telephone number problem:

Select any telephone number. Find as many ways as you can to arrange the 3 area code digits, the 3 exchange digits, and the remaining 4 digits so that all arrangements give the same number.

Suppose that we have the telephone number:

408 – 624 – 8115

(0)......$48 \times 0 = 6 - 2 - 4 = 85 \times (1 - 1)$

(2)......$8.0 \div 4 = 2 \times (.4 + .6) = 8 + 5 - 11$

(4)......$8 - 4 + 0 = 24 \div 6 = 8 - 5 + (1 \times 1)$

(5).....$40 \div 8 = (6 + 4) \div 2 = 5 + [8 \times (1 - 1)]$

(8)......$8 + (4 \times 0) = 6 + 4 - 2 = 11 + 5 - 8$

(12).....$4 + 8 + 0 = 6 + 2 + 4 = 8 + 5 - (1 \times 1)$

(20)....$80 \div 4 = (6 + 4) \times 2 = (15 + 1) \div .8$

(32) $8 \times 4.0 = 64 \div 2 = [8 \times (1 + 1)] \div .5$

(48)....$48 + 0 = 6 \times 2 \times 4 = 8 \times [5 + (1 \times 1)]$

You might find arrangements which give decimal fractions:

(8.4)....$8.0 + .4 = 2.4 + 6 = 8.5 - (.1 \times 1)$

and common fractions:

$$\frac{1}{2} \cdots\cdots \frac{4 + 0}{8} = \frac{4}{6 + 2} = \frac{5 - 1}{8 \times 1}$$

You might make up a game based on the idea above that uses an automobile license plate, a Social Security number, the first two and last three digits of your zip code. Games that you invent are the ones that are the most fun.

Select a Problem

Select the problem you wish to tackle first, and plunge into it. Please share all the interesting facts you uncover with others. Perhaps you can organize your work as a Science Fair project.

23 Problems XVIII-XX - expressing numbers from given digits

Tasks: Students try their hand at each problem and choose one or more that interest them for further study.

Purpose: To help students think about and pursue strategies for solving open-ended problems.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson:

Select a problem. We hope that each student will find at least one problem in this chapter that he would enjoy working at. Perhaps he can invent his own problem or find one that interests him in some other mathematics book.

We also hope that students can have reasonable assurance in advance that the results of their work will be considered carefully and perhaps made available to the entire group or to some wider audience. Several teachers may wish to work together so that the more interesting results will be reported in all of the classes.

Most reports of mathematical investigations tend toward brevity, because a great deal can be said with very few words. The statement:

> By inspection I have found that all whole numbers less than 200 can be obtained as the sum of at most four square numbers.

may summarize the results of hours of computations and search. (Of course, one might present a listing for each number less than 200 or attach the scratch paper showing the original calculations.)

To take another example, a student, in working Problem IV, might come up with the following statement:

1.4142135^2 is less than 2, and

1.4142136^2 is more than 2.

This speaks rather eloquently about work that may have taken hours.

Problems that seem interesting at first may lose their appeal on further investigation. There is nothing to be gained by studying a problem that one has worked at and found dull, especially if his interest has been truly aroused by another problem.

Students often discuss their school work at home. Some of the problems in this chapter may be fun for the entire family, and you may wish to allow students to report results of the family's work. We believe that solving mathematical problems can be enjoyable for everyone and that it can become for many people an exciting new aspect of living.

FPS-23

WHAT CAN YOU SAY? ________________

A. Bill and Jim together had 7 books. What can you say?

____

Indicate whether the following statements are true T or false F.

1. Each may have had the same number of books as the other. T F
2. If one had an even number of books then the other had an odd number of books. T F
3. One may have had exactly twice as many as the other. T F
4. One boy had more books than the other. T F
5. One boy may have an even number of books more than the other. T F

Complete the following statements.

6. If Bill had 2 books then Jim had 5 books.
7. If Jim had $2\frac{1}{2}$ times as many as Bill then Jim had 5 books and Bill had 2.
8. If one boy had an even number of books then the other had an ODD number of books.
9. If Jim had exactly 3 more than Bill then Jim had 5 books, and Bill had 2 books.
10. There are only 8 ways the books might be distributed. Fill out the table below showing these combinations:

If Bill had	7	6	5	4	3	2	1	0
then Jim had	0	1	2	3	4	5	6	7

Check your answers to questions 1-9 using the information in the completed table.

B. There are fewer than 100 customers on Ned's paper route. He noted that exactly $\frac{1}{3}$ of the customers live on Birch Street, exactly $\frac{1}{5}$ of them live on Ash Street, and exactly $\frac{1}{4}$ of them live on Elm Street. What can you say?

____

$\frac{1}{3}$ on Birch . . . $\frac{1}{5}$ on Ash . . . $\frac{1}{4}$ on Elm

True or False

1. Ned has more customers on Ash Street than on Elm Street. T F
2. Ned has more customers on Birch Street than on Elm Street. T F
3. Ned may have 10 more customers on Birch than on Ash. T F
4. Ned has exactly twice as many customers on one street as on another. T F
5. All of Ned's customers live on these 3 streets because $\frac{1}{3} + \frac{1}{5} + \frac{1}{4} = 1$. T F
6. There are more customers on Birch Street than on the other streets put together. T F

Complete the following statements. (You may find a useful hint as you think about statement 5 above.)

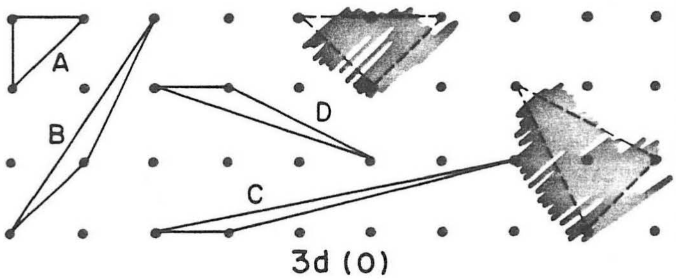
7. Ned has 20 customers on Birch Street.
8. He has 12 customers on Ash Street.
9. He has 15 customers on Elm Street.
10. Ned has a total of 60 customers.

____

Recheck your answers to questions 1-6 above.

A Problem from Geometry
or, Doodler's Delight

What would you make of the following bit of doodling?



All are triangles, but two were erased. Why?

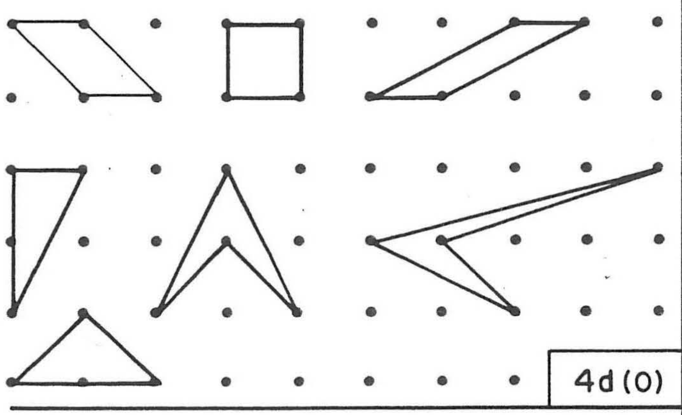
What's the "3d(0)"?

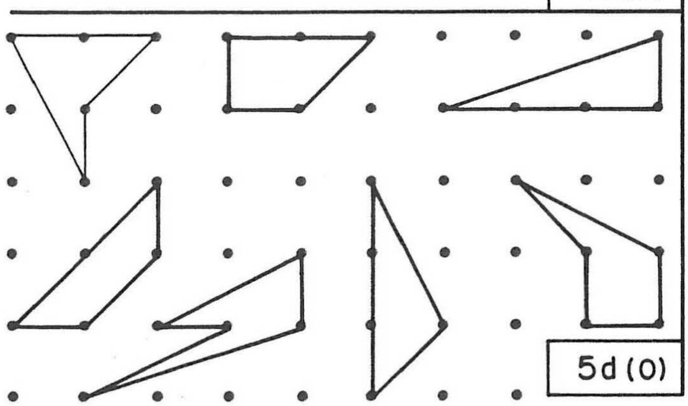
How are A, B, C, and D alike, but different from those erased?

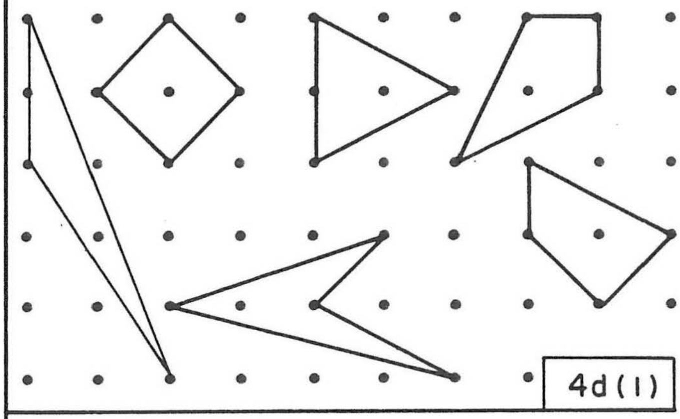
Would you be surprised to find out that "3d(0)" is shorthand for "3-dotter with 0 dots inside"?

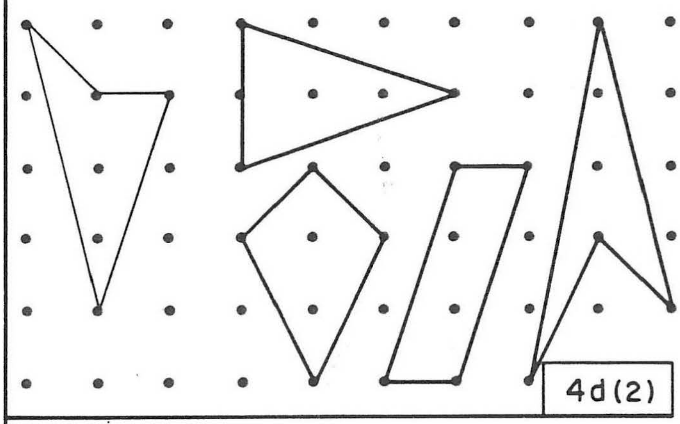
So, one triangle was erased because it touched 4 dots and the other because it had 1 dot inside.

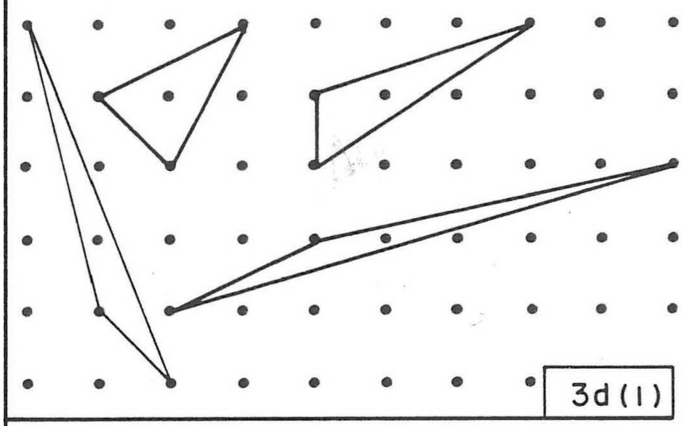
Now it's your turn to doodle. Make several different shapes that belong in each group.







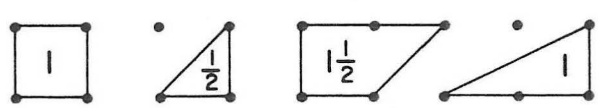




The last doodling — 3d(1) — was probably harder than the others . . . but still fun.

Now for a bit of work! (Every good problem is a mixture of fun and work.)

If we consider  as 1 unit of area, what is the area of each of the doodles? Mark the area of each next to it or inside the shape. (Here's some help — and there is more on the next page.)



Surprise!

24 What do you make of the following bit of doodling: 3d(0)?

Tasks: Students draw shapes on the grids, following directions indicated by the shorthand, and then determine the area of each of the shapes.

Purpose: To enable students to have a rich experience in geometry by exploration of Pick's Theorem.

Unifying Ideas: Structure; sets; number and counting; functions and relations; geometry; measurement.

The Lesson: Pupil pages 24-27 and the corresponding practice sheets lead to a surprising theorem proved by a late nineteenth century mathematician named Pick.

Pupil page 24 introduces a novel way of classifying certain figures. The figures (or shapes) dealt with are those that can be drawn by using segments connecting points in a grid of evenly spaced points. They are classified in terms of

a. the number of dots on the boundary of the figure, and
b. the number of dots in the interior of the figure.

Examples are used to explain the classification, and appropriate notation is introduced. The triangles A, B, C, and D are all labeled $3d(0)$. The number before the letter d tells how many dots are on the boundary, and the number in parentheses after the d tells how many dots are in the interior. Similarly, the label $4d(1)$ is used for a figure with 4 dots on its boundary and 1 dot inside, and so on.

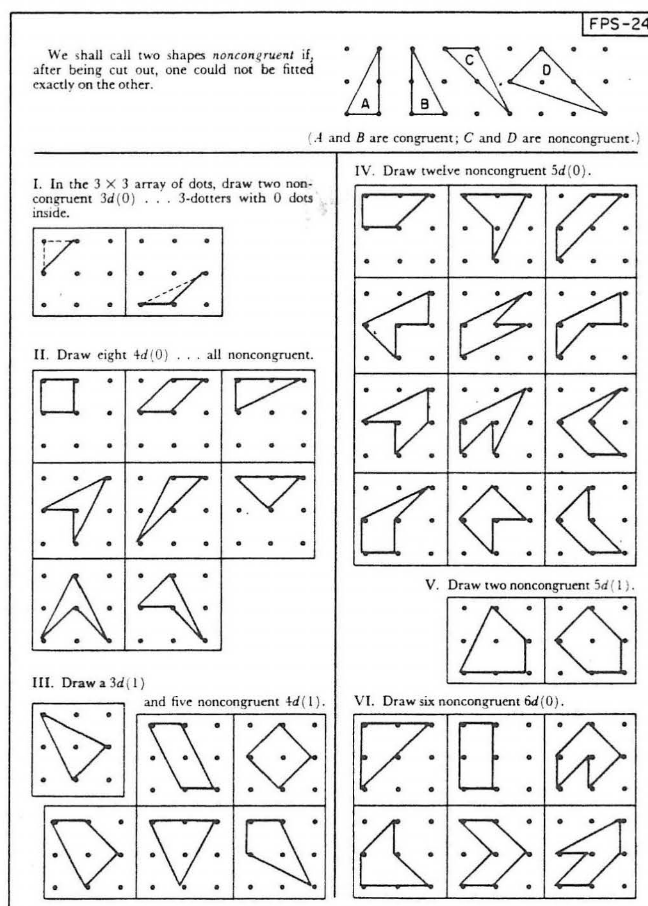
Students may think this way of describing a shape is silly. It appears at first to have little meaning, since many different shapes may have the same label. (Exercise IV of practice sheet 24 shows that even within the limits of a 3-by-3 array of dots, there are twelve basically different 5-dotters with 0 dots inside.)

What the students do not know—and let it come as a surprise to them on the last two pages—is that if one knows the number of dots on the boundary of the shape and the number of dots inside the shape, he can compute its area very quickly and without even looking at the shape. Few if any students will guess this until they get well into the development. Surprises such as this help to make mathematics an exciting subject.

Given a large enough grid, we can find as many noncongruent $3d(0)$ shapes as we want. This is easy to see from the following drawing, which can be extended in both directions:

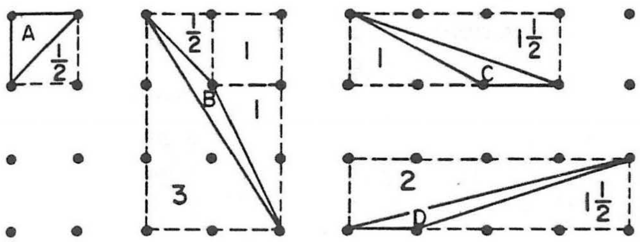
The same is true regardless of the number of dots on the boundary or inside. One of the hardest is $3d(1)$. One $3d(1)$ is shown on pupil page 24, and three others are shown in red.

The more students realize that a variety of shapes fit each label, the more surprised they will be by the final result.



You may have some trouble finding the area of some of the doodles.

It's helpful to remember that the diagonal of a rectangle divides the rectangle in half.

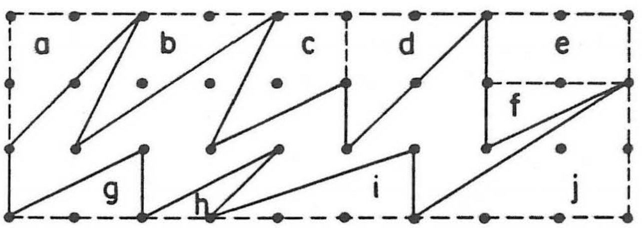


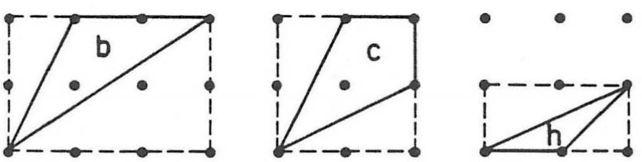
(all numbers refer to units of area)

$A = 1 - \frac{1}{2} = \frac{1}{2}$ $\qquad B = 6 - 3 - \frac{1}{2} - 1 - 1 =$ $\frac{1}{2}$

$C = 3 - 1 - 1\frac{1}{2} =$ $\frac{1}{2}$ $\qquad D =$ $4 - 2 - 1\frac{1}{2}$ $=$ $\frac{1}{2}$

Can you find the area of this streak of lightning? *





a = 2 $\qquad$ b = 2 $\qquad$ c = 2 $\qquad$ d = 2

e = 2 $\qquad$ f = 1 $\qquad$ g = 1 $\qquad$ h = $\frac{1}{2}$

i = $1\frac{1}{2}$ $\qquad$ j = 3

a + b + c + d + e + f + g + h + i + j = 17

The area of the big rectangle around the lightning streak is 27 units. The part shaded blue is 17 units.

So, the area of the unshaded part must be 10 units.

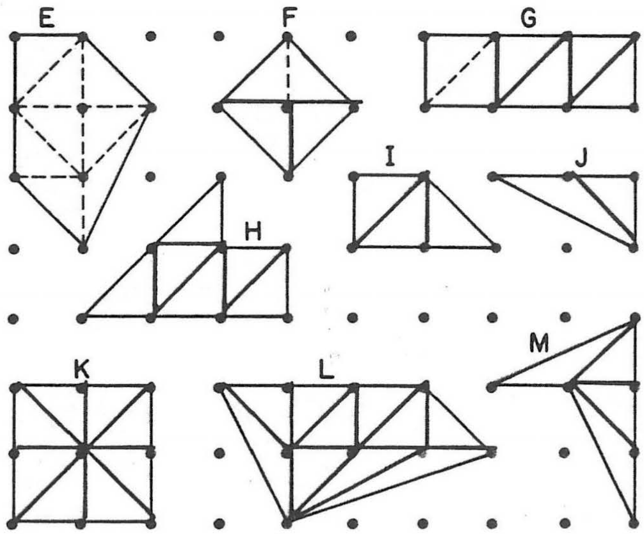
Now, perhaps, you have some ideas that will help you find the area of any doodle on the previous page.

---

* The lightning streak is a 20-dotter with 1 dot inside, a $20d(1)$.

Suppose that you tried to draw segments inside a shape to divide it up into 3-dotters with 0 dots inside.

Remember that the segments must connect dots and must not cross another segment except at a dot. The first one is completed. Others may be started. Please complete them all.

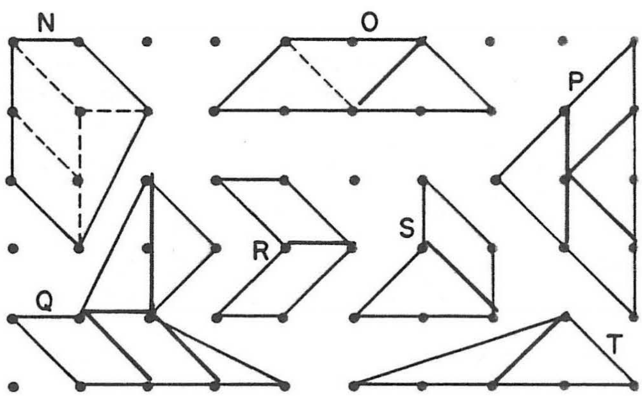


Does this suggest another way to find the areas of the doodles? Remember, all $3d(0)$ have the same area.

Write the areas of the shapes above in this chart:

E	F	G	H	I	J	K	L	M
4	2	3	3	$1\frac{1}{2}$	1	4	5	2

Or, you might decide to break up shapes into 4-dotters with 0 dots inside, with a $3d(0)$ left over sometimes.



N	O	P	Q	R	S	T
4	3	4	$4\frac{1}{2}$	2	2	2

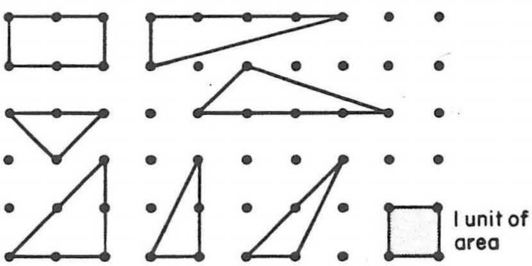
25 Can you find the area of a streak of lightning?

Tasks: Students find the areas of geometric shapes.

Purpose: To enable students to have a rich experience in geometry by exploration of Pick's Theorem.

Unifying Ideas: Structure; sets; addition and subtraction; functions and relations; geometry; measurement.

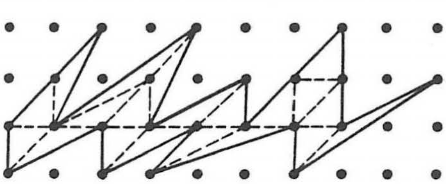
The Lesson: Most students have had experience in finding areas of squares, triangles, and other figures. To get a feeling for what they already know, you might make a grid of dots on the board and draw figures such as:



If the small shaded square is considered to have 1 unit of area, what is the area of the rectangle and the triangles? Unless students know how to approach this question, they will have trouble following the first part of the development on pupil page 25. In this case we suggest the use of practice sheet 25 to provide experiments in finding areas.

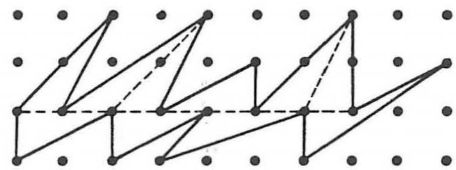
That every 3-dotter with 0 dots inside has ½ unit of area may be the students' first inkling of the surprising result to come.

After the student has used triangulation to find the area of a very complicated lightning streak, a second rather surprising idea is suggested—perhaps every shape can be broken up into 3-dotters with 0 dots inside. If so, then the area of the lightning streak could have been found in this way:

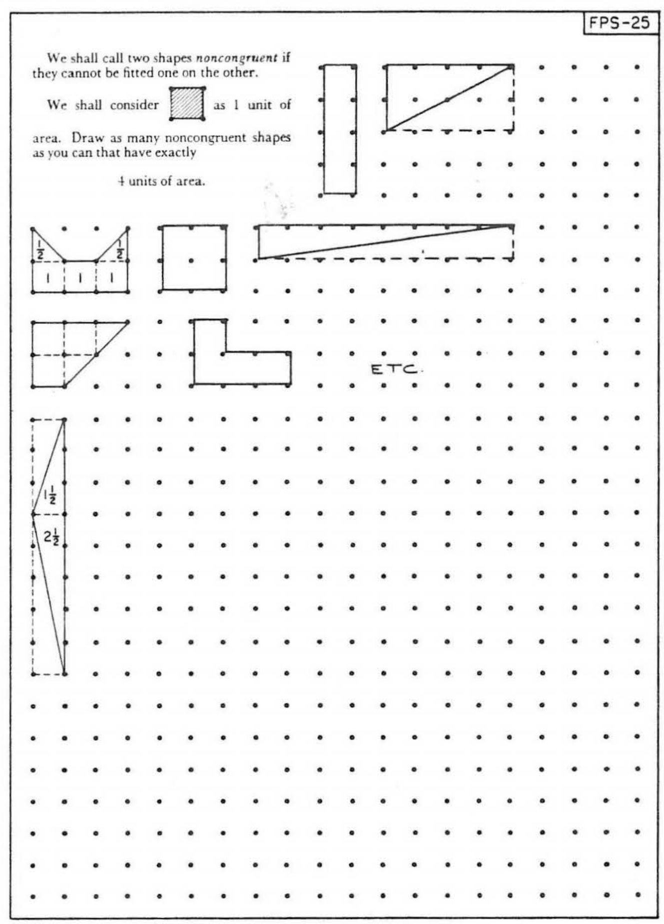


There are twenty 3-dotters with 0 dots inside. Since each of these has ½ unit of area, the area of the streak must be half of 20 units, or 10 units. This, of course, agrees with the result found by triangulation.

Since a 4-dotter with 0 dots inside can be seen as a pair of 3-dotters with 0 dots inside, the former must have 1 unit of area. Try the lightning streak again:

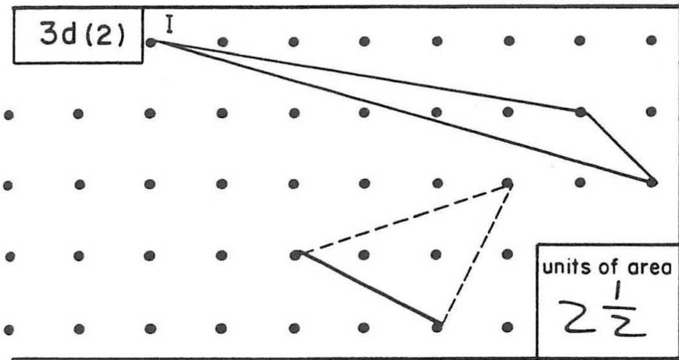


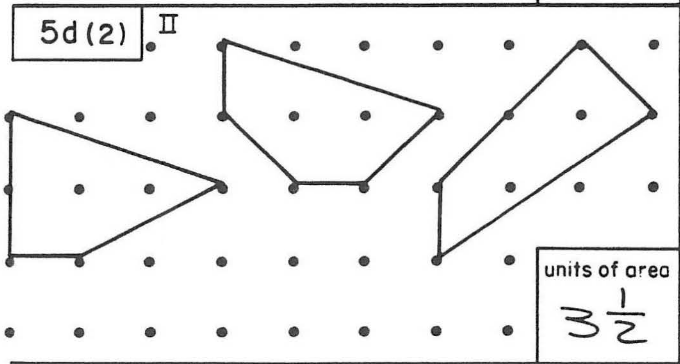
Ten 4-dotters with 0 dots inside must have an area of 10 units.

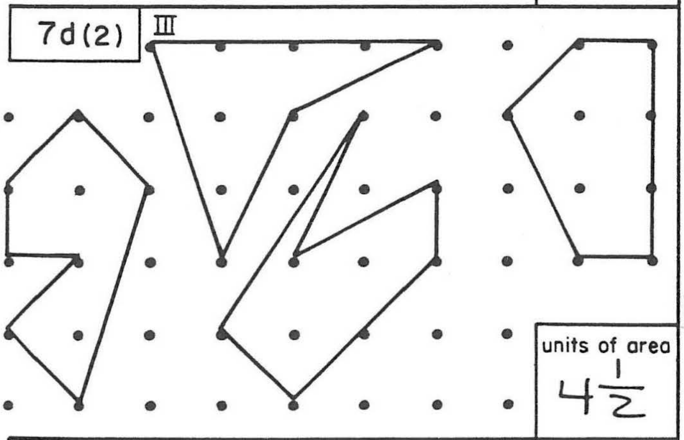


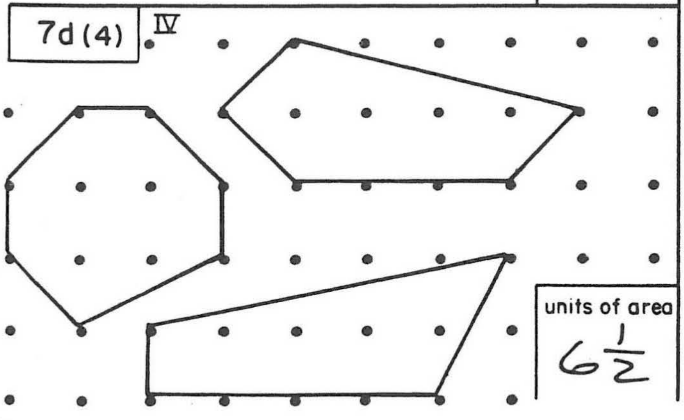
In each section draw at least two of the shapes indicated by our doodler's shorthand. Find the area of each by any method you like.

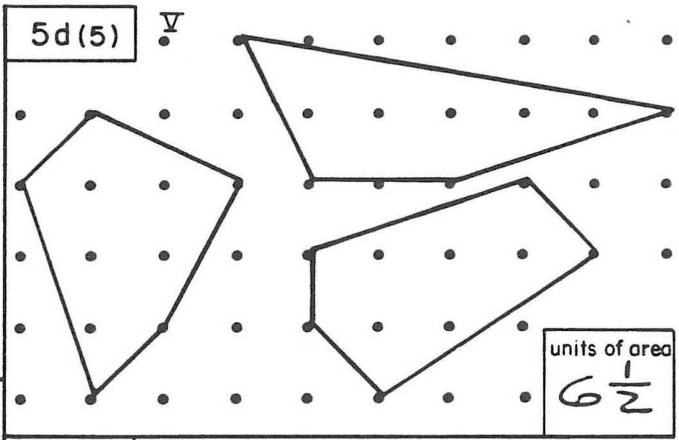
You probably are already convinced that all shapes described by the same piece of shorthand have the same area.

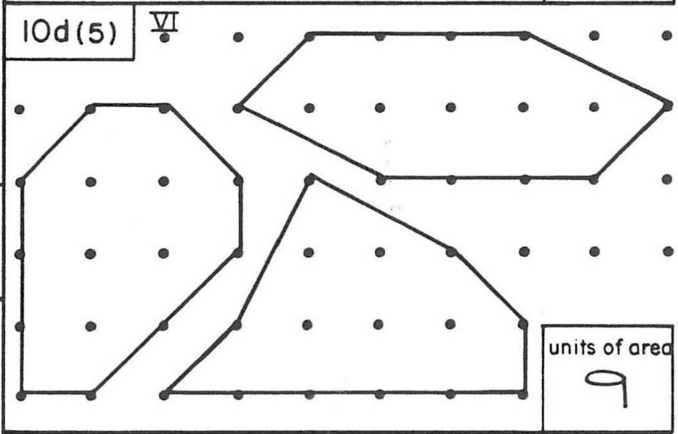












VII

In the blue tinted blocks in the chart below, record the areas you found on this page and on page 24.

Number of dots inside (columns); Number of dots on the boundary (rows)

	(0)	(1)	(2)	(3)	(4)	(5)
3	$\frac{1}{2}$	$1\frac{1}{2}$	$2\frac{1}{2}$	$3\frac{1}{2}$	$4\frac{1}{2}$	$5\frac{1}{2}$
4	1	2	3	4	5	6
5	$1\frac{1}{2}$	$2\frac{1}{2}$	$3\frac{1}{2}$	$4\frac{1}{2}$	$5\frac{1}{2}$	$6\frac{1}{2}$
6	2	3	4	5	6	7
7	$2\frac{1}{2}$	$3\frac{1}{2}$	$4\frac{1}{2}$	$5\frac{1}{2}$	$6\frac{1}{2}$	$7\frac{1}{2}$
8	3	4	5	6	7	8
9	$3\frac{1}{2}$	$4\frac{1}{2}$	$5\frac{1}{2}$	$6\frac{1}{2}$	$7\frac{1}{2}$	$8\frac{1}{2}$
10	4	5	6	7	8	9

Do you notice a pattern in the entries that suggests a way to complete the chart without further experiments?

If so, take a chance — and then check some of these new entries by experiments. Did the pattern hold up?

26 Drawing shapes specified by a doodler's shorthand notes

Tasks: Students draw shapes on the grids provided and then determine the areas of these shapes. They next fill in the blue-tinted sections of the chart and look for a pattern that suggests a way of completing the chart without drawing shapes and measuring areas.

Purpose: To enable students to have a rich experience in geometry by exploration of Pick's Theorem.

Unifying Ideas: Structure; sets; addition and subtraction; functions and relations; geometry; measurement.

The Lesson: Experiments on pupil page 26 lead us to believe that all shapes described by the same piece of doodler's shorthand will have the same area.

How do we know that our result is *always* true? Is it possible that with a larger grid we might find two shapes described by the same shorthand but having different areas? Pick proved that our result always holds. Our experiments do not prove the result; they only provide a great deal of evidence in its favor. Students should be aware of this distinction between proof and the absence of known exceptions.

In exercise 7, students are asked to summarize in a chart the results of their experiments.

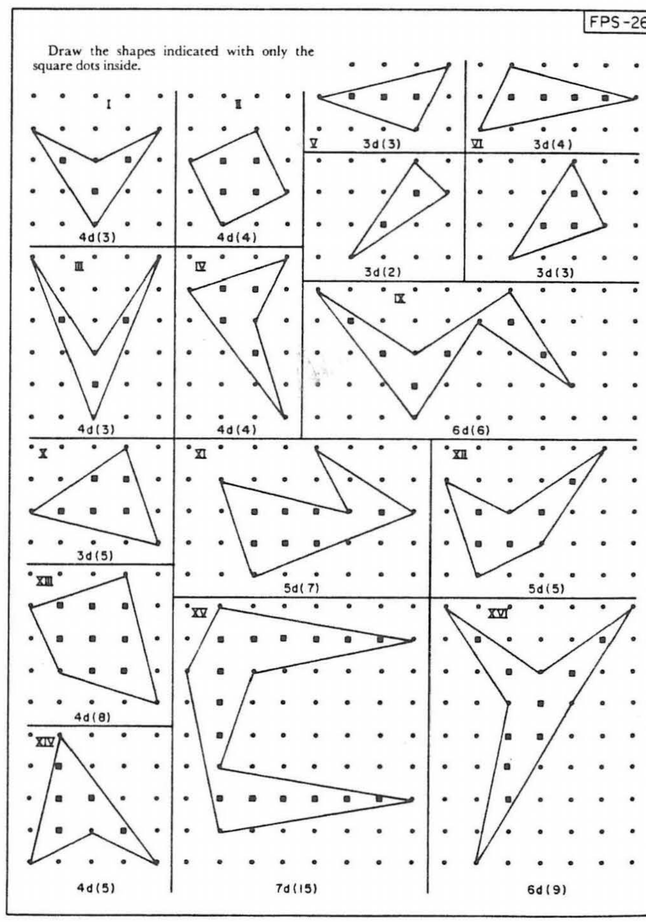
A pattern quickly emerges, and the blanks can be filled in quickly—every step down is ½ and every step across is 1; that is, each additional dot for the boundary adds ½ unit of area, and each additional dot inside adds 1 unit of area.

Be sure that several of the chart entries predicted by the pattern are checked by experiment. Perhaps all entries can be checked by students working together.

Several of the shapes resulting from completion of practice sheet 26 can be used to check entries in the chart.

The tasks outlined in practice sheet 26 are not as easy as they may seem at first glance. In fact, several of them are quite difficult and will require that segments be drawn very carefully with a ruler.

Urge students to make up problems similar to those in the practice sheet for the class to work at. Larger grids lead to more difficult problems, and a cleverly designed shape may require a lot of trial and error. Working at such problems will give students a better feeling for the properties of grids and will be helpful in the study of coordinate systems to be encountered later on in Level F and in mathematics beyond Level F.



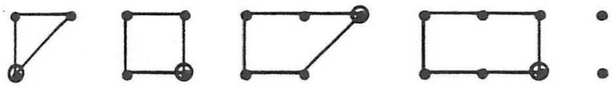
If you can find the entry in the first column of the chart on page 26 — under the (0) — the rest is easy. For example:

	(0)	(1)	(2)	(3)	(4)
5	$1\frac{1}{2}$	$1\frac{1}{2}+1$	$1\frac{1}{2}+2$	$1\frac{1}{2}+3$	$1\frac{1}{2}+4$

So, let's turn to that column of results for shapes with 0 dots inside.

If we have 1 dot or 2 dots, the area is 0.

After the second dot, each additional dot leads to the addition of $\frac{1}{2}$ unit of area.



This suggests a way to find any entry in the left-hand column.

Subtract 2 from the number of dots on the boundary and divide the result by 2.

If we use b as a placeholder for the number of dots on the boundary, we can express the idea by writing:

$$[(b-2) \div 2]$$

To move across the chart, we know we must add as many units of area as there are dots inside. If we use i as a placeholder for the number of dots inside, we can express that idea by writing:

$$+\ i$$

And putting them together, we can find any entry in the chart:

$$[(b-2) \div 2] + i = \text{units of area}$$

That one expression is a summary of all our investigations of this problem. It is a way of describing all the entries in our chart and in extensions of that chart.

If you know the number of dots on the boundary and the number of dots inside, you can find the area without even looking at the shape.

The "streak of lightning" on page 25 is a $20d(1)$. What is its area?

$$[(20-2) \div 2] + \underline{1} = \underline{10}$$

Check your answer on page 25.

Complete the following statements. Use any method you wish, including making sketches.

a $\underline{4d(2)}$ has $\underline{3}$ units of area
a $\underline{7d(1)}$ has $\underline{3\frac{1}{2}}$ units of area
a $\underline{5d(7)}$ has $\underline{8\frac{1}{2}}$ units of area
a $\underline{20d(13)}$ has $\underline{22}$ units of area
a $\underline{3d(3)}$ has $3\frac{1}{2}$ units of area
a $\underline{6d(3)}$ has 5 units of area
a $\underline{12d(4)}$ has 9 units of area
a $\underline{13d(4)}$ has $9\frac{1}{2}$ units of area
a $\underline{4d(16)}$ has 17 units of area
a $\underline{3d(3)}$ has $3\frac{1}{2}$ units of area
a $\underline{7d(2)}$ has $4\frac{1}{2}$ units of area
a $\underline{4d(8)}$ has 9 units of area
a $\underline{11d(5)}$ has $9\frac{1}{2}$ units of area

Complete the following in four different ways:

a $\underline{3d(8)}$ has $8\frac{1}{2}$ units of area
a $\underline{5d(7)}$ has $8\frac{1}{2}$ units of area
a $\underline{11d(4)}$ has $8\frac{1}{2}$ units of area
a $\underline{19d(0)}$ has $8\frac{1}{2}$ units of area

In the late 1800's, a mathematician discovered the formula you have worked out. His name was Pick — and the formula is called *Pick's theorem.*

27 Discovering Pick's Theorem

Tasks: Students study a method for summarizing in a formula the findings of their investigations on pupil pages 24-26. They then use any method—drawing or formula—to complete mathematical statements.

Purpose: To enable students to have a rich experience in geometry by exploration of Pick's Theorem.

Unifying Ideas: Structure; functions and relations; geometry; measurement.

The Lesson: A simple pattern can usually be expressed in a simple formula—and this is the case with Pick's Theorem.

The pattern in the chart can be described as follows: Each additional dot inside adds a full unit of area, and each additional dot on the boundary adds a half unit of area. If d is the number of dots on the boundary and i is the number of dots inside, we might be tempted to write:

$$\frac{d}{2} + i$$

for the number of units of area. But this would lead us, in the case of $3d(0)$, to write:

$$\frac{3}{2} + 0 = 1\frac{1}{2}$$

which is 1 too much, since $3d(0)$ has an area of only ½ unit.

If we applied our obviously incorrect formula to a $4d(2)$, we would write:

$$\frac{4}{2} + 2 = 4$$

which, according to the chart, is again 1 too much since a $4d(2)$ has an area of 3 units.

This suggests that we modify our formula to:

$$\frac{d}{2} + i - 1 = A \text{ (number of units of area)}$$

which, as experiments will show, would give us the same results as the chart.

Thus, one can begin with an educated guess; if it turns out to be inadequate, try it on several examples and look for a common aspect in the errors produced. After this, the remedy is obvious.

We suggest that you consider discussing this approach with the class. It is very different than the development followed on pupil page 27; even the formulas look different. However, both formulas seem to produce the same results. Thus, we are led to the conclusion that the expressions:

$$(d-2) \div 2 + i \quad \text{and:} \quad \frac{d}{2} + i - 1$$

are equivalent.

A student who wishes to remember one or the other of these formulas can make his own choice. However, we see no reason to memorize them, since anytime one wishes to reconstruct them, he can do so in a few minutes by drawing a small grid of dots and carrying out a few experiments.

Pick's Theorem is an excellent example of the kind of surprise that can help students see mathematics as the lively, creative subject that it is.

FPS-27

Janet said: "I've drawn all the squares I can (by connecting dots in the grid) that have an area of 30 units or less. The squares for some areas are easy to find, others hard, the rest impossible. Here's my summary."

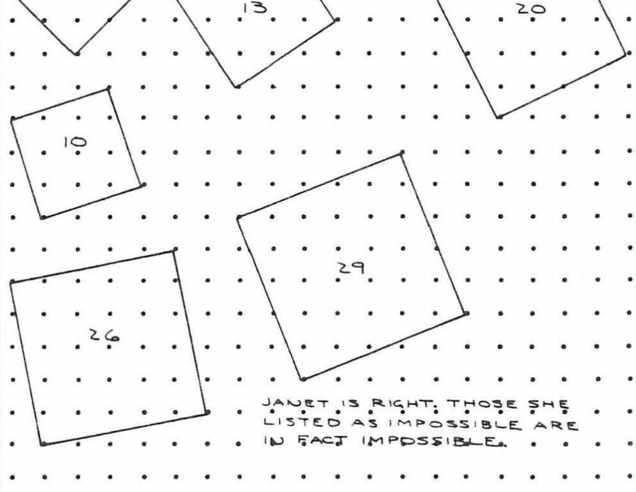
Easy: squares with 1, 4, 9, 16, and 25 units of area.

Hard: squares with 2, 5, 8, 10, 13, 18, 20, 26, and 29 units of area.

Impossible: squares with 3, 6, 7, 11, 12, 14, 17, 19, 21, 22, 23, 24, 27, 28, 30 units.

Can you draw all those in Janet's "Hard" list?

Can you find one she has listed as "Impossible?"



Chapter FIVE — A Problem from Arithmetic

Legend has it that the Grand Vizier invented the game of chess for King Shirham of India. After learning the rules, the King was fascinated by the game and wished to reward the Grand Vizier (who was also a mathematician).

"Oh King," said the inventor, "give me two grains of wheat to put on one square of the chessboard, four for another square, eight for another, sixteen for another — and keep doubling the number of grains until we have accounted for the 64 squares on the board.

"That is a modest request," the King replied. "It is granted!" And he ordered his servants to bring in a bag of wheat.

The legend is told in many different ways. However, you can be sure that in no version is the King able to fulfill his promise.

You might guess that a chart something like the following was found in the Grand Vizier's notebook.

No. of squares	No. of grains for each square	Cumulative total
1st	2	2
2nd	4	6
3rd	8	14
4th	16	30
5th	32	62
6th	64	126
7th	128	254
8th	256	510
9th	512	1,022
10th	1,024	2,046
11th	2,048	4,094
12th	4,096	8,190
13th	8,192	16,382
14th	16,384	32,766
15th	32,768	65,534
16th	65,536	131,070
17th	131,072	262,142
18th	262,144	524,286

A total of more than half a million grains of wheat — and only 18 of the 64 squares accounted for.

As he continued, the Grand Vizier saw no reason to keep the cumulative total. He had stopped adding the previous amount. (Can you see why?)

	Number of grains for each square
19th	524,288
20th	1,048,576
21th	2,097,152
22th	4,194,304
23th	8,388,608
24th	16,777,216
25th	33,554,432
26th	67,108,864
27th	134,217,728
28th	268,435,456
29th	536,870,912
30th	1,073,741,824

Now the cumulative total has grown to 2,147,483,646 GRAINS

28 The Grand Vizier's "modest request"

Tasks: *Students complete the table, giving the number of grains for the first 18 squares and the cumulative totals through the 18th square. They then complete the table for the number of grains from the 19th through the 30th square and record the cumulative total for the 30th square.*

Purpose: To help students see how fast a number "grows" when it is repeatedly doubled.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: Repeated doubling and halving quickly produce great changes.

Here's a brief history of man's land speed record on wheels:

37.5	mph	in	1897	
				> 7 yrs.
75	mph	in	1904	
				> 16 yrs.
150	mph	in	1920	
				> 15 yrs.
300	mph	in	1935	
				> 30 yrs.
600	mph	in	1965	
				> ?
1200	mph	in	?	

The next time man is able to double his land speed on wheels he will have to break the sound barrier. Doubling is a rapid kind of growth.

You might have the students look at an inch on a ruler. What is half an inch? What is half of that? We obtain the sequence 1, ½, ¼, ⅛, $\frac{1}{16}$, . . . Very quickly we have exhausted the markings on the ruler. Halving is a rapid kind of growing smaller.

The Grand Vizier stopped keeping the cumulative total because he noticed that it was 2 less than twice the last term. The cumulative total at the 19th square is $(2 \times 524{,}288) - 2$, or 1,048,574.

The sequence 1, 2, 4, 8, 16, etc., is unique in that

a. every whole number can be expressed as the sum of some combination of terms (repeating none), and

b. there is only one such combination for each whole number.

We started with 2 grains rather than 1 because we are anticipating the use of exponents that follows on later pages—and 2 is the 1st power of 2 (2^1), 4 is the 2nd power of 2 (2^2), and so on.

Several situations are explored on practice sheet 28. The second of these demonstrates the halving process dramatically. A piece of notebook paper can be folded reasonably well only about 5 or 6 times (into 32 or 64 regions). The folds soon get too thick for us to continue.

Pass a sheet of paper around the class. Have each student cut off half of the piece he receives and pass the other half on. Unless you have fewer than twenty students, the task will require better instruments than scissors and the naked eye.

This activity should be fun, and it leads us to consider the following question: How much is

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}+\text{, etc.}$$

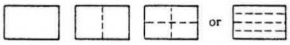
That's not too hard. The first child kept ½ of the sheet of notebook paper, the next kept ¼ of the sheet, the next ⅛, the next $\frac{1}{16}$; the 19th would keep 1/524,288, and so on. Together the class would have 1 sheet of paper—and to keep the cutting going doesn't change the amount of paper. It is easy to see that the sum gets as close to 1 as we might please. We say the sum "approaches 1 as a limit."

FPS-28

I. Write in each block the sum of the two numbers above it — one on the left and the other on the right. Note the sum of all the numbers in each row.

	Sum
1	1
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16
1 5 10 10 5 1	32
1 6 15 20 15 6 1	64

II. Let's say that a sheet of paper with no folds has 1 region; and a piece with 1 fold has 2 regions. A piece of paper folded in half and then folded again has how many regions?



0 folds, 1 region; 1 fold, 2 regions; 2 folds, 4 regions

Continue putting in one more fold each time and counting the regions to find the entries to extend the table.

Folds	0	1	2	3	4	5	6
Regions	1	2	4	8	16	32	64

III. How many half-pints?

1 half-pint is 1 half-pint
1 pint is 2 half-pints
1 quart is 4 half-pints
1 half-gallon is 8 half-pints
1 gallon is 16 half-pints

How could you buy 31 half-pints of milk in the fewest cartons? ONE EACH OF ALL THE CARTONS LISTED.

IV. Extend the sequence below:

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, -----

Complete the following statements using only numbers from the list above:

14 = 2 + 4 + 8
536 = 8 + 16 + 512
2314 = 2 + 8 + 256 + 2,048
832 = 64 + 256 + 512
63 = 1 + 2 + 4 + 8 + 16 + 32
2152 = 8 + 32 + 64 + 2,048
3328 = 256 + 1,024 + 2,048

V. True or False

In the sequence above (IV):

a. Each term larger than 1 is twice as large as the next smaller term in the sequence. T F

b. All terms are even numbers. T F

c. There are no prime numbers in this sequence. T F

d. No number in this sequence is exactly divisible by 3. T F

e. The sum of the terms in the sequence from 1 through 2048 is 4095. T F

f. There is 1 term in the sequence which is exactly divisible by 12. T F

VI. In a reference book, find out what the following weights and lengths mean:

a *rod* is $5\frac{1}{2}$ yd. or $16\frac{1}{2}$ ft.
a *furlong* is 220 YD., 66 FT., 40 RODS.
a *gill* is $\frac{1}{4}$ OF A PINT
a *hundredweight* is 100 LB.
a *meter* is 39.37 IN.
an *acre* is 160 SQ. RODS, 43,560 SQ. FT.
a *peck* is $\frac{1}{4}$ BU. OR 32 QT.
a *league* is 3 MILES

	Number of grains for the square indicated	Cumulative total of grains
64th	18,446,744,073,709,551,616	36,893,488,147,419,103,230

It has been estimated that there are about 5,000,000 grains of wheat in a bushel. The total wheat production of the world in a year is about 2,000,000,000 bushels.

Can you estimate the number of years it would take to fill the Grand Vizier's request if all wheat production were available?

We can only hope that the King and the Grand Vizier had a good laugh and then noticed some interesting facts about all those computations, such as:

1.

1st square	2
2nd square	4
3rd square	8

2.

2nd square	4
4th square	16
6th square	64

3.

2nd square	4
3rd square	8
5th square	32

4.

3rd square	8
4th square	16
7th square	128

5.

2nd	4
2nd	4
4th	16

6.

9th	512
1st	2
10th	1,024

7.

10th	1,024
3rd	8
13th	8,192

8.

12th	4,096
2nd	4
14th	16,384

9.

20th	1,048,576
3rd	8
23rd	8,388,608

10.

6th	64
3RD	8
9th	512

11.

5th	32
5th	32
10TH	1,024

12.

9TH	512
2ND	4
11TH	2,048

13.

6TH	64
6TH	64
12TH	4,096

14.

3RD	8
5TH	32
8TH	256

15.

7TH	128
5TH	32
12TH	4,096

16.

9TH	512
9TH	512
18TH	262,144

17.

2		5		7
4	x	32	=	128

18.

5		6		11
32	x	64	=	2,048

19.

5		8		13
32	x	256		8,192

20.

8		3		5
256	÷	8	=	32

21.

10		4		6
1,024	÷	16	=	64

22.

14		7		7
16,384	÷	128	=	128

23. 16 x 16 = 256

24. 32 x 64 = 2,048

25. 8,192 ÷ 128 = 64

26. 16 x 1,024 = 16,384

27. 256 x 256 = 65,536

28. 16,384 x 32 = 524,288

Tasks: Students complete records of data from the table on pupil page 28, looking for relationships that will simplify computations.

Purpose: To help students discover patterns in the table on pupil page 28 which anticipate later work with logarithms.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Doubling certainly got out of hand for King Shirham. Let's go back and study the records we kept.

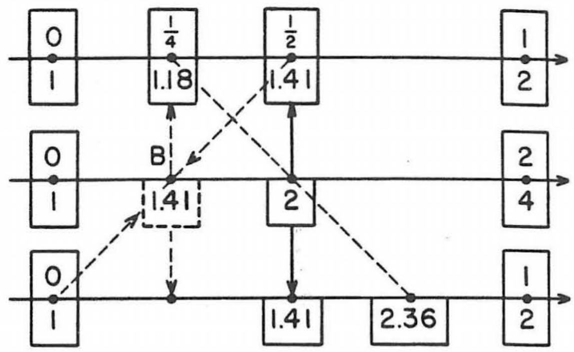
We suggest that, before discussing the page, you put on the board some of the examples at the bottom of pupil page 29—long multiplication and long division examples—with a side comment that the students can do them later.

If one entry from the grain record is multiplied by another, will the product also be in the record (or in an extension of the record)? If one entry is divided by another, will the quotient also be an entry in the record? Can an entry in the record be expressed as the sum of two or more other entries? (Not if repetition of addends is prohibited, because each is 2 more than the sum of all earlier entries.)

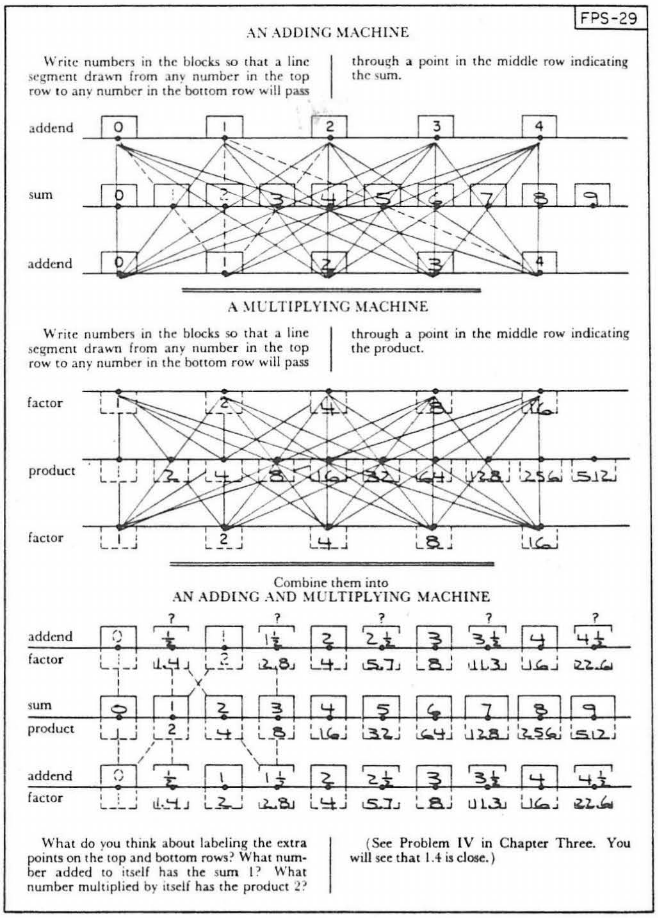
Thus, addition and subtraction take us out of the record. But multiplication and division cannot get us out of the record (or the record extended). See practice sheet 30.

(In this chapter we are exploring an idea that students will meet later in mathematics—the idea of logarithms. This mathematical idea was developed to meet the needs of astronomers who were called upon to perform many multiplication and division computations with very large numbers. To multiply two numbers one looks up one of the factors in a "log table" to find its logarithm, which is another number. Next, he looks up the logarithm of the other factor. He then adds the two logarithms and finds the number whose logarithm is that sum. The procedure shown on pupil page 29 is identical to the one described. The difference is that we have limited ourselves to whole-number powers of 2, whereas standard log tables use fractional powers of 10. We are not suggesting that students are ready for this discussion.)

Note that on the multiplying scales on the practice sheet, the labels are evenly spaced but they grow at the rapid rate of powers of 2. So, finding more points in between is not so easy:



We could expand on this idea and eventually have a clumsy but much more useful computer. The ultimate solution is to buy a slide rule on which the work has already been done.



Because the Grand Vizier was also a skillful mathematician, there is hope that his interest turned to a study of the fascinating results on the previous page.

Mathematicians have extended the results.

		Shorthand		
2 used as a factor 0 times		2^0	is equal to	1
2 used as a factor 1 time	2	2^1		2
2 used as a factor 2 times	$2 \cdot 2$	2^2		4
2 used as a factor 3 times	$2 \cdot 2 \cdot 2$	2^3		8
2 used as a factor 4 times	$2 \cdot 2 \cdot 2 \cdot 2$	2^4		16
2 used as a factor 5 times	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^5		32
2 used as a factor 6 times	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	2^6		64
		2^7		128
		2^8		256
		2^9		512
		2^{10}		1,024
		2^{11}		2,048
		2^{12}		4,096
		2^{13}		8,192
		2^{14}		16,384
		2^{15}		32,768
		2^{16}		65,536
		2^{17}		131,072
		2^{18}		262,144
		2^{19}		524,288
		2^{20}		1,048,576

2^1 is read as "2 to the 1st power"

2^2 is read as "2 to the 2nd power" or "2 squared"

2^3 is read as "2 to the 3rd power" or "2 cubed"

2^4 is read as "2 to the 4th power"

2^5 is read as "2 to the 5th power"

. . . . etc.

2^0 is read as "2 to the zeroth power" but what does it mean? It certainly fits the pattern in the "shorthand" column.

In the column on the right, each number is half of the number below it. So, we hope that 2^0 is half of 2, or

$$\boxed{2^0 = 1} \; .$$

The following examples use 2^0 as 1. Does it cause any trouble?

$2^2 \cdot 2^3 = 2^5$ because $4 \times 8 = 32$

$2^1 \cdot 2^6 = 2^7$ because $2 \times 64 = 128$

$2^0 \cdot 2^3 = 2^3$ because $1 \times 8 = 8$

$2^4 \cdot 2^3 = 2^7$ because $16 \times 8 = 128$

$2^2 \cdot 2^4 = 2^6$ because $4 \times 16 = 64$

$2^5 \cdot 2^0 = 2^5$ because $32 \times 1 = 32$

$2^0 \cdot 2^0 = 2^0$ because $1 \times 1 = 1$

Letting $2^0 = 1$ passes every test.

$2^5 \div 2^3 = 2^2$ because $32 \div 8 = 4$

$2^6 \div 2^1 = 2^5$ because $64 \div 2 = 32$

$2^5 \div 2^4 = 2^1$ because $32 \div 16 = 2$

$2^6 \div 2^0 = 2^6$ because $64 \div 1 = 64$

$2^6 \div 2^4 = 2^2$ because $64 \div 16 = 4$

$2^8 \cdot 2^5 = 2^{13} = 8{,}192$

$2^{17} \cdot 2^3 = 2^{20} = 1{,}048{,}576$

$2^2 \cdot 8 = 2^5 = 32$

$256 \cdot 64 = 2^{14} = 16{,}384$

30-32 Multiplying and dividing powers of 2, 3, 5, and 10 by adding and subtracting exponents

Tasks: *Students extend the results of their work on pupil page 29 and consider the meaning of "2 to the zeroth power."*

Purpose: To provide further practice with exponents. To help students realize that the notation $2^0 = 1$ is meaningful because it helps complete an established pattern, and it leads to reasonable results.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Students may be quite surprised to see us write:

$$2^0 = 1 \qquad 3^0 = 1 \qquad 17^0 = 1 \quad \text{etc.}$$

Most people say to themselves that in expressions such as 2^7 and 2^4, the exponent tells how many 2's "get into the act." But how can one interpret "2 used as a factor 0 times"? If it will help dispel any reluctance to accept this definition of 2^0, one may use this interpretation:

2^3	means	$2\times2\times2\times1$
2^2	means	$2\times2\times1$
2^1	means	2×1
2^0	means	1

The agreement to say that any number to the zeroth power is equal to 1 extends the pattern already established and always leads to reasonable results.

When negative exponents are introduced, the definition of zeroth power fills a troublesome gap in the middle of an otherwise complete system.

Negative exponents are considered on practice sheet 30. You may decide that this topic should be postponed until students are more familiar with negative numbers. Or you may feel that this is an excellent opportunity to introduce or reintroduce signed numbers as a useful system of numbers. We introduce negative numbers as exponents merely as a shorthand:

$$\frac{1}{2} = \frac{1}{2^1} = 2^{-1} \qquad \frac{1}{8} = \frac{1}{2^3} = 2^{-3}$$

And when multiplication or division involves such expressions, one does not need to know any rules to handle the computation; he simply writes the question in a form with which he is more familiar:

I

$$2^{-4} \times 2^{-1} =$$
$$\frac{1}{2^4} \times \frac{1}{2^1} =$$
$$\frac{1}{16} \times \frac{1}{2} = \frac{1}{32}$$
$$= \frac{1}{2^5} = 2^{-5}$$

II

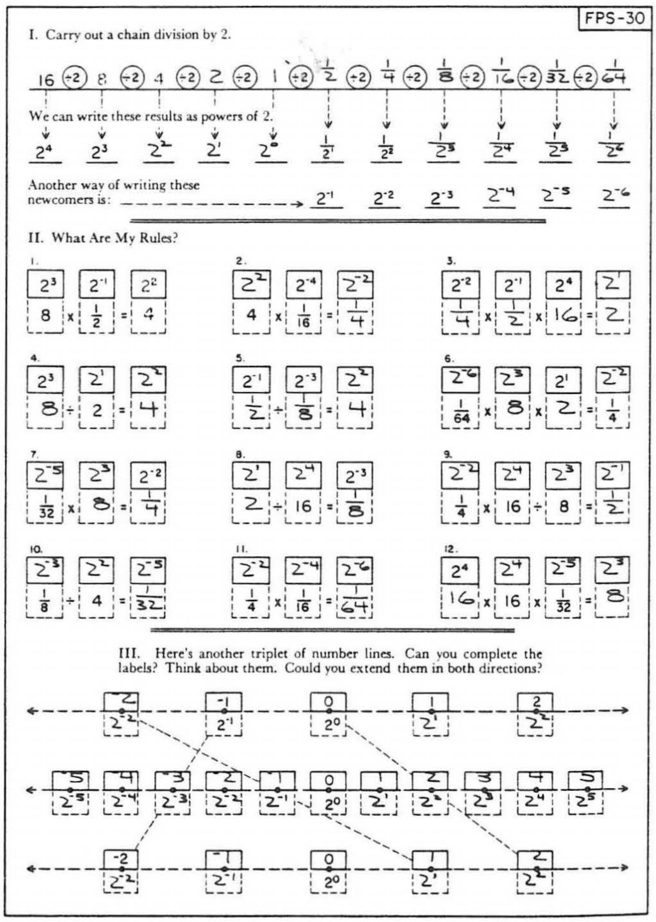
$$2^4 \times 2^{-1} =$$
$$2^4 \times \frac{1}{2^1} =$$
$$16 \times \frac{1}{2} = 8 = 2^3$$

III

$$2^2 \div 2^{-3} =$$
$$2^2 \div \frac{1}{2^3} =$$
$$4 \div \frac{1}{8} = 32 = 2^5$$

IV

$$2^{-2} \div 2^{-3} =$$
$$\frac{1}{2^2} \div \frac{1}{2^3} =$$
$$\frac{1}{4} \div \frac{1}{8} = 2 = 2^1$$



Of course, the Grand Vizier, being a mathematician, would wonder if other tables could be built that would have a similar pattern.

Suppose that we triple each time instead of doubling.

I.

3 as a factor 0 times		3^0	1
3 as a factor 1 time	3	3^1	3
3 as a factor 2 times	$3 \cdot 3$	3^2	9
3 as a factor 3 times	$3 \cdot 3 \cdot 3$	3^3	27
3 as a factor 4 times	$3 \cdot 3 \cdot 3 \cdot 3$	3^4	81
3 as a factor 5 times	$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$	3^5	243
3 as a factor 6 times	$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$	3^6	729

II.

3^3	27
3^2	9
3^5	243

3^2	9
3^1	3
3^3	27

3^4	81
3^0	1
3^4	81

3^4	81
3^2	9
3^6	729

3^3	27
3^3	27
3^6	729

3^4	81
3^1	3
3^5	243

III.

Let's extend the table.

3^7	2,187
3^8	6,561
3^9	19,683
3^{10}	59,049
3^{11}	177,147
3^{12}	531,441
3^{13}	1,594,323

IV.

3^5	243
3^3	27
3^8	6,561

3^6	729
3^4	81
3^{10}	59,049

3^4	81
3^3	27
3^7	2,187

3^4	81
3^7	2,187
3^{11}	177,147

3^6	729
3^6	729
3^{12}	531,441

3^8	6,561
3^5	243
3^{13}	1,594,323

V.

5		5		10
243	x	243	=	59,049
7		0		7
2,187	x	1	=	2,187
9		5		4
19,683	÷	243	=	81
8		12		4
6,561	=	531,441	÷	81
13		6		7
1,594,323	=	729	x	2,187

VI.

6,561 x 27 = 177,147

2,187 ÷ 81 = 27

VII.

$$243 \overline{)531{,}441} \quad = 2{,}187$$

$$\begin{array}{r} 729 \\ \times\ 243 \\ \hline 177{,}147 \end{array}$$

Tasks: Students extend the results of their work on pupil pages 29 and 30 by using powers of 3.

Purpose: To provide students with experience in investigating and using mathematical patterns and mathematical notations involving exponents.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: A chain of doubling tends to get out of hand very fast; tripling does so even faster. Compare the rates at which numbers grow as a result of doubling and tripling:

$2^0 = 1$	$3^0 = 1$
$2^1 = 2$	$3^1 = 3$
$2^2 = 4$	$3^2 = 9$
$2^3 = 8$	$3^3 = 27$
$2^4 = 16$	$3^4 = 81$
$2^5 = 32$	$3^5 = 243$
$2^6 = 64$	$3^6 = 729$

As with powers of 2, will multiplying a pair of entries in the list of triples lead to another entry? Will dividing a larger entry by itself or a smaller entry lead to an entry in the list?

Does addition and subtraction of entries lead to another entry? We noticed that addition for powers of 2 was impossible (without repetition) because each entry was 2 larger than the sum of all smaller entries (starting with 2). What can be said of powers of 3?

Each power of 3 is 1 more than twice the sum of preceding entries:

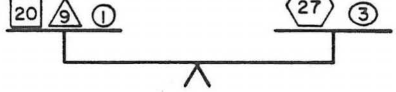
$$3 = 2 \times 1 + 1$$
$$9 = 2 \times (1 + 3) + 1$$
$$27 = 2 \times (1 + 3 + 9) + 1$$
$$81 = 2 \times (1 + 3 + 9 + 27) + 1$$

This observation leads to the discovery that all numbers can be obtained by adding powers of 3, provided we allow each power to be used twice:

1 = 1	6 = 3+3	11 = 9+1+1
2 = 1+1	7 = 3+3+1	12 = 9+3
3 = 3	8 = 3+3+1+1	13 = 9+3+1
4 = 3+1	9 = 9	14 = 9+3+1+1
5 = 3+1+1	10 = 9+1	15 = 9+3+3...etc.

A slightly different use of this relationship is explored on practice sheet 31. Instead of allowing each power of 3 to be used twice, the same results can be obtained by using each power of 3 not more than once, but allowing subtraction as well as addition.

This idea is used by the merchant who has four weights, a 1-pounder, a 3-pounder, a 9-pounder, and a 27-pounder and combines them on a balance scale to make all whole number weights lighter than 41 pounds. Suppose that he wants to weigh out 20 pounds of flour. He puts the 3-pounder and the 27-pounder on the right side of the scale and the 1-pounder and the 9-pounder on the left side. Now, when he has added 20 pounds of flour to the left side, the scale will balance:



Putting weights on one side is equivalent to adding; putting them on the other is equivalent to subtracting; thus $20 = 27 + 3 - 9 - 1$.

FPS-31

I. Extend this sequence in which each term after 1 is three times the preceding term.

1 3 9 27 81 243 729 2,187 6,561 19,683 ......

II. Find a combination of terms from this sequence and use addition or subtraction, or both, to obtain each of the numbers indicated below. (No term can be used more than once in any combination.)

1 = 1	14 = 27-9-3-1	27 = 27
2 = 3-1	15 = 27-9-3	28 = 27+1
3 = 3	16 = 27-9-3+1	29 = 27+3-1
4 = 3+1	17 = 27-9-1	30 = 27+3
5 = 9-3-1	18 = 27-9	31 = 27+3+1
6 = 9-3	19 = 27-9+1	32 = 27+9-3-1
7 = 9-3+1	20 = 27-9+3-1	33 = 27+9-3
8 = 9-1	21 = 27-9+3	34 = 27+9-3+1
9 = 9	22 = 27-9+3+1	35 = 27+9-1
10 = 9+1	23 = 27-3-1	36 = 27+9
11 = 9+3-1	24 = 27-3	37 = 27+9+1
12 = 9+3	25 = 27-3+1	38 = 27+9+3-1
13 = 9+3+1	26 = 27-1	39 = 27+9+3

Skipping around:

40 = 27+9+3+1	85 = 81+3+1	2,106 = 2,187-81
41 = 81-27-9-3-1	86 = 81+9-3-1	4,374 = 6,561-2,187
42 = 81-27-9-3	87 = 81+9-3	405 = 729-243-81
66 = 81-27+9+3	99 = 81+27-9	279 = 243+27+9
67 = 81-27+9+3+1	100 = 81+27-9+1	689 = 729-27-9-3-1
68 = 81-9-3-1	101 = 81+27-9+3-1	20,412 = 19,683+729

III. A merchant had a balance-scale with a tray on each side.

He had a set of 4 weights. They were 1 pound, 3 pounds, 9 pounds, and 27 pounds.

He said he never needed fractions of pounds and he could use the weights and the scale to weigh out any whole number of pounds less than 41 pounds.

Do you believe him? YES

More tables of powers

I.

	5^0	1
5	5^1	5
5·5	5^2	25
5·5·5	5^3	125
5·5·5·5	5^4	625
5·5·5·5·5	5^5	3,125
5·5·5·5·5·5	5^6	15,625
5·5·5·5·5·5·5	5^7	78,125
5·5·5·5·5·5·5·5	5^8	390,625

II.

	7^0	1
7	7^1	7
7·7	7^2	49
7·7·7	7^3	343
7·7·7·7	7^4	2,401
7·7·7·7·7	7^5	16,807
7·7·7·7·7·7	7^6	117,649
7·7·7·7·7·7·7	7^7	823,543

III.

	10^0	1
10	10^1	10
10·10	10^2	100
10·10·10	10^3	1,000
10·10·10·10	10^4	10,000
10·10·10·10·10	10^5	100,000
10·10·10·10·10·10	10^6	1,000,000

In the examples below, use the tables for shortcuts if you wish to.

IV.

2		3		5
25	x	125	=	3,125
6		4		2
15,625	÷	625	=	25
8		3		5
390,625	=	125	x	3,125
4		7		3
625	=	78,125	÷	125

V.

3		3		6
343	x	343	=	117,649
4		3		1
2,401	÷	343	=	7
7		2		5
823,543	÷	49	=	16,807
5		0		5
16,807	x	1	=	16,807

VI.

3		4		1
1,000	=	10,000	÷	10
6		2		4
1,000,000	÷	100	=	10,000
0		3		3
1	x	1,000	=	1,000
2		4		6
100	x	10,000	=	1,000,000

Tasks: Students extend the results of pupil pages 29-31, using powers of 5, 7, and 10.

Purpose: To provide students with experience in investigating and using mathematical patterns and notations involving powers and exponents.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: On pupil page 32 and practice sheet 32, brief explorations are made with powers of 4, 5, 6, 7, 8, 9, and 10. In each case, a table of powers is constructed. Then entries can be multiplied by adding exponents and checking back to the list for the entry with that sum as its exponent. In summary: $a^m \times a^n = a^{m+n}$.

In dividing one entry by another, the difference between the exponents is noted, and the appropriate power is selected from the list. In summary: $a^m \div a^n = a^{m-n}$.

Nothing new develops as lists are created and used until we try powers of 10. The difference here is that the lists are so easy to make:

$$10^0 = 1$$
$$10^1 = 10$$
$$10^2 = 100$$
$$10^3 = 1000$$

We only have to count zeroes! The results we obtain are like those obtained in previous cases:

$$100 \times 100 = 10^2 \times 10^2 = 10^4 = 10{,}000$$
$$1000 \times 10{,}000 = 10^3 \times 10^4 = 10^7 = 10{,}000{,}000$$

and yet this idea grows into the powerful tool of *common logarithms* with its carefully worked-out tables of fractional exponents.

You might wish to discuss ways in which some of the tables we have built could have been developed from others:

$$4^2 = 4 \times 4 = (2 \times 2) \times (2 \times 2) = 2^4$$
$$4^3 = 4 \times 4 \times 4 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) = 2^6$$
$$4^4 = 2^8$$
$$4^5 = 2^{10}$$

These are in the Grand Vizier's list. And again:

$$8^1 = 8 = 2 \times 2 \times 2 = 2^3$$
$$8^2 = 8 \times 8 = (2 \times 2 \times 2) \times (2 \times 2 \times 2) = 2^6$$
$$8^3 = 2^9$$
$$8^4 = 2^{12}$$

These are also in his list. And again:

$$9^1 = 9 = 3 \times 3 = 3^2$$
$$9^2 = 9 \times 9 = (3 \times 3) \times (3 \times 3) = 3^4$$
$$9^3 = 3^6$$

And these are on pupil page 31. Also:

$$6^1 = 6 = 2 \times 3 = 2^1 \times 3^1$$
$$6^2 = 6 \times 6 = (2 \times 3) \times (2 \times 3) = 2^2 \times 3^2$$
$$6^3 = 6 \times 6 \times 6 = (2 \times 3) \times (2 \times 3) \times (2 \times 3) = 2^3 \times 3^3$$
$$6^4 = 2^4 \times 3^4$$

These could be computed from tables already worked out—with some saving of time. And finally:

$$10^1 = 10 = 2 \times 5 = 2^1 \times 5^1$$
$$10^2 = 10 \times 10 = (2 \times 5) \times (2 \times 5) = 2^2 \times 5^2$$
$$10^3 = 10 \times 10 \times 10 = (2 \times 5) \times (2 \times 5) \times (2 \times 5) = 2^3 \times 5^3$$
$$10^4 = 2^4 \times 5^4$$

These could be computed from tables already worked out, but that would be a waste of time.

FPS-32

A		B		C		D	
4^2	16	6^0	1	8^0	1	9^0	1
4^3	64	6^1	6	8^1	8	9^1	9
4^4	256	6^2	36	8^2	64	9^2	81
4^5	1,024	6^3	216	8^3	512	9^3	729
4^6	4,096	6^4	1,296	8^4	4,096	9^4	6,561
4^7	16,384	6^5	7,776	8^5	32,768	9^5	59,049
4^8	65,536	6^6	46,656	8^6	262,144	9^6	531,441
4^9	262,144	6^7	279,936	8^7	2,097,152	9^7	4,782,969
4^{10}	1,048,576	6^8	1,679,616	8^8	16,777,216	9^8	43,046,721
4^{11}	4,194,304	6^9	10,077,696	8^9	134,217,728	9^9	387,420,489

If you complete the tables, the arithmetic below will be quite simple.

1. 1024 x 256 = 262,144
2. 4096 x 512 = 2,097,152
3. 59,049 x 729 = 43,046,721
4. 1296 x 1296 = 1,679,616
5. 65,536) 1,048,576 = 16
6. 81) 4,782,969 = 59,049
7. 216) 46,656 = 216
8. $655.36 x 64 = $41,943.04
9. 6.561 x 7.29 = 47.82969
10. $327.68 x .64 = $209.7152
11. 7776 x 1296 = 10,077,696
12. 8.1) 531.441 = 65.61
13. 327.68) 16,777.216 = 51.2
14. 36) $16,796.16 = $466.56

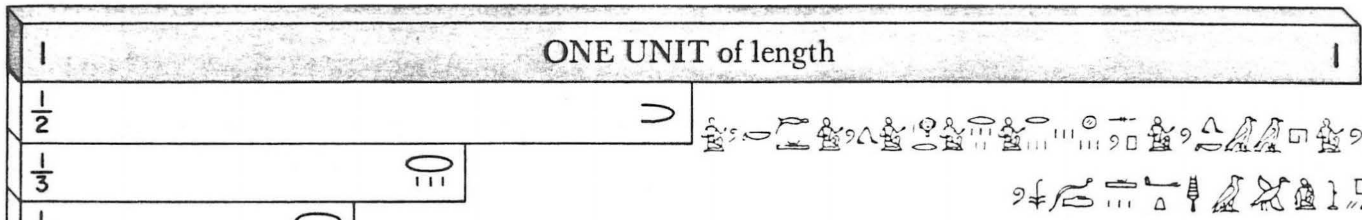
Rather useful tables!

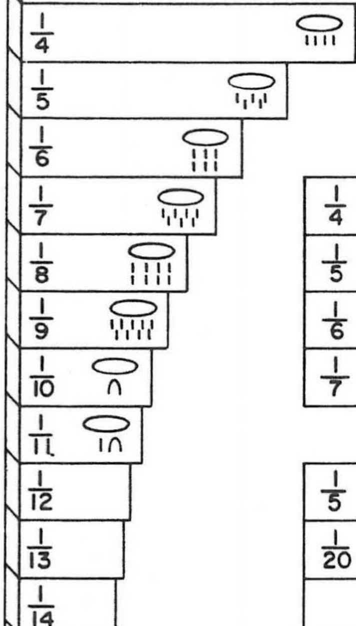
This suggests just a few of the very powerful tools mathematicians have invented to speed up long, tiring computations.

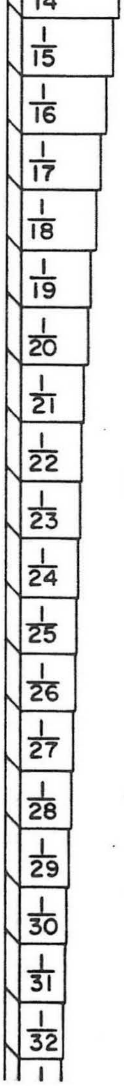
While the tables seem very limited, you will learn later of much more useful ones called *logarithm tables.* They turn hard multiplication problems into easy addition problems.

Use the tables to give *approximate* answers to the following:

15. 8775 x 214 — about 1,679,616
16. 32,765 x 4,103 — approximately 134,217,728
17. 6547 x 6576 — in the neighborhood of 43,046,721
18. 16,375 x 257 — approximately 4,194,304



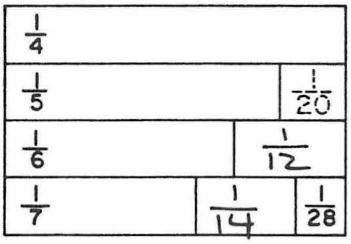


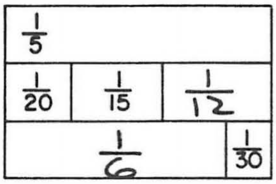


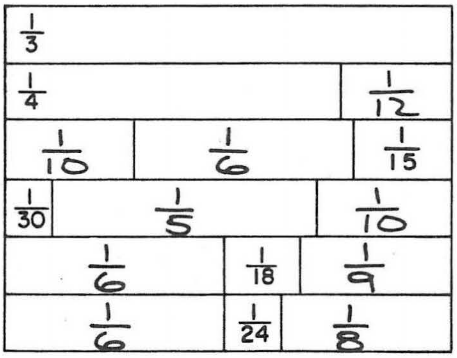
Translated from Egyptian hieroglyphics, this says:

I am half of my friend who is a third of one.

What is the quantity saying it?







Think of a ruler:

$$\frac{3}{8} = \frac{1}{4} + \frac{1}{8} \qquad \frac{9}{16} = \frac{1}{2} + \frac{1}{16}$$

$$\frac{15}{16} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$

Think of U.S. coins:

$$\frac{3}{4} = \frac{1}{4} + \frac{1}{2} \qquad \frac{3}{10} = \frac{1}{4} + \frac{1}{20}$$

$$\frac{90}{100} = \frac{1}{2} + \frac{1}{4} + \frac{1}{10} + \frac{1}{20}$$

In 1858, a young Scotsman, Mr. A. Henry Rhind, reportedly found a piece of papyrus in a ruins near Thebes. It had originally been about 18 feet long and 13 inches high, but parts were missing.

Surprisingly enough, some of those missing parts turned up at the New York Historical Society in about 1900.

The scroll was written about 1700 B.C. When finally translated, it turned out to be a practical handbook showing how the Egyptians counted, computed, and measured more than 3500 years ago.

They used fractions in measurement, but for some reason they followed peculiar rules. They limited themselves to what we call *unit fractions,* fractions with 1 as the numerator. (There was one exception: they had a way of writing $\frac{2}{3}$, but we shall overlook this exception.) And, they would not repeat a unit fraction in an expression. They would not express $\frac{3}{4}$ by writing:

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$$

Instead, they would use two different unit fractions and write:

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$$

Thus, they would choose a unit — such as the green bar above — and consider as smaller units only $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc.

Try your hand at some of the problems suggested by the sketches.

33 Unit fractions

Tasks: Students convert common fractions into expressions involving the sum of unit fractions.

Purpose: To remind students that arithmetic has a history of its own. To consider fractions in an unfamiliar setting. To focus on a physical representation of fractions. To introduce fractions in a problem-solving setting.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: The expression translated from hieroglyphics (which are read from right to left) is a typical ending used by the ancient Egyptians in stating a problem. "I am half of my friend who is a third of one. *What is the quantity saying it?*"

Chapter Six has several purposes. Students need to be reminded that arithmetic has a history of its own—a history of man's inventions of more efficient ways to describe quantities and relationships of quantities. The class might well discuss the question: Why do you think the Egyptians confined themselves in their notation to unit fractions?—noting, of course, that they did use notation for ⅔. Or: Why did they never repeat a unit fraction, thus, for example, writing $\frac{1}{5}+\frac{1}{6}+\frac{1}{30}$ rather than $\frac{1}{5}+\frac{1}{5}$? (One can imagine an Official Measurer with a bag of sticks—1 unit, ½ unit, ⅓ unit, etc.—but only one of each. He would put sticks end to end to describe a length, but once he had used the ⅕-unit stick he would have to search through those left to make a train long enough.)

Another purpose of Chapter Six is to consider fractions in an unfamiliar setting where the usual algorithms are only of incidental concern. It would be unfortunate for a student to think of fractions only in terms of rules for handling denominators and numerators.

Yet another purpose is to focus attention on a physical representation of fractions which will help students develop a feeling for comparing the "sizes" of fractions.

Finally, the restriction to unit fractions introduces an important problem-solving aspect. The student has few rules he can resort to. Rather, he is confronted with new situations that he must think his way through.

Although the use of unit fractions dates back thousands of years, the divisions we use on our rulers follow a very similar system: ½, ¼, ⅛, $\frac{1}{16}$, $\frac{1}{32}$, and sometimes $\frac{1}{64}$. Ordinary measurements (no finer than $\frac{1}{64}$) can be expressed as a combination of these fractions:

$$\frac{43}{64} = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{64}$$

$$(64\text{ths}) \rightarrow (32) + (8) + (2) + (1)$$

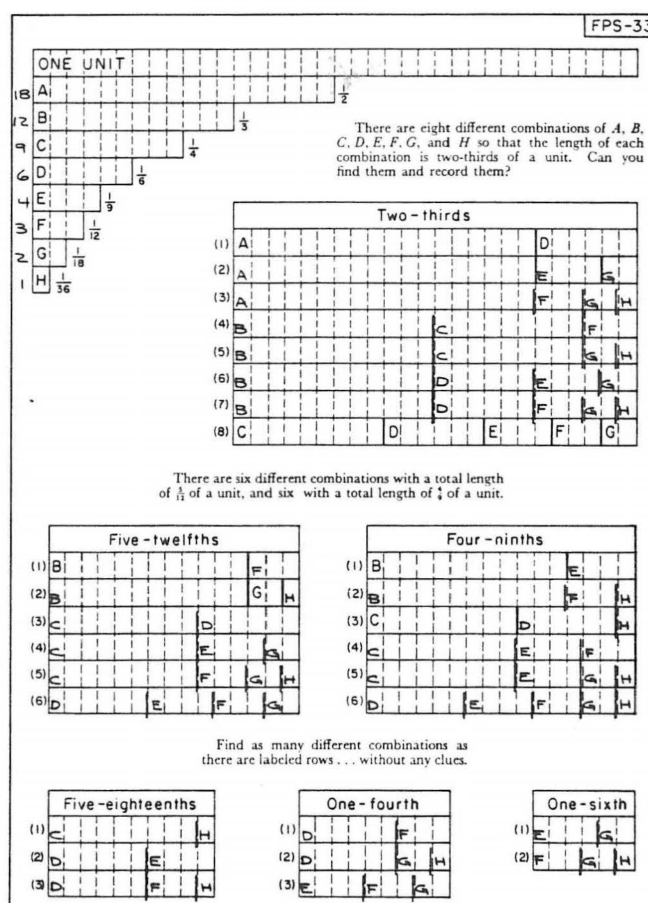
When we make change we often use the same system. We use the fewest coins possible, using several of the same kind only when necessary. Our coins are ½, ¼, $\frac{1}{10}$, $\frac{1}{20}$, and $\frac{1}{100}$ of a dollar:

$$\frac{43}{100} = \frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \frac{3}{100} \text{ or}$$

$$\frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \frac{1}{100} + \frac{1}{100} + \frac{1}{100}$$

$$(25¢) + (10¢) + (5¢) + (3¢)$$

Thus unit fractions give us keener insight into some of the most familiar uses of arithmetic.



Egyptians had fraction troubles.

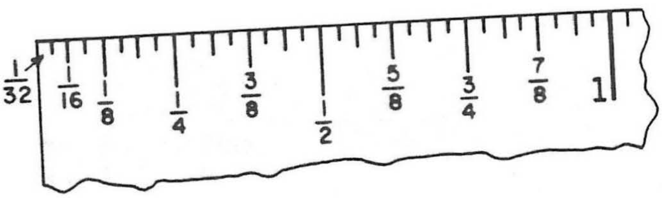
This 3500-year-old way of thinking about fractions is certainly difficult. And, on top of that, the Egyptian method of writing fractions was clumsy. (See the old Egyptian labels on fractions of the unit on the previous page.) The Egyptians would describe $\frac{3}{7}$ by thinking:

$$\frac{3}{7} = \frac{1}{28} + \frac{1}{7} + \frac{1}{4}$$

and they would express $\frac{3}{7}$ this way, without addition signs:

I. Let's see if we can overcome some of these problems, without getting involved in their clumsy notation.

If you think about the divisions on a ruler, you won't have much trouble with the examples below. Through a magnifying glass, an inch on a ruler looks like this:



a.	$\frac{5}{8}$	$\frac{1}{2}+\frac{1}{8}$	j.	$\frac{13}{16}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{16}$
b.	$\frac{3}{8}$	$\frac{1}{4}+\frac{1}{8}$	k.	$\frac{3}{32}$	$\frac{1}{16}+\frac{1}{32}$
c.	$\frac{9}{16}$	$\frac{1}{2}+\frac{1}{16}$	l.	$\frac{5}{32}$	$\frac{1}{8}+\frac{1}{32}$
d.	$\frac{3}{4}$	$\frac{1}{2}+\frac{1}{4}$	m.	$\frac{7}{32}$	$\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$
e.	$\frac{7}{8}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$	n.	$\frac{13}{32}$	$\frac{1}{4}+\frac{1}{8}+\frac{1}{32}$
f.	$\frac{3}{16}$	$\frac{1}{8}+\frac{1}{16}$	o.	$\frac{15}{16}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$
g.	$\frac{7}{16}$	$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$	p.	$\frac{15}{32}$	$\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$
h.	$\frac{9}{16}$	$\frac{1}{2}+\frac{1}{16}$	q.	$\frac{21}{32}$	$\frac{1}{2}+\frac{1}{8}+\frac{1}{32}$
i.	$\frac{11}{16}$	$\frac{1}{2}+\frac{1}{8}+\frac{1}{16}$	r.	$\frac{27}{32}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\frac{1}{32}$

II. Circle those fractions below which can be reduced to unit fractions and cross out those which can't.

$\frac{3}{6}$ $\cancel{\frac{6}{9}}$ $\cancel{\frac{6}{8}}$ $\frac{4}{12}$ $\frac{6}{12}$ $\cancel{\frac{8}{12}}$ $\frac{2}{10}$ $\cancel{\frac{4}{10}}$

$\cancel{\frac{12}{16}}$ $\cancel{\frac{6}{15}}$ $\frac{25}{100}$ $\cancel{\frac{22}{33}}$ $\frac{7}{28}$ $\frac{9}{99}$ $\frac{13}{52}$

III. In the top row of each chart below, complete the fractions so they are already or can be reduced to unit fractions. Indicate the unit fraction in the bottom row.

	a.	b.	c.	d.	e.	f.	g.	h.	i.
1.	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{1}{10}$
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{10}$

	a.	b.	c.	d.	e.	f.	g.	h.	i.
2.	$\frac{2}{10}$	$\frac{5}{10}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{4}{12}$	$\frac{6}{12}$	$\frac{1}{15}$	$\frac{3}{15}$
	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{15}$	$\frac{1}{5}$

	a.	b.	c.	d.	e.	f.	g.	h.	i.
3.	$\frac{5}{15}$	$\frac{1}{18}$	$\frac{2}{18}$	$\frac{3}{18}$	$\frac{6}{18}$	$\frac{9}{18}$	$\frac{1}{20}$	$\frac{2}{20}$	$\frac{4}{20}$
	$\frac{1}{3}$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{20}$	$\frac{1}{10}$	$\frac{1}{5}$

	a.	b.	c.	d.	e.	f.	g.	h.	i.
4.	$\frac{1}{24}$	$\frac{2}{24}$	$\frac{3}{24}$	$\frac{4}{24}$	$\frac{6}{24}$	$\frac{8}{24}$	$\frac{12}{24}$	$\frac{5}{20}$	$\frac{10}{20}$
	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$

IV. How can we obtain $\frac{7}{12}$ as a sum of unit fractions? Look in row 2 above.

$$\frac{7}{12} = \frac{6}{12}+\frac{1}{12} \text{ or } \frac{4}{12}+\frac{3}{12} \text{ or } \frac{4}{12}+\frac{2}{12}+\frac{1}{12}$$

Now reduce all fractions to unit fractions:

$$\frac{7}{12} = \frac{1}{2}+\frac{1}{12} \text{ or } \frac{1}{3}+\frac{1}{4} \text{ or } \frac{1}{3}+\frac{1}{6}+\frac{1}{12}$$

V. Find two ways to obtain $\frac{11}{12}$ as a sum of three unit fractions.

$$\frac{11}{12} = \frac{1}{2}+\frac{1}{4}+\frac{1}{6} \text{ or } \frac{1}{2}+\frac{1}{3}+\frac{1}{12}$$

Tasks: Students convert common fractions into expressions involving the sum of unit fractions.

Purpose: To remind students that arithmetic has a history of its own. To consider fractions in an unfamiliar setting. To focus on a physical representation of fractions. To introduce fractions in a problem-solving setting.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: Unit fractions are explored in the context of divisions of an inch on pupil page 34 and in the context of U.S. coins on pupil page 35.

The class might consider parts of a dozen eggs within the context of unit fractions. We begin by expressing 1, 2, 3, 4, and 6 as unit fractions of 1 dozen:

	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$
eggs	6	4	3	2	1

But we have none to indicate 5, 7, 8, 9, 10, or 11 eggs. However, each can be indicated by a combination of two or more of the unit fractions above:

$$5 = 4 + 1 = 3 + 2$$

$$7 = 6 + 1 = 4 + 3 = 4 + 2 + 1$$

$$8 = 6 + 2$$

$$9 = 6 + 3 = 6 + 2 + 1 = 4 + 3 + 2$$

$$10 = 6 + 4 = 6 + 3 + 1 = 4 + 3 + 2 + 1$$

$$11 = 6 + 4 + 1 = 6 + 3 + 2$$

which leads us to write, in terms of dozens:

$$\frac{5}{12} = \frac{1}{3} + \frac{1}{12} = \frac{1}{4} + \frac{1}{6}$$

$$\frac{7}{12} = \frac{1}{2} + \frac{1}{12} = \frac{1}{3} + \frac{1}{4} = \frac{1}{3} + \frac{1}{6} + \frac{1}{12}$$

$$\frac{2}{3} = \frac{8}{12} = \frac{1}{2} + \frac{1}{6}$$

$$\frac{3}{4} = \frac{9}{12} = \frac{1}{2} + \frac{1}{4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3} + \frac{1}{4} + \frac{1}{6}$$

$$\frac{5}{6} = \frac{10}{12} = \frac{1}{2} + \frac{1}{3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{12} = \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}$$

$$\frac{11}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{12} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6}$$

A similar result follows from consideration of inches and feet, since the foot as well as the dozen is divided into twelve subunits.

Practice sheet 34, in effect, explores all such subdivisions that lead to using denominators less than 100. Consider the fractions with denominator 30:

$$\frac{15}{30} \quad \frac{10}{30} \quad \frac{6}{30} \quad \frac{5}{30} \quad \frac{3}{30} \quad \frac{2}{30} \quad \frac{1}{30}$$

Their unit fraction forms are:

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{10} \quad \frac{1}{15} \quad \frac{1}{30}$$

With this information we write other fractions with denominator 30 as sums of unit fractions:

$$\frac{4}{30} = \frac{3}{30} + \frac{1}{30} = \frac{1}{10} + \frac{1}{30}$$

$$\frac{29}{30} = \frac{15}{30} + \frac{10}{30} + \frac{3}{30} + \frac{1}{30} = \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{30}$$

By inspection, we find that all whole numbers less than 30 can be expressed as some combination of 1, 2, 3, 5, 6, 10, and 15; these, of course, are all the whole-number factors of 30 less than 30 itself.

Notice that each prime denominator has but a single unit fraction in its family, with itself as denominator and 1 as numerator.

UNIT FRACTIONS

FPS-34

How many fractions smaller than 1 and with denominators less than 100 are, or can be reduced to, unit fractions? 374

$\boxed{\frac{1}{2}}$	$\boxed{\frac{1}{3}}$	$\frac{2}{4}$	$\frac{1}{4}$	$\boxed{\frac{1}{5}}$	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\boxed{\frac{1}{7}}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{3}{9}$	$\frac{1}{9}$	$\frac{5}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\boxed{\frac{1}{11}}$
$\frac{6}{12}$	$\frac{4}{12}$	$\frac{3}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\boxed{\frac{1}{13}}$	$\frac{7}{14}$	$\frac{2}{14}$	$\frac{1}{14}$	$\frac{5}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{8}{16}$	$\frac{4}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\boxed{\frac{1}{17}}$	$\frac{9}{18}$
$\frac{6}{18}$	$\frac{3}{18}$	$\frac{2}{18}$	$\frac{1}{18}$	$\boxed{\frac{1}{19}}$	$\frac{10}{20}$	$\frac{5}{20}$	$\frac{4}{20}$	$\frac{2}{20}$	$\frac{1}{20}$	$\frac{7}{21}$	$\frac{3}{21}$	$\frac{1}{21}$	$\frac{11}{22}$	$\frac{2}{22}$	$\frac{1}{22}$	$\boxed{\frac{1}{23}}$	$\frac{12}{24}$
$\frac{8}{24}$	$\frac{6}{24}$	$\frac{4}{24}$	$\frac{3}{24}$	$\frac{2}{24}$	$\frac{1}{24}$	$\frac{5}{25}$	$\frac{1}{25}$	$\frac{13}{26}$	$\frac{2}{26}$	$\frac{1}{26}$	$\frac{9}{27}$	$\frac{3}{27}$	$\frac{1}{27}$	$\frac{14}{28}$	$\frac{7}{28}$	$\frac{4}{28}$	$\frac{2}{28}$
$\frac{1}{28}$	$\boxed{\frac{1}{29}}$	$\frac{15}{30}$	$\frac{10}{30}$	$\frac{6}{30}$	$\frac{5}{30}$	$\frac{3}{30}$	$\frac{2}{30}$	$\frac{1}{30}$	$\boxed{\frac{1}{31}}$	$\frac{16}{32}$	$\frac{8}{32}$	$\frac{4}{32}$	$\frac{2}{32}$	$\frac{1}{32}$	$\frac{11}{33}$	$\frac{3}{33}$	$\frac{1}{33}$
$\frac{17}{34}$	$\frac{2}{34}$	$\frac{1}{34}$	$\frac{7}{35}$	$\frac{5}{35}$	$\frac{1}{35}$	$\frac{18}{36}$	$\frac{12}{36}$	$\frac{9}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\boxed{\frac{1}{37}}$	$\frac{19}{38}$	$\frac{2}{38}$	$\frac{1}{38}$
$\frac{13}{39}$	$\frac{3}{39}$	$\frac{1}{39}$	$\frac{20}{40}$	$\frac{10}{40}$	$\frac{8}{40}$	$\frac{5}{40}$	$\frac{4}{40}$	$\frac{2}{40}$	$\frac{1}{40}$	$\boxed{\frac{1}{41}}$	$\frac{21}{42}$	$\frac{14}{42}$	$\frac{7}{42}$	$\frac{6}{42}$	$\frac{3}{42}$	$\frac{2}{42}$	$\frac{1}{42}$
$\boxed{\frac{1}{43}}$	$\frac{22}{44}$	$\frac{11}{44}$	$\frac{4}{44}$	$\frac{2}{44}$	$\frac{1}{44}$	$\frac{15}{45}$	$\frac{9}{45}$	$\frac{5}{45}$	$\frac{3}{45}$	$\frac{1}{45}$	$\frac{23}{46}$	$\frac{2}{46}$	$\frac{1}{46}$	$\boxed{\frac{1}{47}}$	$\frac{24}{48}$	$\frac{16}{48}$	$\frac{12}{48}$
$\frac{8}{48}$	$\frac{6}{48}$	$\frac{4}{48}$	$\frac{3}{48}$	$\frac{2}{48}$	$\frac{1}{48}$	$\frac{7}{49}$	$\frac{1}{49}$	$\frac{25}{50}$	$\frac{10}{50}$	$\frac{5}{50}$	$\frac{2}{50}$	$\frac{1}{50}$	$\frac{17}{51}$	$\frac{3}{51}$	$\frac{1}{51}$	$\frac{26}{52}$	$\frac{13}{52}$
$\frac{4}{52}$	$\frac{2}{52}$	$\frac{1}{52}$	$\boxed{\frac{1}{53}}$	$\frac{27}{54}$	$\frac{18}{54}$	$\frac{9}{54}$	$\frac{6}{54}$	$\frac{3}{54}$	$\frac{2}{54}$	$\frac{1}{54}$	$\frac{11}{55}$	$\frac{5}{55}$	$\frac{1}{55}$	$\frac{28}{56}$	$\frac{14}{56}$	$\frac{8}{56}$	$\frac{7}{56}$
$\frac{4}{56}$	$\frac{2}{56}$	$\frac{1}{56}$	$\frac{19}{57}$	$\frac{3}{57}$	$\frac{1}{57}$	$\frac{29}{58}$	$\frac{2}{58}$	$\frac{1}{58}$	$\boxed{\frac{1}{59}}$	$\frac{30}{60}$	$\frac{20}{60}$	$\frac{15}{60}$	$\frac{12}{60}$	$\frac{10}{60}$	$\frac{6}{60}$	$\frac{5}{60}$	$\frac{4}{60}$
$\frac{3}{60}$	$\frac{2}{60}$	$\frac{1}{60}$	$\boxed{\frac{1}{61}}$	$\frac{31}{62}$	$\frac{2}{62}$	$\frac{1}{62}$	$\frac{21}{63}$	$\frac{9}{63}$	$\frac{7}{63}$	$\frac{3}{63}$	$\frac{1}{63}$	$\frac{32}{64}$	$\frac{16}{64}$	$\frac{8}{64}$	$\frac{4}{64}$	$\frac{2}{64}$	$\frac{1}{64}$
$\frac{13}{65}$	$\frac{5}{65}$	$\frac{1}{65}$	$\frac{33}{66}$	$\frac{22}{66}$	$\frac{11}{66}$	$\frac{6}{66}$	$\frac{3}{66}$	$\frac{2}{66}$	$\frac{1}{66}$	$\boxed{\frac{1}{67}}$	$\frac{34}{68}$	$\frac{17}{68}$	$\frac{4}{68}$	$\frac{2}{68}$	$\frac{1}{68}$	$\frac{23}{69}$	$\frac{3}{69}$
$\frac{1}{69}$	$\frac{35}{70}$	$\frac{14}{70}$	$\frac{10}{70}$	$\frac{7}{70}$	$\frac{5}{70}$	$\frac{2}{70}$	$\frac{1}{70}$	$\boxed{\frac{1}{71}}$	$\frac{36}{72}$	$\frac{24}{72}$	$\frac{18}{72}$	$\frac{12}{72}$	$\frac{9}{72}$	$\frac{8}{72}$	$\frac{6}{72}$	$\frac{4}{72}$	$\frac{3}{72}$
$\frac{2}{72}$	$\frac{1}{72}$	$\boxed{\frac{1}{73}}$	$\frac{37}{74}$	$\frac{2}{74}$	$\frac{1}{74}$	$\frac{25}{75}$	$\frac{15}{75}$	$\frac{5}{75}$	$\frac{3}{75}$	$\frac{1}{75}$	$\frac{38}{76}$	$\frac{19}{76}$	$\frac{4}{76}$	$\frac{2}{76}$	$\frac{1}{76}$	$\frac{11}{77}$	$\frac{7}{77}$
$\frac{1}{77}$	$\frac{39}{78}$	$\frac{26}{78}$	$\frac{13}{78}$	$\frac{6}{78}$	$\frac{3}{78}$	$\frac{2}{78}$	$\frac{1}{78}$	$\boxed{\frac{1}{79}}$	$\frac{40}{80}$	$\frac{20}{80}$	$\frac{16}{80}$	$\frac{10}{80}$	$\frac{8}{80}$	$\frac{5}{80}$	$\frac{4}{80}$	$\frac{2}{80}$	$\frac{1}{80}$
$\frac{27}{81}$	$\frac{9}{81}$	$\frac{3}{81}$	$\frac{1}{81}$	$\frac{41}{82}$	$\frac{2}{82}$	$\frac{1}{82}$	$\boxed{\frac{1}{83}}$	$\frac{42}{84}$	$\frac{28}{84}$	$\frac{21}{84}$	$\frac{14}{84}$	$\frac{12}{84}$	$\frac{7}{84}$	$\frac{6}{84}$	$\frac{4}{84}$	$\frac{3}{84}$	$\frac{2}{84}$
$\frac{1}{84}$	$\frac{17}{85}$	$\frac{5}{85}$	$\frac{1}{85}$	$\frac{43}{86}$	$\frac{2}{86}$	$\frac{1}{86}$	$\frac{29}{87}$	$\frac{3}{87}$	$\frac{1}{87}$	$\frac{44}{88}$	$\frac{22}{88}$	$\frac{11}{88}$	$\frac{8}{88}$	$\frac{4}{88}$	$\frac{2}{88}$	$\frac{1}{88}$	$\boxed{\frac{1}{89}}$
$\frac{45}{90}$	$\frac{30}{90}$	$\frac{18}{90}$	$\frac{15}{90}$	$\frac{10}{90}$	$\frac{9}{90}$	$\frac{6}{90}$	$\frac{5}{90}$	$\frac{3}{90}$	$\frac{2}{90}$	$\frac{1}{90}$	$\frac{13}{91}$	$\frac{7}{91}$	$\frac{1}{91}$	$\frac{46}{92}$	$\frac{23}{92}$	$\frac{4}{92}$	$\frac{2}{92}$
$\frac{1}{92}$	$\frac{31}{93}$	$\frac{3}{93}$	$\frac{1}{93}$	$\frac{47}{94}$	$\frac{2}{94}$	$\frac{1}{94}$	$\frac{19}{95}$	$\frac{5}{95}$	$\frac{1}{95}$	$\frac{48}{96}$	$\frac{32}{96}$	$\frac{24}{96}$	$\frac{16}{96}$	$\frac{12}{96}$	$\frac{8}{96}$	$\frac{6}{96}$	$\frac{4}{96}$
$\frac{3}{96}$	$\frac{2}{96}$	$\frac{1}{96}$	$\boxed{\frac{1}{97}}$	$\frac{49}{98}$	$\frac{14}{98}$	$\frac{7}{98}$	$\frac{2}{98}$	$\frac{1}{98}$	$\frac{33}{99}$	$\frac{11}{99}$	$\frac{9}{99}$	$\frac{3}{99}$	$\frac{1}{99}$				

ONE DOLLAR will be our unit.

I. Suppose that we have 1 each of the regular U.S. coins less than a dollar. Indicate with unit fractions the fractional part that each coin is of the unit.

50¢	25¢	10¢	5¢	1¢
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{100}$

Remember that we have only 1 of each. Indicate a combination of those coins whose sum is the amount on the left.

a.	75¢	$\frac{1}{2}+\frac{1}{4}$
b.	36¢	$\frac{1}{4}+\frac{1}{10}+\frac{1}{100}$
c.	81¢	$\frac{1}{2}+\frac{1}{4}+\frac{1}{20}+\frac{1}{100}$
d.	40¢	$\frac{1}{4}+\frac{1}{10}+\frac{1}{20}$
e.	16¢	$\frac{1}{10}+\frac{1}{20}+\frac{1}{100}$
f.	$\$\frac{3}{10}$	$\frac{1}{4}+\frac{1}{20}$
g.	$\$\frac{3}{20}$	$\frac{1}{10}+\frac{1}{20}$
h.	$\$\frac{31}{100}$	$\frac{1}{4}+\frac{1}{20}+\frac{1}{100}$
i.	$\$\frac{41}{100}$	$\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{100}$
j.	$\$\frac{7}{20}$	$\frac{1}{4}+\frac{1}{10}$
k.	$\$\frac{3}{5}$	$\frac{1}{2}+\frac{1}{10}$
l.	$\$\frac{9}{25}$	$\frac{1}{4}+\frac{1}{10}+\frac{1}{100}$
m.	$\$\frac{4}{5}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{20}$
n.	$\$\frac{14}{25}$	$\frac{1}{2}+\frac{1}{20}+\frac{1}{100}$
o.	$\$\frac{13}{20}$	$\frac{1}{2}+\frac{1}{10}+\frac{1}{20}$
p.	$.91	$\frac{1}{2}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{100}$

II. Now suppose that we invent three new coins — a 2¢ piece, a 4¢ piece, and a 20¢ piece — all unit fractions of our dollar unit.

50¢	25¢	20¢	10¢	5¢	4¢	2¢	1¢
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{25}$	$\frac{1}{50}$	$\frac{1}{100}$

Remember that we have only 1 of each. Indicate a combination of those coins whose sum is the amount on the left. (Some sums can be made in more than one way. Show any one you like.)

a.	35¢	$\frac{1}{4}+\frac{1}{10}=\frac{1}{5}+\frac{1}{10}+\frac{1}{20}=\frac{1}{5}+\frac{1}{10}+\frac{1}{25}+\frac{1}{100}$
b.	$.19	$\frac{1}{10}+\frac{1}{20}+\frac{1}{25}$
c.	$\$\frac{9}{10}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}$
d.	$.49	$\frac{1}{4}+\frac{1}{5}+\frac{1}{25}$
e.	$.33	$\frac{1}{4}+\frac{1}{20}+\frac{1}{50}+\frac{1}{100}$
f.	$\$\frac{17}{50}$	$\frac{1}{4}+\frac{1}{20}+\frac{1}{25}$
g.	$\$\frac{4}{5}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{20}$
h.	$\$\frac{26}{40}$	$\frac{1}{2}+\frac{1}{10}+\frac{1}{20}$
i.	99¢	$\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{25}$
j.	$\$\frac{6}{8}$	$\frac{1}{2}+\frac{1}{4}$
k.	$\$\frac{48}{100}$	$\frac{1}{4}+\frac{1}{5}+\frac{1}{50}+\frac{1}{100}$
l.	$.83	$\frac{1}{2}+\frac{1}{4}+\frac{1}{20}+\frac{1}{50}+\frac{1}{100}$
m.	$\$\frac{14}{35}$	$\frac{1}{4}+\frac{1}{10}+\frac{1}{20}$
n.	$\$\frac{51}{75}$	$\frac{1}{2}+\frac{1}{10}+\frac{1}{20}+\frac{1}{50}+\frac{1}{100}$
o.	$\$\frac{19}{20}$	$\frac{1}{2}+\frac{1}{4}+\frac{1}{5}$
p.	$1.15	$\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{10}+\frac{1}{20}+\frac{1}{25}+\frac{1}{100}$

You know more about the ancient Egyptian's fractions than you thought you did.

35 One dollar will be our unit

Tasks: Students express amounts of money as sums of unit fractions of a dollar.

Purpose: To remind students that arithmetic has a history of its own. To consider fractions in an unfamiliar setting. To focus on a physical representation of fractions. To introduce fractions in a problem-solving setting.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: As mentioned earlier, our system of coins contains the germ of the unit fraction idea, but we cannot enforce the rule against using coins of the same denomination if we are to make amounts such as 2¢, 3¢, 4¢, 45¢, etc.

So, after exploring our limited system using $\frac{1}{100}$, $\frac{1}{20}$, $\frac{1}{10}$, $\frac{1}{4}$, and $\frac{1}{2}$, we imagine new coins—a 20¢ piece, a 4¢ piece, and a 2¢ piece. Thus, we have a coin for each fraction which is smaller than 1, has 100 as denominator, and is a unit fraction or can be reduced to a unit fraction:

$$\frac{1}{100},\ \frac{2}{100},\ \frac{4}{100},\ \frac{5}{100},\ \frac{10}{100},\ \frac{20}{100},\ \frac{25}{100},\ \frac{50}{100}$$

or as unit fractions:

$$\frac{1}{100},\ \frac{1}{50},\ \frac{1}{25},\ \frac{1}{20},\ \frac{1}{10},\ \frac{1}{5},\ \frac{1}{4},\ \frac{1}{2}$$

Notice that the numerators in the first list are all the factors of 100 less than 100 and the denominators in the second list are all the factors of 100 larger than 1.

Can every whole number less than 100 be expressed as the sum of a combination of those factors —1, 2, 4, 5, 10, 20, 25, and 50—without using any one of them twice in a single combination? Let the students discuss the question. They will soon conclude that the answer is yes and that some can be expressed by several combinations of coins.

Practice sheet 35 is designed to challenge each student's ingenuity in devising ways of working with unit fractions. There are, of course, many algorithms that were useful to the Egyptians, but we do not intend to burden the students with any such shortcuts unless they have direct application to our present economical methods. On the other hand, the problems they encounter in work such as that on practice sheet 35 helps develop a facility in handling simple fractions without resort to rules.

All of the examples on practice sheet 35 can be solved in the same way (except the first, in which the answer is already given). Consider the last example, exercise 39: Write $\frac{19}{20}$ as the sum of unit fractions. We can write out all fractions with denominator 20 that can be reduced to unit fractions:

$$\frac{10}{20}\quad \frac{5}{20}\quad \frac{4}{20}\quad \frac{2}{20}\quad \frac{1}{20}$$

$$\text{or}\quad \frac{1}{2}\quad \frac{1}{4}\quad \frac{1}{5}\quad \frac{1}{10}\quad \frac{1}{20}$$

Since $19 = 10 + 5 + 4$, we know that

$$\frac{19}{20} = \frac{1}{2} + \frac{1}{4} + \frac{1}{5}$$

On closer examination, we see that that is the only way we can express it without using denominators larger than 20.

FPS-35

Express the given fractions as the sum of the indicated number of unit fractions. Some may have several answers. Use practice sheet 34 and all the shortcuts you can invent or learn from others.

1. $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$
2. $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$
3. $\frac{5}{6}\quad \frac{1}{2} + \frac{1}{3}$
4. $\frac{3}{8} = \frac{1}{4} + \frac{1}{8}$
5. $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$
6. $\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$
7. $\frac{4}{9} = \frac{1}{3} + \frac{1}{9}$
8. $\frac{3}{10} = \frac{1}{5} + \frac{1}{10}$
9. $\frac{6}{10} = \frac{1}{2} + \frac{1}{10}$
10. $\frac{7}{10} = \frac{1}{2} + \frac{1}{5}$
11. $\frac{8}{10} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$
12. $\frac{5}{12} = \frac{1}{3} + \frac{1}{12}$
13. $\frac{7}{12} = \frac{1}{2} + \frac{1}{12}$
14. $\frac{11}{12} = \frac{1}{2} + \frac{1}{3} + \frac{1}{12}$
15. $\frac{3}{14} = \frac{1}{7} + \frac{1}{14}$
16. $\frac{8}{14} = \frac{1}{2} + \frac{1}{14}$
17. $\frac{9}{14} = \frac{1}{2} + \frac{1}{7}$
18. $\frac{10}{14} = \frac{1}{2} + \frac{1}{7} + \frac{1}{14}$
19. $\frac{4}{15} = \frac{1}{5} + \frac{1}{15}$
20. $\frac{6}{15} = \frac{1}{3} + \frac{1}{15}$
21. $\frac{8}{15} = \frac{1}{3} + \frac{1}{5}$
22. $\frac{9}{15} = \frac{1}{3} + \frac{1}{5} + \frac{1}{15}$
23. $\frac{3}{16} = \frac{1}{8} + \frac{1}{16}$
24. $\frac{5}{16} = \frac{1}{4} + \frac{1}{16}$
25. $\frac{7}{16} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$
26. $\frac{9}{16} = \frac{1}{2} + \frac{1}{16}$
27. $\frac{11}{16} = \frac{1}{2} + \frac{1}{8} + \frac{1}{16}$
28. $\frac{13}{16} = \frac{1}{2} + \frac{1}{4} + \frac{1}{16}$
29. $\frac{15}{16} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$
30. $\frac{5}{18} = \frac{1}{6} + \frac{1}{9}$
31. $\frac{7}{18} = \frac{1}{3} + \frac{1}{18}$
32. $\frac{11}{18} = \frac{1}{2} + \frac{1}{9}$
33. $\frac{3}{20} = \frac{1}{10} + \frac{1}{20}$
34. $\frac{7}{20} = \frac{1}{4} + \frac{1}{10}$
35. $\frac{8}{20} = \frac{1}{4} + \frac{1}{10} + \frac{1}{20}$
36. $\frac{9}{20} = \frac{1}{4} + \frac{1}{5}$
37. $\frac{11}{20} = \frac{1}{2} + \frac{1}{20}$
38. $\frac{17}{20} = \frac{1}{2} + \frac{1}{4} + \frac{1}{10}$
39. $\frac{19}{20} = \frac{1}{2} + \frac{1}{4} + \frac{1}{5}$

a. What is the largest fraction that can be written as the sum of two different unit fractions?

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$$

b. Is there more than one way to express that fraction as the sum of two different unit fractions? No

c. Make each of the following into fractions that are smaller than 1 and whose numerators are greater than 1.

MANY ANSWERS POSSIBLE.

$$\frac{2}{3}\quad \frac{9}{11}\quad \frac{3}{5}\quad \frac{4}{13}\quad \frac{4}{19}\quad \frac{5}{29}\quad \frac{6}{7}\quad \frac{7}{17}$$

$$\frac{8}{31}\quad \frac{9}{61}\quad \frac{10}{71}\quad \frac{11}{37}\quad \frac{12}{59}\quad \frac{13}{73}\quad \frac{14}{67}\quad \frac{15}{53}$$

How many can be reduced? NONE

How many could have been reduced if you carefully selected the numerators and followed the rules? NONE

d. Are all the denominators listed above prime numbers? YES

Subtracting one unit fraction from another.

I. Try a few: remember that the difference must be a unit fraction or a sum of unit fractions.

a. $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

b. $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

c. $\frac{1}{9} - \frac{1}{10} = \frac{1}{90}$

d. $\frac{1}{7} - \frac{1}{8} = \frac{1}{56}$

e. $\frac{1}{4} - \frac{1}{5} = \frac{1}{20}$

f. $\frac{1}{6} - \frac{1}{7} = \frac{1}{42}$

g. $\frac{1}{5} - \frac{1}{6} = \frac{1}{30}$

h. $\frac{1}{20} - \frac{1}{21} = \frac{1}{420}$

Easy! . . . after you have completed a few.

But those are all examples of a special case — the denominators are consecutive whole numbers.

II. In the green-tinted blocks below, use only unit fractions:

a. $\frac{1}{4} - \frac{1}{6} = \frac{3}{12} - \frac{2}{12} = \frac{1}{12}$

b. $\frac{1}{6} - \frac{1}{8} = \frac{4}{24} - \frac{3}{24} = \frac{1}{24}$

c. $\frac{1}{6} - \frac{1}{9} = \frac{3}{18} - \frac{2}{18} = \frac{1}{18}$

d. $\frac{1}{6} - \frac{1}{10} = \frac{5}{30} - \frac{3}{30} = \frac{1}{15}$

e. $\frac{1}{8} - \frac{1}{10} = \frac{5}{40} - \frac{4}{40} = \frac{1}{40}$

f. $\frac{1}{8} - \frac{1}{12} = \frac{3}{24} - \frac{2}{24} = \frac{1}{24}$

g. $\frac{1}{10} - \frac{1}{15} = \frac{3}{30} - \frac{2}{30} = \frac{1}{30}$

Do you suspect that the difference between unit fractions is always a unit fraction?

III. It is either very hard or impossible to express the difference as a unit fraction in all the starred examples that follow:

a. $\frac{1}{9} - \frac{1}{12} = \frac{4}{36} - \frac{3}{36} = \frac{1}{36}$

b. $\frac{1}{4} - \frac{1}{9} = \frac{9}{36} - \frac{4}{36} = \frac{1}{}$ ☆ IMPOSSIBLE

c. $\frac{1}{10} - \frac{1}{14} = \frac{7}{70} - \frac{5}{70} = \frac{1}{35}$

d. $\frac{1}{9} - \frac{1}{15} = \frac{5}{45} - \frac{3}{45} = \frac{1}{}$ ☆ IMPOSSIBLE

e. $\frac{1}{30} - \frac{1}{45} = \frac{3}{90} - \frac{2}{90} = \frac{1}{90}$

f. $\frac{1}{5} - \frac{1}{12} = \frac{12}{60} - \frac{5}{60} = \frac{1}{}$ ☆ IMPOSSIBLE

g. $\frac{1}{18} - \frac{1}{24} = \frac{4}{72} - \frac{3}{72} = \frac{1}{72}$

h. $\frac{1}{34} - \frac{1}{51} = \frac{3}{102} - \frac{2}{102} = \frac{1}{102}$

IV. Perhaps it would be easier in the three starred examples to write the difference as a sum of two different unit fractions.

$\frac{1}{4} - \frac{1}{9} = \frac{9}{36} - \frac{4}{36} = \frac{5}{36} = \frac{1}{9} + \frac{1}{36}$

$\frac{1}{9} - \frac{1}{15} = \frac{5}{45} - \frac{3}{45} = \frac{2}{45} = \frac{1}{30} + \frac{1}{90}$

$\frac{1}{5} - \frac{1}{12} = \frac{12}{60} - \frac{5}{60} = \frac{7}{60} = \frac{1}{10} + \frac{1}{60}$

V. None of the above differences can be expressed as a single unit fraction. The same is true of the following, but each can be expressed as the sum of two unit fractions.

*

a. $\frac{1}{3} - \frac{1}{7} = \frac{7}{21} - \frac{3}{21} = \frac{4}{21} = \frac{1}{7} + \frac{1}{21}$

b. $\frac{1}{3} - \frac{1}{8} = \frac{8}{24} - \frac{3}{24} = \frac{5}{24} = \frac{1}{8} + \frac{1}{12}$

c. $\frac{1}{4} - \frac{1}{7} = \frac{7}{28} - \frac{4}{28} = \frac{3}{28} = \frac{1}{14} + \frac{1}{28}$

d. $\frac{1}{8} - \frac{1}{13} = \frac{13}{104} - \frac{8}{104} = \frac{5}{104} = \frac{1}{26} + \frac{1}{104}$

* MANY POSSIBLE ANSWERS.

Tasks: Students calculate differences between unit fractions and look for useful patterns.

Purpose: To remind students that arithmetic has a history of its own. To consider fractions in an unfamiliar setting. To introduce fractions in a problem-solving setting.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: In the Notes on Pupil Page 33, we mentioned that an Official Measurer might carry around a bag of sticks, each being a whole number subdivision of the official unit. He might very well wonder how much longer one stick was than another. Thus, he would be confronted with a subtraction situation.

The first few examples ask the question about sticks that are "next" to each other (see the sketch on pupil page 33): "How far down the set of sticks must I go to find a stick that tells me the difference between two neighboring sticks?"

The outcome may be a bit of a surprise. If $1/a$ and $1/b$ are "neighboring" sticks then

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a \times b}$$

It's as easy as that.

But when the sticks are not neighbors, this system does not work. The second set of examples (II) might suggest the conclusion that finding the least common denominator leads straightaway to a unit fraction as the difference between the given unit fractions. But this hasty generalization is wrong, as shown by some of the examples in group III. For instance:

$$\frac{1}{4} - \frac{1}{9} = \frac{9}{36} - \frac{4}{36} = \frac{5}{36} \text{ (not a unit fraction)}$$

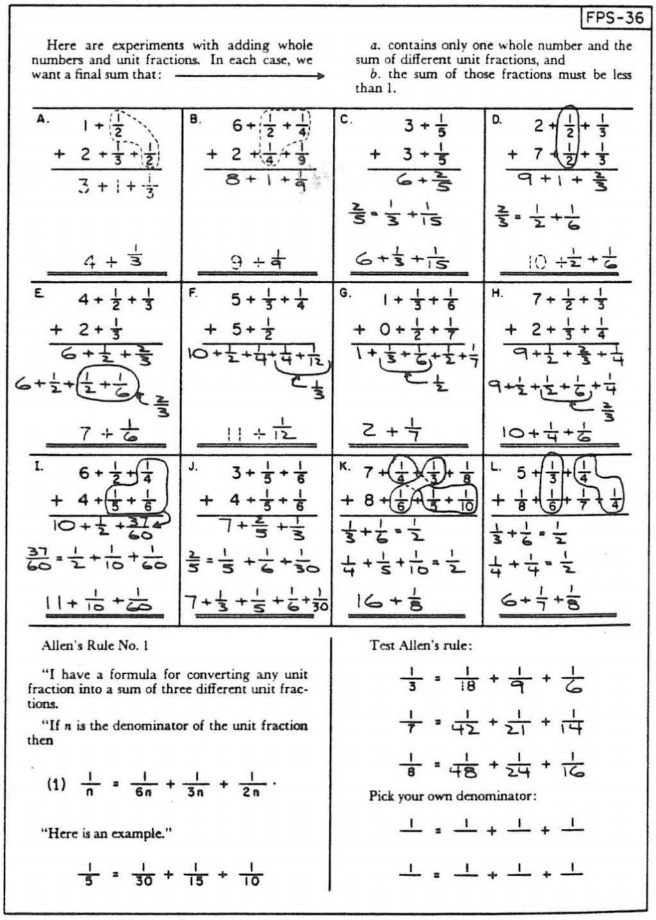
Some differences cannot be measured with any single stick in the Official Measurer's kit. In the example above, we are forced to use two sticks:

$$\frac{1}{4} - \frac{1}{9} = \left(\frac{5}{36} = \frac{4}{36} + \frac{1}{36}\right) = \frac{1}{9} + \frac{1}{36}$$

$$\text{(or } \frac{1}{18} + \frac{1}{12}\text{, and many others)}$$

The last two groups of examples (IV and V) require that the difference be expressed as the sum of two unit fractions. Some students might like to explore the question: Will more than two unit fractions ever be required to express the difference between two unit fractions?

Practice sheet 36 considers "mixtures" of whole numbers and unit fractions. If students can work their way across this unfamiliar ground, you can be sure they have a firm grasp on the usual algorithm for computing sums of "mixed numbers." The best way to test for understanding of an algorithm is to confront the student with a basically similar situation for which his algorithm was not intended. If he falters then work in such an unusual surrounding will help develop a better understanding of the algorithm.


FPS-36

Here are experiments with adding whole numbers and unit fractions. In each case, we want a final sum that: → *a.* contains only one whole number and the sum of different unit fractions, and *b.* the sum of those fractions must be less than 1.

A. $1 + \frac{1}{2}$ + $2 + \frac{1}{3} + \frac{1}{2}$ = $3 + 1 + \frac{1}{3}$ → $4 + \frac{1}{3}$

B. $6 + \frac{1}{2} + \frac{1}{4}$ + $2 + \frac{1}{4} + \frac{1}{9}$ = $8 + 1 + \frac{1}{9}$ → $9 + \frac{1}{9}$

C. $3 + \frac{1}{5}$ + $3 + \frac{1}{5}$ = $6 + \frac{2}{5}$; $\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$ → $6 + \frac{1}{3} + \frac{1}{15}$

D. $2 + \frac{1}{2} + \frac{1}{3}$ + $7 + \frac{1}{2} + \frac{1}{3}$ = $9 + 1 + \frac{2}{3}$; $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$ → $10 + \frac{1}{2} + \frac{1}{6}$

E. $4 + \frac{1}{2} + \frac{1}{3}$ + $2 + \frac{1}{3}$ → $7 + \frac{1}{6}$

F. $5 + \frac{1}{3} + \frac{1}{4}$ + $5 + \frac{1}{2}$ → $11 + \frac{1}{12}$

G. $1 + \frac{1}{3} + \frac{1}{6}$ + $0 + \frac{1}{2} + \frac{1}{7}$ → $2 + \frac{1}{7}$

H. $7 + \frac{1}{2} + \frac{1}{3}$ + $2 + \frac{1}{3} + \frac{1}{4}$ → $10 + \frac{1}{4} + \frac{1}{6}$

I. → $11 + \frac{1}{10} + \frac{1}{60}$

J. → $7 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{30}$

K. → $16 + \frac{1}{8}$

L. → $6 + \frac{1}{7} + \frac{1}{8}$

Allen's Rule No. 1

"I have a formula for converting any unit fraction into a sum of three different unit fractions.

"If n is the denominator of the unit fraction then

(1) $\frac{1}{n} = \frac{1}{6n} + \frac{1}{3n} + \frac{1}{2n}$.

"Here is an example."

$\frac{1}{5} = \frac{1}{30} + \frac{1}{15} + \frac{1}{10}$

Test Allen's rule:

$\frac{1}{3} = \frac{1}{18} + \frac{1}{9} + \frac{1}{6}$

$\frac{1}{7} = \frac{1}{42} + \frac{1}{21} + \frac{1}{14}$

$\frac{1}{8} = \frac{1}{48} + \frac{1}{24} + \frac{1}{16}$

Pick your own denominator:

$\frac{1}{} = \frac{1}{} + \frac{1}{} + \frac{1}{}$

$\frac{1}{} = \frac{1}{} + \frac{1}{} + \frac{1}{}$

Peter's Rule

"While I was doing those subtraction examples in which the denominators are consecutive whole numbers, I had an idea. I looked at the examples for a while, and I can prove that . . .

". . . every unit fraction can be obtained by adding two unit fractions."

Here is Peter's argument:

"If $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ then $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$

and the same goes for every similar example. So, I have a rule that will work for every unit fraction, no matter what the denominator is. I'll let *n* hold a place for whatever number you pick as the denominator.

Since $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$ it follows

that $\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$

and that's my rule! Since *n* can hold a place for any whole number (except 0), this is a proof that every unit fraction can be expressed as the sum of two unit fractions."

Do you agree that Peter has proved his rule? Has he proved that the two unit fractions in the sum must be different? When questioned about this, Peter said, "The largest unit fraction is $\frac{1}{2}$, all have denominator 2 or larger, and

if $n + 1 = n(n+1)$ then $n = 1$."

Try Peter's rule:

$\frac{1}{8} = \frac{1}{9} + \frac{1}{72}$ $\quad$ $\frac{1}{13} = \frac{1}{14} + \frac{1}{182}$

$\frac{1}{10} = \frac{1}{11} + \frac{1}{110}$ $\quad$ $\frac{1}{25} = \frac{1}{26} + \frac{1}{650}$

Mabel pointed out that this rule would help express any fraction with 2 as numerator as a sum of different unit fractions. She demonstrated her idea:

$$\frac{2}{3} = \frac{1}{3} + \underbrace{\frac{1}{3}}_{\text{by Peter's rule}} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}$$

$$\frac{2}{5} = \frac{1}{5} + \frac{1}{5} = \frac{1}{5} + \frac{1}{6} + \frac{1}{30}$$

$$\frac{2}{11} = \frac{1}{11} + \frac{1}{11} = \frac{1}{11} + \frac{1}{12} + \frac{1}{132}$$

$$\frac{2}{13} = \frac{1}{13} + \frac{1}{13} = \frac{1}{13} + \frac{1}{14} + \frac{1}{182}$$

"I soon began eliminating the middle step," Mabel explained and gave more examples:

$\frac{2}{9} = \frac{1}{9} + \frac{1}{10} + \frac{1}{90}$ $\quad$ $\frac{2}{15} = \frac{1}{15} + \frac{1}{16} + \frac{1}{240}$

$\frac{2}{7} = \frac{1}{7} + \frac{1}{8} + \frac{1}{56}$ $\quad$ $\frac{2}{19} = \frac{1}{19} + \frac{1}{20} + \frac{1}{380}$

Peter's Theorem

"I can use my rule to prove that every unit fraction can be expressed as the sum of any given number of unit fractions.

"I start with a unit fraction and apply my rule. Next I apply it again to the last unit fraction of the expression, and so on as long as I wish. For example:

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{6} \quad \left(\text{but } \frac{1}{6} = \frac{1}{7} + \frac{1}{42}\right), \text{so}$$

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{7} + \frac{1}{42} \left(\text{but } \frac{1}{42} = \frac{1}{43} + \frac{1}{1806}\right), \text{so}$$

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1806}$$

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \frac{1}{3{,}263{,}442}$$

and I could keep on going until I had as many unit fractions as I like!"

What do you think of Peter's theorem and his proof?

37 Expressing any unit fraction as the sum of unit fractions

Tasks: Students examine Peter's Rule and Peter's Theorem and complete examples illustrating Peter's Rule.

Purpose: To help students explore the proof of a rule. To deepen understanding of fractions by exploring several patterns for obtaining a unit fraction by adding two or more unit fractions.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: Peter starts with the pattern of the first set of exercises on pupil page 36:

$$\frac{1}{2}-\frac{1}{2+1}=\frac{1}{2\times(2+1)}$$

He can express this pattern by writing:

$$\frac{1}{n}-\frac{1}{n+1}=\frac{1}{n(n+1)}$$

He knows that, for all numbers a, b, and c,

$$\text{if } a-b=c$$

$$\text{then } a=b+c.$$

This is true for fractions as well as whole numbers. So, he argues that

$$\text{if } \frac{1}{n}-\frac{1}{n+1}=\frac{1}{n(n+1)}$$

$$\text{then } \frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)}.$$

But there is a loose end that needs to be tied up. Will these unit fractions obey the Egyptians' requirement that they be different? He must show that

$$n+1\neq n(n+1)$$

He argues that, for $n>0$, the only case in which $n+1=n(n+1)$ is that in which $n=1$. Consider several cases to be sure. Let n be 1, 2, 3, and 4:

$$1+1 = 1\times(1+1) \qquad 1=1$$

$$2+1 \neq 2\times(2+1) \qquad 3\neq 6$$

$$3+1 \neq 3\times(3+1) \qquad 4\neq 12$$

$$4+1 \neq 4\times(4+1) \qquad 5\neq 20$$

But n is the denominator of a unit fraction (smaller than 1); so the smallest number n can be is 2 and, for all numbers n which are at least as large as 2, $n+1\neq n(n+1)$. Thus, the unit fractions:

$$\frac{1}{n+1} \text{ and } \frac{1}{[n(n+1)]}$$

must be different. In essence, his argument is a mathematical proof.

Mabel puts Peter's idea to work in a special case that enables her to write any fraction smaller than 1 with numerator 2 as the sum of three unit fractions.

Mabel's special use of Peter's general idea leads Peter to the idea that any unit fraction can be expressed as the sum of any desired number of unit fractions. He could push this idea further. He could say that since every fraction can be obtained by adding unit fractions, every such sum can be expressed as the sum of any larger number of unit fractions.

Practice sheet 37 explores subtractions in which whole numbers and unit fractions are involved. This work, like that of practice sheet 36, will test each student's understanding of the usual algorithm for subtracting "mixed numbers" and give him keener insight into why it works.

FPS-37

Here are experiments with subtracting whole numbers and unit fractions. In each case we are looking for a final answer that:

a. contains no more than one whole number and a unit fraction or a sum of unit fractions;

b. the sum of the unit fractions must be less than 1.

M. $8+\frac{1}{3} - (5+\frac{1}{2})$; $7+1+\frac{1}{3} - (5+\frac{1}{2})$; $2+\frac{1}{2}+\frac{1}{3}$

N. $6 - (4+\frac{1}{4})$; $1+\frac{3}{4}$; $1+\frac{1}{2}+\frac{1}{4}$

O. $4+\frac{1}{3} - (1+\frac{1}{2})$; $3+1+\frac{1}{3} - (1+\frac{1}{2})$; $2+\frac{1}{2}+\frac{1}{3}$

P. $4+\frac{1}{3}+\frac{1}{7} - (1+\frac{1}{7})$; $3+\frac{1}{3}$

Q. $7 - (\frac{1}{4}+\frac{1}{2})$; $7-\frac{3}{4}=6\frac{1}{4}$; $6+\frac{1}{4}$

R. $5+\frac{1}{4}+\frac{1}{3} - (2+\frac{1}{3}+\frac{1}{5})$; $\frac{1}{4}-\frac{1}{5}=\frac{1}{20}$; $3+\frac{1}{20}$

S. $8+\frac{1}{6} - (\frac{1}{3}+\frac{1}{6})$; $8-\frac{1}{3}=7\frac{2}{3}$; $\frac{2}{3}=\frac{1}{2}+\frac{1}{6}$; $7+\frac{1}{2}+\frac{1}{6}$

T. $1+\frac{1}{2} - (\frac{1}{3}+\frac{1}{4}+\frac{1}{6})$; $\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$; $1-\frac{1}{4}=\frac{3}{4}$; $\frac{1}{2}+\frac{1}{4}$

U. $10+\frac{1}{2}+\frac{1}{10} - (3+\frac{1}{10}+\frac{1}{5})$; $\frac{1}{2}-\frac{1}{5}=\frac{3}{10}$; $\frac{3}{10}=\frac{1}{5}+\frac{1}{10}$; $7+\frac{1}{5}+\frac{1}{10}$

V. $8+\frac{1}{3}+\frac{1}{4}+\frac{1}{5} - (7+\frac{1}{2})$; $8+(\frac{1}{4}+\frac{1}{12})+\frac{1}{4}+\frac{1}{5}$; $7+\frac{1}{2}$; $1+\frac{1}{12}+\frac{1}{5}$

W. $8+\frac{1}{2}+\frac{1}{4} - (3+\frac{1}{3})$; $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$; $5+\frac{1}{4}+\frac{1}{6}$

X. $17+\frac{1}{2}+\frac{1}{4} - (2+\frac{1}{3}+\frac{1}{4})$; $1\frac{1}{4}-\frac{1}{3}=\frac{11}{12}$; $\frac{11}{12}=\frac{1}{2}+\frac{1}{3}+\frac{1}{12}$; $14+\frac{1}{2}+\frac{1}{3}+\frac{1}{12}$

Allen's Rules, No. 2 and No. 3

"Here are rules to show that I can write any unit fraction as a sum of four different unit fractions in two different ways."

(2) $\frac{1}{n}=\frac{1}{12n}+\frac{1}{6n}+\frac{1}{4n}+\frac{1}{2n}$

(3) $\frac{1}{n}=\frac{1}{18n}+\frac{1}{9n}+\frac{1}{3n}+\frac{1}{2n}$

(2) $\frac{1}{3}=\frac{1}{36}+\frac{1}{18}+\frac{1}{12}+\frac{1}{6}$

(3) $\frac{1}{3}=\frac{1}{54}+\frac{1}{27}+\frac{1}{9}+\frac{1}{6}$

Test Allen's rules.

(a) $\frac{1}{5}=\frac{\ }{60}+\frac{\ }{30}+\frac{\ }{20}+\frac{\ }{10}$; $\frac{1}{5}=\frac{\ }{90}+\frac{\ }{45}+\frac{\ }{15}+\frac{\ }{10}$

(b) $\frac{1}{2}=\frac{\ }{24}+\frac{\ }{12}+\frac{\ }{8}+\frac{\ }{4}$; $\frac{1}{2}=\frac{\ }{36}+\frac{\ }{18}+\frac{\ }{6}+\frac{\ }{4}$

(c) $\frac{1}{7}=\frac{\ }{84}\ \frac{\ }{42}\ \frac{\ }{28}\ \frac{\ }{14}$; $\frac{1}{7}=\frac{\ }{126}+\frac{\ }{63}+\frac{\ }{21}+\frac{\ }{14}$

A Side-trip to Ancient Rome

The Egyptians used 1 as the numerator of all fractions (except $\frac{2}{3}$) and then varied the denominators to express different amounts.

The Romans of ancient times did the reverse. They divided everything into twelfths (using 12 as the denominator) and used the numerator to indicate the number of 12ths.

I. Let's explore where this system leads. We shall begin by limiting ourselves to 12ths and to whole numbers. We shall use eleven or fewer 12ths in reporting any fractional part of the results and state our answer in simplest terms.

a. $\frac{7}{12} + \frac{8}{12} = \frac{15}{12} = 1\frac{3}{12}$ b. $\frac{3}{12} + \frac{7}{12} = \frac{10}{12}$

c. $1\frac{4}{12} + 3\frac{7}{12} = 4\frac{11}{12}$ d. $2\frac{6}{12} + 7\frac{6}{12} = 10$

e. $2\frac{8}{12} + \frac{9}{12} = 2\frac{17}{12} = 3\frac{5}{12}$ f. $1\frac{10}{12} + 4\frac{11}{12} = 5\frac{21}{12} = 6\frac{9}{12}$

g. $\frac{8}{12} + \frac{10}{12} + \frac{6}{12} = 2$ h. $\frac{7}{12} + 1\frac{7}{12} = 1\frac{14}{12} = 2\frac{2}{12}$

i. $\frac{9}{12} + \frac{9}{12} + \frac{9}{12} = \frac{27}{12} = 2\frac{3}{12}$ j. $75 + 28\frac{6}{12} = 103\frac{6}{12}$

Easy! So let's try subtraction:

k. $3 - \frac{7}{12} = 2\frac{5}{12}$ l. $7\frac{9}{12} - 2\frac{3}{12} = 5\frac{6}{12}$

m. $5 - \frac{4}{12} = 4\frac{8}{12}$ n. $6 - 5\frac{1}{12} = \frac{11}{12}$

o. $1\frac{8}{12} - \frac{9}{12} = \frac{11}{12}$ p. $10\frac{8}{12} - 7\frac{8}{12} = 3$

q. $71\frac{11}{12} - 28 = 43\frac{11}{12}$ r. $100\frac{1}{12} - \frac{2}{12} = 99\frac{11}{12}$

s. $9\frac{3}{12} - 2\frac{6}{12} = 6\frac{9}{12}$ t. $196 - 8\frac{9}{12} = 187\frac{3}{12}$

Also easy! The denominators are always alike — always 12!

II. Multiplying by a whole number and dividing by a whole number can also be quite easy:

a. $\frac{5}{12} \times 3 = \frac{15}{12} = 1\frac{3}{12}$ b. $6\frac{8}{12} \div 2 = 3\frac{4}{12}$

c. $2\frac{2}{12} \times 7 = 14\frac{14}{12} = 15\frac{2}{12}$ d. $1\frac{2}{12} \div 2 = \frac{7}{12}$

e. $3\frac{5}{12} \times 5 = 15\frac{25}{12} = 17\frac{1}{12}$ f. $1\frac{3}{12} \div 3 = \frac{5}{12}$

g. $8\frac{7}{12} \times 6 = 48\frac{42}{12} = 51\frac{6}{12}$ h. $1\frac{3}{12} \div 5 = \frac{3}{12}$

i. $4 \times 9\frac{5}{12} = 36\frac{20}{12} = 37\frac{8}{12}$ j. $2\frac{4}{12} \div 7 = \frac{4}{12}$

k. $8 \times 12\frac{6}{12} = 96\frac{48}{12} = 100$ l. $43\frac{2}{12} \div 7 = 6\frac{2}{12}$

Look back through these examples. Think of each as a question involving feet and inches.

a. How much is 5 inches $\times$ 3?

b. How much is (6 feet and 8 inches) $\div$ 2?

c. How much is (2 feet and 2 inches) $\times$ 7?

Etc.

Any difficulty you may have had soon disappears.

III. Take half of it . . . or $\frac{6}{12}$ of it . . . or $\times \frac{6}{12}$.

a. $10 \times \frac{6}{12} = 5$ b. $1 \times \frac{6}{12} = \frac{6}{12}$

c. $\frac{8}{12} \times \frac{6}{12} = \frac{4}{12}$ d. $1\frac{6}{12} \times \frac{6}{12} = \frac{9}{12}$

IV. Take a third of it . . . or $\frac{4}{12}$ of it . . . or $\times \frac{4}{12}$.

a. $1 \times \frac{4}{12} = \frac{4}{12}$ b. $\frac{9}{12} \times \frac{4}{12} = \frac{3}{12}$

c. $2\frac{3}{12} \times \frac{4}{12} = \frac{9}{12}$ d. $\frac{3}{12} \times \frac{4}{12} = \frac{1}{12}$

V. Take two-thirds of it . . . or $\frac{8}{12}$ of it . . . or $\times \frac{8}{12}$.

a. $3 \times \frac{8}{12} = 2$ b. $\frac{9}{12} \times \frac{8}{12} = \frac{6}{12}$

c. $1\frac{3}{12} \times \frac{8}{12} = \frac{10}{12}$ d. $2\frac{6}{12} \times \frac{8}{12} = 1\frac{8}{12}$

e. $6\frac{6}{12} \times \frac{8}{12} = 4\frac{4}{12}$ f. $4\frac{9}{12} \times \frac{8}{12} = 3\frac{2}{12}$

A little strange — but still easy.

38-39 A side trip to ancient Rome - twelfths and percents

Tasks: *Students complete examples of addition, subtraction, multiplication, and division of fractions involving the denominator 12.*

Purpose: To remind students that arithmetic has a history of its own. To consider fractions in an unfamiliar setting. To introduce fractions in a problem-solving setting.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: We have anticipated this excursion to ancient Rome on pupil page 34, particularly by dividing the foot ruler into inches and the dozen eggs into parts (see comments in the Notes on Pupil Page 34).

Practice sheet 38 suggests some of the background for dividing units into twelfths. We hope this bit of history will lead students to read some of the interesting books and articles that have been written about systems of measurement.

The Egyptian system used 1 as the numerator and varied the denominator; the Romans used 12 as the denominator and varied the numerator. Our present system allows us to vary both numerator and denominator.

Students may wish to discuss the advantages and disadvantages of each of these systems. Which system involves the least complicated arithmetic? Which is the most useful? What would our system be like if we had to write numerators and denominators in Roman numerals?

If any difficulties arise in exploring the Roman system, urge students to think in terms of feet and inches or in terms of dozens of eggs and numbers of eggs.

Why did the Romans select 12 as their standard denominator? Quite probably because 12 has so many factors: 1, 2, 3, 4, 6, and 12. Our choice of 10 with factors limited to 1, 2, 5, and 10 clearly had other motivations—such as the fact that we have 10 fingers. If we take the next step up in our system —from 10 to 100—we still cannot divide that many into thirds without getting into fractions. When we think in terms of percent, we have to change $\frac{1}{3}$ to the very clumsy $33\frac{1}{3}\%$.

How many factors are there for 10×10, or 100? There are nine of them (1, 2, 4, 5, 10, 20, 25, 50, and 100).

How many for 12×12, or 144? There are thirteen (1, 2, 3, 4, 6, 8, 12, 18, 24, 32, 36, 72, and 144).

FPS-38

A Bit of History of Weights and Measures

Because the history of weights and measures is so fascinating, many books, articles, and chapters of books have been written about it. Some are in almost every school library. We recommend a little browsing in your spare time.

We can only suggest a bit of this history, especially the contribution of the Romans with their practice of dividing units into 12ths.

The foot was one of the earliest units of measurement. It represented the length of a man's foot. So, its length varied, depending on whose foot was used. You can be sure that this led to many arguments.

One inch was the width of a thumb — a man's thumb — and had no direct relationship to the length of his feet.

But the Romans changed that. They decided that an inch should be $\frac{1}{12}$ of a foot. That agreement is still with us.

And the Roman pound, as you might expect, was divided into 12ths. There were 12 ounces in the Roman pound.

When the Romans occupied England, they introduced their system of weights and measures. Since the English basic unit of money is a pound, they still have 12 ounces in the pound used for weighing precious metals. This is called the Troy system of weight. They also have a pound with 16 ounces for weighing in general. This is called the avoirdupois system of weight.

The idea of foot and its division into twelve inches was brought to England by the Romans.

By this time, a foot was considered as a third of a yard. But what was a yard? Legend has it that King Henry I of England ruled that a yard is the distance between the tip of his nose and the end of his thumb.

The mile was considered 1,000 paces — and a pace was a double step. As they walked, a Roman counted the number of times he walked on his right foot — and at 1,000, he had gone a mile. Later, the Roman pace was considered as 5 feet — and 5,000 feet in a mile.

The Roman mile of 5,000 feet was introduced into England and remained until the 16th century. At that time it was changed to 5,280 feet.

Would you be surprised to find in your browsing through the library that the Romans invented the dozen? How about 24 hours to a day (perhaps 12 hours of daylight and 12 hours of darkness)? A circle is divided into 360 degrees — and that is 30×12.

Why does 12 crop up so often?

There is the Duodecimal Society of America that publishes the Duodecimal Bulletin. It carries many interesting articles arguing for changing our whole decimal system based on ten into a duodecimal system based on twelve.

After all, 1, 2, 3, 4, 6, and 12 are all factors of 12.

And 1, 2, 5, and 10 are the only factors of 10.

Do you wonder why we have 7 days in a week and then months that vary from 28 to 31 days?

We hope these few bits of history of weights and measures may lead you to some interesting reading on your own.

VI. How many inches in 2 feet, 1 foot, half a foot, a third of a foot? How many 12ths of a foot are there?

a. $1 \div \frac{1}{12} = 12$ b. $2 \div \frac{1}{12} = 24$

c. $\frac{6}{12} \div \frac{1}{12} = 6$ d. $1\frac{6}{12} \div \frac{1}{12} = 18$

e. $\frac{7}{12} \div \frac{1}{12} = 7$ f. $3\frac{5}{12} \div \frac{1}{2} = 6\frac{10}{12}$

VII. How many 2-inch lengths in a foot? How many $\frac{2}{12}$-of-a-foot lengths are there in 10 inches?

a. $1 \div \frac{2}{12} = 6$ b. $2 \div \frac{2}{12} = 12$

c. $\frac{6}{12} \div \frac{2}{12} = 3$ d. $1\frac{8}{12} \div \frac{2}{12} = 10$

VIII. How many 3-inch lengths in 2 feet, in 1 foot?

a. $2 \div \frac{3}{12} = 8$ b. $1 \div \frac{3}{12} = 4$

c. $\frac{6}{12} \div \frac{3}{12} = 2$ d. $1\frac{9}{12} \div \frac{3}{12} = 7$

IX. How many 6-inch strips in 1 foot, in 6 inches, in 3 inches, in 9 inches?

a. $1 \div \frac{6}{12} = 2$ b. $\frac{6}{12} \div \frac{6}{12} = 1$

c. $\frac{3}{12} \div \frac{6}{12} = \frac{6}{12}$ d. $\frac{9}{12} \div \frac{6}{12} = 1\frac{6}{12}$

e. $\frac{2}{12} \div \frac{6}{12} = \frac{4}{12}$ f. $\frac{1}{12} \div \frac{6}{12} = \frac{2}{12}$

X. How many 1 foot 6 inch lengths are there in 3 feet, in 6 feet, in 9 inches, in 6 inches?

a. $3 \div 1\frac{6}{12} = 2$ b. $6 \div 1\frac{6}{12} = 4$

c. $\frac{9}{12} \div 1\frac{6}{12} = \frac{6}{12}$ d. $\frac{6}{12} \div 1\frac{6}{12} = \frac{4}{12}$

A little harder? A little, but not much.

Of course, we've kept the examples on the simple side so you could see more clearly how the system works.

What's inconvenient about this system? Aren't you happy you don't have to say "six-twelfths of an hour"?

But how would you refer to a minute? Five minutes is $\frac{1}{12}$ of an hour.

Suppose that you wished to measure a length which was more than $\frac{6}{12}$ of a foot but less than $\frac{7}{12}$ of a foot. It would seem logical to divide each inch into 12 parts so that you could say that the length was $\frac{6}{12}$ plus, say, $\frac{3}{12}$ of $\frac{1}{12}$, or plus $\frac{3}{144}$.

$$\frac{6}{12} + \frac{3}{144} = \frac{72}{144} + \frac{3}{144} = \frac{75}{144}$$

Perhaps you would like to think about such arithmetic and explore the problems you would encounter. You might, in a reference book, look up the "duodecimal system," a system based on twelve and powers of twelve as our decimal system is based on ten and powers of ten.

But we wish to complete this side-trip at this point.

Before we leave it completely, we would like to mention that we do have a system of fractions that, like the system of early Romans, has just one denominator. That denominator is 100 and we use the word *percent* or the symbol % to indicate that we are using that system.

In this system we change everything to hundredths and then use % instead of writing a denominator. You are probably familiar with it, but for those who aren't, here are some examples:

$\frac{1}{2} = \frac{50}{100} = 50\%$ $\frac{1}{4} = \frac{25}{100} = 25\%$

$\frac{3}{4} = \frac{75}{100} = 75\%$ $\frac{1}{5} = \frac{20}{100} = 20\%$

$\frac{2}{5} = \frac{40}{100} = 40\%$ $\frac{7}{10} = \frac{70}{100} = 70\%$

$1\frac{1}{2} = \frac{150}{100} = 150\%$ $2 = \frac{200}{100} = 200\%$

$1\frac{3}{10} = \frac{130}{100} = 130\%$ $\frac{3}{25} = \frac{12}{100} = 12\%$

Notes on Pupil Page 39

Tasks: Students complete mathematical sentences involving division of fractions with denominator 12 and change common fractions into decimal fractions and percents.

Purpose: To remind students that arithmetic has a history of its own. To consider fractions in an unfamiliar setting. To focus on a physical representation of fractions. To introduce fractions in a problem-solving setting.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: The development in the first column and the work on practice sheet 39 anticipate the detailed discussion in Chapter Ten of fractions as divisors.

With the approach used here, we must keep clearly in mind the units and subunits we are thinking about.

How many inches are there in 1 foot? There is a tendency to write:

$$12 \div \frac{1}{12} = ?$$

which asks: How many one-twelfths of a foot are there in *12 feet?* or: How many twelfths of an *inch* are there in *12 inches?*

In using comparisons of measurements to help shed light on the arithmetic involved, we need to emphasize the importance of taking care that all numbers referring to units of measure refer to the same unit of measure.

Practice sheet 39 is a collection of exercises about units and subunits and the relationships between them. The fractions and whole numbers used in the statements usually refer to different units, so failure to think of the relationship will lead to nonsense instead of reasonable statements.

The side trip to Rome is completed by noticing that we often use a system very much like the one the Romans used, one that requires the constant denominator 100.

In any expression involving percent, there is the implication that we are going to work within this system that requires a constant denominator. And that is all the percent system is.

In fact, there is a bit of the Roman limitation on denominators in our own method of recording decimal fractions. Here we require that every fraction have a denominator that is a power of 10: 10, 100, 1,000, etc. And, in a sense, we have included more of the Egyptian system than is at first apparent. The Egyptians might point out that we let numerators range from 0 through 9 and limit denominators to powers of 10. We have a shorthand for writing every decimal fraction as the sum of such fractions:

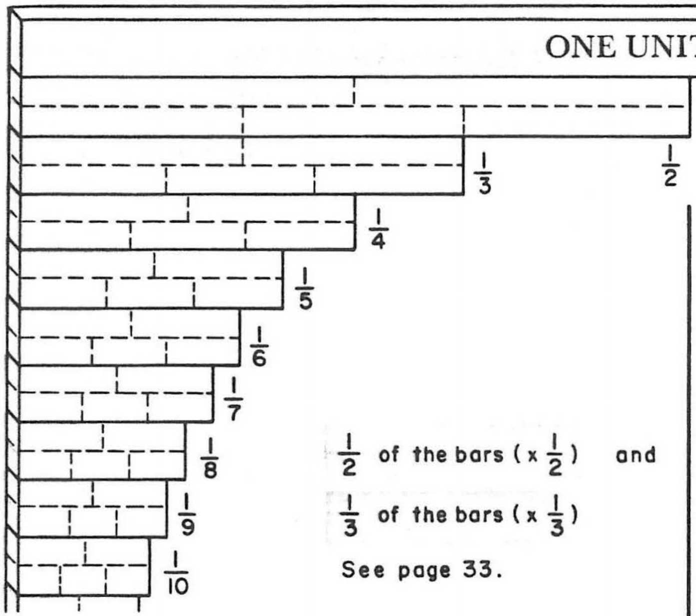
$$.3049 \text{ means } \frac{3}{10} + \frac{0}{100} + \frac{4}{1{,}000} + \frac{9}{10{,}000}$$

FPS-39

Complete each sentence by filling in the blanks with whole numbers or unit fractions.

1. There are 24 hours in a day.
2. An hour is $\frac{1}{24}$ of a day.
3. There are 48 half-hours in a day.
4. A half-hour is $\frac{1}{48}$ of a day.
5. A second is $\frac{1}{60}$ of a minute.
6. A minute is $\frac{1}{60}$ of an hour.
7. A second is $\frac{1}{3{,}600}$ of an hour.
8. An inch is $\frac{1}{36}$ of a yard.
9. A foot is $\frac{1}{3}$ of a yard.
10. An inch is $\frac{1}{18}$ of a half-yard.
11. There are 6 half-feet in a yard.
12. There are 9 square feet in a sq. yd.
13. There are 144 sq. in. in a sq. ft.
14. There are 1,296 sq. in. in 9 sq. ft.
15. A sq. ft. is $\frac{1}{9}$ of a sq. yd.
16. A sq. in. is $\frac{1}{1{,}296}$ of a sq. yd.
17. A sq. in. is $\frac{1}{144}$ of a sq. ft.
18. A dime is $\frac{1}{20}$ of two dollars.
19. A nickel is $\frac{1}{40}$ of two dollars.
20. A penny is $\frac{1}{200}$ of two dollars.
21. A nickel is $\frac{1}{10}$ of a half-dollar.
22. A quarter is $\frac{1}{12}$ of three dollars.
23. A quarter is $\frac{1}{2}$ of a half-dollar.
24. A dime is $\frac{1}{5}$ of a half-dollar.
25. There is $\frac{1}{10}$ of a dollar in a dime.
26. There is $\frac{1}{60}$ of an hour in a minute.
27. There is $\frac{1}{3}$ of an hour in 20 minutes.
28. There is $\frac{1}{4}$ of an hour in 15 minutes.
29. There is $\frac{1}{5}$ of an hour in 12 minutes.
30. There is $\frac{1}{6}$ of an hour in 10 minutes.
31. There is $\frac{1}{10}$ of an hour in 6 minutes.
32. There is $\frac{1}{12}$ of an hour in 5 minutes.
33. There is $\frac{1}{15}$ of an hour in 4 minutes.
34. There is $\frac{1}{20}$ of an hour in 3 minutes.
35. There is $\frac{1}{30}$ of an hour in 2 minutes.
36. There is $\frac{1}{60}$ of an hour in 1 minute.
37. There is $\frac{1}{120}$ of an hour in $\frac{1}{2}$ minute.
38. There is $\frac{1}{180}$ of an hour in $\frac{1}{3}$ minute.
39. There is $\frac{1}{300}$ of an hour in $\frac{1}{5}$ minute.
40. There is $\frac{1}{8}$ of 2 hours in 15 minutes.
41. There is $\frac{1}{24}$ of 2 hours in 5 minutes.
42. There is $\frac{1}{120}$ of 2 hours in 1 minute.
43. There is $\frac{1}{6}$ of a day in 4 hours.
44. There is $\frac{1}{48}$ of a day in a half-hour.
45. There is $\frac{1}{24}$ of a half-day in a half-hour.
46. There is $\frac{1}{240}$ of a half-day in 3 minutes.
47. There is $\frac{1}{1{,}440}$ of a day in a minute.
48. There is $\frac{1}{86{,}400}$ of a day in a second.
49. There is $\frac{1}{604{,}800}$ of a week in a second.

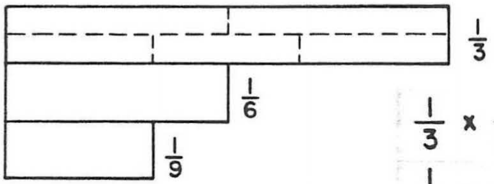
Back from our side trip to ancient Rome, let's take a brief look at multiplication and division involving unit fractions and whole numbers.



$\frac{1}{2}$ of the bars (x $\frac{1}{2}$) and

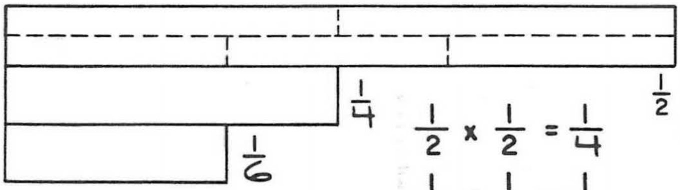
$\frac{1}{3}$ of the bars (x $\frac{1}{3}$)

See page 33.



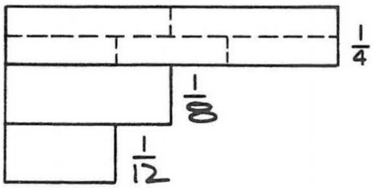
$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

$\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$



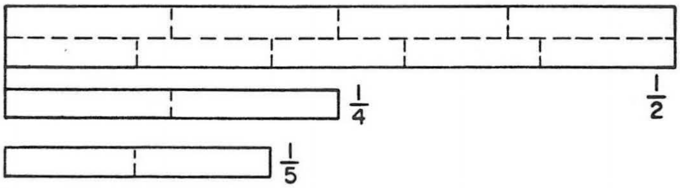
$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$



$\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

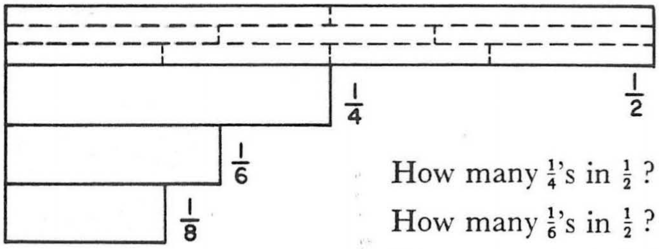
$\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$



$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ $\quad \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

$\frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$ $\quad \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$

There is certainly nothing new yet in multiplying a pair of unit fractions. All multiplication should be as simple!

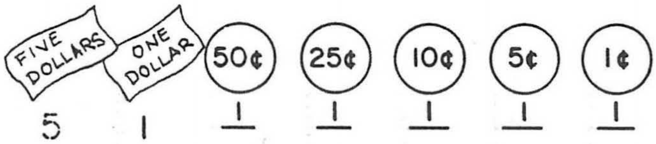


How many $\frac{1}{4}$'s in $\frac{1}{2}$?

How many $\frac{1}{6}$'s in $\frac{1}{2}$?

How many $\frac{1}{8}$'s in $\frac{1}{2}$?

$\frac{1}{2} \div \frac{1}{4} = 2 \quad \frac{1}{2} \div \frac{1}{6} = 3 \quad \frac{1}{2} \div \frac{1}{8} = 4$



How many quarters in a half?
How many halves in a quarter?

$\frac{1}{2} \div \frac{1}{4} = 2 \quad \frac{1}{4} \div \frac{1}{2} = \frac{1}{2}$

How many dimes in a half?
How many halves in a dime?

$\frac{1}{2} \div \frac{1}{10} = 5 \quad \frac{1}{10} \div \frac{1}{2} = \frac{1}{5}$

How many pennies in a quarter?
How many quarters in a penny?

$\frac{1}{4} \div \frac{1}{100} = 25 \quad \frac{1}{100} \div \frac{1}{4} = \frac{1}{25}$

How many quarters in five dollars?
How many five dollars in a quarter?

$5 \div \frac{1}{4} = 20 \quad \frac{1}{4} \div 5 = \frac{1}{20}$

How many dimes in a dollar?
How many dollars in a dime?

$1 \div \frac{1}{10} = 10 \quad \frac{1}{10} \div 1 = \frac{1}{10}$

40-41 Multiplication and division of unit fractions and whole numbers

Tasks: *Students complete mathematical sentences involving multiplication and division of fractions.*

Purpose: To help students relate multiplication and division of fractions to everyday situations.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: We return to unit fractions to consider multiplication and division.

The first thing we need to do is to define what we mean by $1/3 \times 1/2$ and $1/2 \div 1/6$. Let's go back to the way our Official Measurer made his set of rods. To make his $1/2$-rod and his $1/3$-rod, he started with a unit rod in each case. He divided one unit rod into 2 parts of the same length and used one of them as his $1/2$-rod. Then he divided another unit rod into 3 parts of the same length and used one of them as his $1/3$-rod.

Instead of starting with unit rods, he could have performed the same operation on $1/2$-unit rods, $1/3$-unit rods, etc. Suppose that he started with a $1/2$-unit rod, divided it into 3 parts of the same length, and called one of them his "$1/2 \times 1/3$"-rod. But he already has a rod of the same length, his $1/6$-rod. So he could decide to summarize all that work this way:

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

He has defined multiplication of unit fractions.

How many $1/6$-rods can be made out of a $1/2$-rod? Or, how many $1/6$-rods placed end to end make a rod of the same length as the $1/2$-rod? He found that he could make three $1/6$-rods out of a $1/2$-rod, and he summarized this activity and its results by writing:

$$\frac{1}{2} \div \frac{1}{6} = 3$$

He also noticed that if he had a rod 3 units long and divided it into 6 parts of the same length, he could use each one of them as a $1/2$-rod, or:

$$3 \times \frac{1}{6} = \frac{1}{2}$$

The relationship between multiplication and division is clear:

$$3 \times \frac{1}{6} = \frac{1}{2} \text{ and } \frac{1}{2} \div \frac{1}{6} = 3$$

Dividing by $1/6$ "undoes" a multiplication by $1/6$ and a multiplication by $1/6$ "undoes" a division by $1/6$.

Unit fractions of a dollar are considered as a second application of these ideas.

Practice sheet 40 is a "discovery" sequence—a search for a simple method for comparing fractions. This anticipates the whole idea of ratio, which is considered in Chapter Eleven.

Comparing Fractions

FPS-40

$\frac{a}{b}$ and $\frac{c}{d}$ ➡ Is $\frac{a}{b} = \frac{c}{d}$? (equal to) Is $\frac{a}{b} > \frac{c}{d}$? (greater than) Is $\frac{a}{b} < \frac{c}{d}$? (less than)

$\frac{3}{8} > \frac{1}{3}$	$\frac{3}{7} < \frac{3}{4}$	$\frac{4}{5} < \frac{5}{6}$	$\frac{2}{3} > \frac{4}{7}$	$\frac{5}{8} > \frac{3}{5}$	$\frac{4}{7} > \frac{5}{9}$	$\frac{1}{3} < \frac{7}{17}$
$\frac{9}{24} > \frac{8}{24}$	$\frac{12}{28}$ $\frac{21}{28}$	24 25	14 12	25 24	36 35	17 21

DENOMINATORS NOT NECESSARY - SEE BELOW.

$\frac{5}{3} > \frac{3}{2}$	$\frac{8}{5} < \frac{16}{9}$	$\frac{100}{2} > \frac{200}{5}$	$\frac{347}{650} > \frac{1}{2}$	$\frac{45}{7} < \frac{54}{7}$	$\frac{7}{45} > \frac{7}{54}$	$\frac{3}{8} < \frac{7}{16}$
10 9	72 80	500 400	694 650	You can tell without any computation.		48 56

Before continuing, consider this. Did you really need to compute a common denominator in those examples above? On the right are three ways of finding which fraction is larger. Use the one you prefer in the rest of the examples.

$\frac{4}{13} < \frac{6}{17}$	$\frac{4}{13} < \frac{6}{17}$	$\frac{4}{13} < \frac{6}{17}$
$\frac{68}{221}$ $\frac{78}{221}$	$\frac{68}{13 \cdot 17}$ $\frac{78}{13 \cdot 17}$	68 78

$\frac{5}{9} < \frac{8}{14}$	$\frac{14}{9} < \frac{8}{5}$	$\frac{7}{17} < \frac{3}{7}$	$\frac{17}{7} > \frac{7}{3}$	$\frac{9}{6} < \frac{11}{7}$	$\frac{7}{6} < \frac{11}{9}$	$\frac{6}{7} > \frac{9}{11}$
70 72	70 72	49 51	51 49	63 66	63 66	66 63
$\frac{3}{25} < \frac{5}{41}$	$\frac{25}{3} > \frac{41}{5}$	$\frac{25}{41} > \frac{3}{5}$	$\frac{2}{3} < \frac{4}{5}$	$\frac{2}{4} < \frac{3}{5}$	$\frac{4}{2} > \frac{5}{3}$	$\frac{a}{b}$? $\frac{c}{d}$
123 125	125 123	125 123	10 12	10 12	12 10	$\frac{ad}{bd}$ $\frac{bc}{bd}$

$\frac{a}{b} > \frac{c}{d}$ if $ad > bc$

$\frac{a}{b} < \frac{c}{d}$ if $ad < bc$

$\frac{a}{b} = \frac{c}{d}$ if $ad = bc$

Can you complete the last sentence? Try some examples:

$\frac{4}{6} = \frac{10}{15}$ $\quad 4 \times 15 = 6 \times 10$

True or False

If two fractions have the same denominator,

$\frac{a}{b} > \frac{c}{b}$ if $a > c$ ------------ T F

$\frac{a}{b} < \frac{c}{b}$ if $a < c$ ------------ T F

If two fractions have the same numerator,

$\frac{a}{b} > \frac{a}{c}$ if $b < c$ ------------ T F

$\frac{a}{b} < \frac{a}{c}$ if $b > c$ ------------ T F

Again . . . nothing new, so let's use some sums of unit fractions and whole numbers.

$12+\frac{1}{2}+\frac{1}{5}$ $\times\frac{1}{3}$ $4+\frac{1}{6}+\frac{1}{15}$	$35+\frac{1}{6}+\frac{1}{9}+\frac{1}{13}$ $\times\frac{1}{7}$ $5+\frac{1}{42}+\frac{1}{63}+\frac{1}{91}$
$6+\frac{1}{8}+\frac{1}{15}$ $\times\frac{1}{5}$ $1+\frac{1}{5}+\frac{1}{40}+\frac{1}{75}$	$20+\frac{1}{18}+\frac{1}{64}$ $\times\frac{1}{10}$ $2+\frac{1}{180}+\frac{1}{640}$

No trouble at all, if you thought of 6 as $5+1$ and noticed:

$$\begin{array}{r} 5+1 \\ \times\frac{1}{5} \\ \hline 1+\frac{1}{5} \end{array}$$

In fact, it's usually easier to multiply by a unit fraction than by a whole number. Try the next two on your own and without many clues.

$3+\frac{1}{4}+\frac{1}{6}+\frac{1}{20}$ $\times 2$ $6+\frac{2}{4}+\frac{2}{6}+\frac{2}{20}$ $6+\frac{1}{2}+\frac{1}{3}+\frac{1}{10}$	$3+\frac{1}{4}+\frac{1}{6}+\frac{1}{20}$ $\times 3$ $9+\frac{3}{4}+\frac{3}{6}+\frac{3}{20}$?

The first example is too simple. The second presents real trouble. Only one of the "partial products" can be reduced to a unit fraction. The other two need more work:

$$\frac{3}{4}=\frac{1}{2}+\frac{1}{4} \qquad \frac{3}{20}=\frac{1}{10}+\frac{1}{20}$$

So, we write out:

$$9+\frac{1}{2}+\frac{1}{4}+\frac{1}{2}+\frac{1}{10}+\frac{1}{20}$$

But we still are in trouble. The unit fractions are not all different, and their sum is greater than 1. There's an easy way out because the sum of the two fractions with denominator 2 is a whole number. So we can write the product according to all the rules:

$$10+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}$$

Try two examples completely on your own. Use the space not tinted green to keep track of your work.

$7+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}$ $\times 2$ $14+\frac{2}{3}+\frac{1}{2}+\frac{1}{3}$ $\frac{2}{3}=\frac{1}{2}+\frac{1}{6}$ $\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$ $15+\frac{1}{3}+\frac{1}{6}$ OR $15+\frac{1}{2}$	$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ $\times 3$ $\left(1\frac{1}{2}+1\right)+\frac{3}{4}+\frac{3}{5}$ $\frac{3}{4}=\boxed{\frac{1}{2}}+\frac{1}{4}$ $\quad\frac{3}{5}=\frac{1}{2}+\frac{1}{10}$ $3+\frac{1}{2}+\frac{1}{4}+\frac{1}{10}$ *

* MANY POSSIBILITIES.

Division

It's quite the opposite with division. Dividing by a whole number is not hard at all. The difficulty arises when we divide by a fraction. Let's try a few examples with the help of some clues.

$3+\frac{1}{2}+\frac{1}{4}+\frac{1}{3}$ $\div\frac{1}{2}$ $6+1+\frac{1}{2}+?$ Think of $\frac{1}{3}$ as $\frac{1}{6}+\frac{1}{6}$. $\frac{1}{6}+\frac{1}{6}$ $\div\frac{1}{2}$ $\frac{1}{3}+\frac{1}{3}=\frac{1}{2}+\frac{1}{6}$ $6+1+\frac{1}{2}+\frac{1}{2}+\frac{1}{6}$ $8+\frac{1}{6}$	$\frac{1}{8}+\frac{1}{12}+\frac{1}{16}+\frac{1}{20}$ $\div\frac{1}{4}$ $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ But that sum is more than 1. $\frac{1}{3}+\frac{1}{4}=\frac{1}{2}+\frac{1}{12}$ $\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{12}\right)+\frac{1}{5}$ $1+\frac{1}{12}+\frac{1}{5}$
$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$ $\div\frac{1}{3}$	$7+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}$ $\div\frac{1}{2}$

Before working, look above for clues.

$3+\frac{1}{2}+\frac{1}{4}+\frac{1}{10}$	$15+\frac{1}{2}$

Tasks: *Students complete examples involving multiplication and division by a fraction.*

Purpose: To provide practice in multiplication (and division) where factors may be whole numbers and fractions.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: When the multipliers are unit fractions, the immediate results are almost always in the required form of a whole number and a series of *different* unit fractions. There are exceptions; for example:

$$\begin{array}{r} 5 + \frac{1}{2} \\ \times\ \frac{1}{3} \\ \hline \end{array} \qquad \text{or} \qquad \begin{array}{r} 3 + 2 + \frac{1}{2} \\ \times\ \frac{1}{3} \\ \hline 1 + \frac{2}{3} + \frac{1}{6} \\ 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{6} \end{array}$$

In these examples, the rule of no repetition is violated, but an easy way out is to use Peter's Theorem and replace $\frac{1}{6}$ with $\frac{1}{7} + \frac{1}{42}$.

As expected, the problems encountered in division are the reverse. Dividing by a whole number presents no difficulty, and dividing by a fraction can involve situations requiring a few maneuvers, but none that are basically new.

The final pair of examples on the page are so related to the first two at the top of the column that the answers can be written without making computations.

Practice sheets 41 and 42 put to work the means for comparing fractions developed on practice sheet 40.

The same fractions are compared on practice sheet 41 with $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$, and on practice sheet 42 with $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, and $\frac{1}{10}$.

Students are asked first to circle all of the given fractions that are greater than $\frac{1}{2}$. There are three: $\frac{2}{3}$, $\frac{5}{8}$, and $\frac{5}{9}$. The enterprising student will notice that these three fractions appear in each list, since in the later lists they are compared with fractions smaller than $\frac{1}{2}$. So, why not circle them before going on? In that way, the number of new comparisons to be made will be fewer from one list to the next. The two practice sheets are so organized that each time the given fractions come up for comparison, they are compared with a smaller fraction. Thus, as soon as a fraction is circled in any list, it can be circled in all lists that follow, since $\frac{1}{2} > \frac{1}{3} > \frac{1}{4}$, etc.

The lower section of practice sheet 41 considers comparing fractions in yet another context—finding the largest unit fraction less than a given fraction. This problem reduces to that of finding the smallest n which meets the condition:

$$\frac{a}{b} > \frac{1}{n}$$

where a/b is any given fraction. We need only remember that this is true if

$$a \times n > b$$

From here on, the selection of an appropriate n is a matter of simple arithmetic.

FPS-41

Draw a loop around each fraction that *is greater than* (>) the unit fraction given. (Note: Don't tackle this page unless you have completed Practice Sheet 40.)

I

$\frac{3}{7}$	$\frac{4}{25}$	$\frac{2}{3}$	$\frac{3}{11}$	$\frac{4}{9}$	$\frac{2}{11}$	$\frac{2}{15}$	$> \frac{1}{2}$
$\frac{3}{8}$	$\frac{3}{14}$	$\frac{4}{33}$	$\frac{2}{19}$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{2}{13}$	$\frac{3}{16}$
$\frac{4}{13}$	$\frac{3}{22}$	$\frac{2}{100}$	$\frac{3}{17}$	$\frac{4}{29}$	$\frac{2}{9}$	$\frac{5}{8}$	$\frac{3}{40}$
$\frac{2}{7}$	$\frac{5}{33}$	$\frac{3}{25}$	$\frac{5}{9}$	$\frac{2}{17}$	$\frac{3}{32}$	$\frac{3}{28}$	$\frac{3}{13}$

II

$\frac{3}{7}$	$\frac{4}{25}$	$\frac{2}{3}$	$\frac{3}{11}$	$\frac{4}{9}$	$\frac{2}{11}$	$\frac{2}{15}$	$> \frac{1}{3}$
$\frac{3}{8}$	$\frac{3}{14}$	$\frac{4}{33}$	$\frac{2}{19}$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{2}{13}$	$\frac{3}{16}$
$\frac{4}{13}$	$\frac{3}{22}$	$\frac{2}{100}$	$\frac{3}{17}$	$\frac{4}{29}$	$\frac{2}{9}$	$\frac{5}{8}$	$\frac{3}{40}$
$\frac{2}{7}$	$\frac{5}{33}$	$\frac{3}{25}$	$\frac{5}{9}$	$\frac{2}{17}$	$\frac{3}{32}$	$\frac{3}{28}$	$\frac{3}{13}$

III

$\frac{3}{7}$	$\frac{4}{25}$	$\frac{2}{3}$	$\frac{3}{11}$	$\frac{4}{9}$	$\frac{2}{11}$	$\frac{2}{15}$	$> \frac{1}{4}$
$\frac{3}{8}$	$\frac{3}{14}$	$\frac{4}{33}$	$\frac{2}{19}$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{2}{13}$	$\frac{3}{16}$
$\frac{4}{13}$	$\frac{3}{22}$	$\frac{2}{100}$	$\frac{3}{17}$	$\frac{4}{29}$	$\frac{2}{9}$	$\frac{5}{8}$	$\frac{3}{40}$
$\frac{2}{7}$	$\frac{5}{33}$	$\frac{3}{25}$	$\frac{5}{9}$	$\frac{2}{17}$	$\frac{3}{32}$	$\frac{3}{28}$	$\frac{3}{13}$

IV

$\frac{3}{7}$	$\frac{4}{25}$	$\frac{2}{3}$	$\frac{3}{11}$	$\frac{4}{9}$	$\frac{2}{11}$	$\frac{2}{15}$	$> \frac{1}{5}$
$\frac{3}{8}$	$\frac{3}{14}$	$\frac{4}{33}$	$\frac{2}{19}$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{2}{13}$	$\frac{3}{16}$
$\frac{4}{13}$	$\frac{3}{22}$	$\frac{2}{100}$	$\frac{3}{17}$	$\frac{4}{29}$	$\frac{2}{9}$	$\frac{5}{8}$	$\frac{3}{40}$
$\frac{2}{7}$	$\frac{5}{33}$	$\frac{3}{25}$	$\frac{5}{9}$	$\frac{2}{17}$	$\frac{3}{32}$	$\frac{3}{28}$	$\frac{3}{13}$

V

$\frac{3}{7}$	$\frac{4}{25}$	$\frac{2}{3}$	$\frac{3}{11}$	$\frac{4}{9}$	$\frac{2}{11}$	$\frac{2}{15}$	$> \frac{1}{6}$
$\frac{3}{8}$	$\frac{3}{14}$	$\frac{4}{33}$	$\frac{2}{19}$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{2}{13}$	$\frac{3}{16}$
$\frac{4}{13}$	$\frac{3}{22}$	$\frac{2}{100}$	$\frac{3}{17}$	$\frac{4}{29}$	$\frac{2}{9}$	$\frac{5}{8}$	$\frac{3}{40}$
$\frac{2}{7}$	$\frac{5}{33}$	$\frac{3}{25}$	$\frac{5}{9}$	$\frac{2}{17}$	$\frac{3}{32}$	$\frac{3}{28}$	$\frac{3}{13}$

Suppose that you wish to write $\frac{2}{7}$ as a sum of unit fractions. You decide that the first unit fraction in the sum be as large as possible. The problem is this. Find a whole number n such that

$$\frac{2}{7} > \frac{1}{n}$$

with n as small as possible. You can compare these fractions as we compared them on Practice Sheet 40.

$$\frac{2 \times n}{7n} > \frac{7 \times 1}{7n} \quad \text{if} \quad 2 \times n > 7 \times 1$$

and we wish n to be as small as possible. If we replace n with 3, we have:

$$2 \times 3 > 7 \times 1 \qquad \text{(not true)}$$

So we must use a larger number to replace n. We try 4, and have:

$$2 \times 4 > 7 \times 1 \qquad \text{(true)}$$

Now we write:

$$\frac{2}{7} = \frac{1}{4} + \dots\dots$$

knowing that $\frac{1}{4}$ is the largest unit fraction less than $\frac{2}{7}$.

In general: for any fraction $\frac{a}{b}$, we can find the largest unit fraction smaller than $\frac{a}{b}$ in this way. Since

$$\frac{a}{b} > \frac{1}{n} \quad \text{if} \quad a \times n > b \times 1,$$

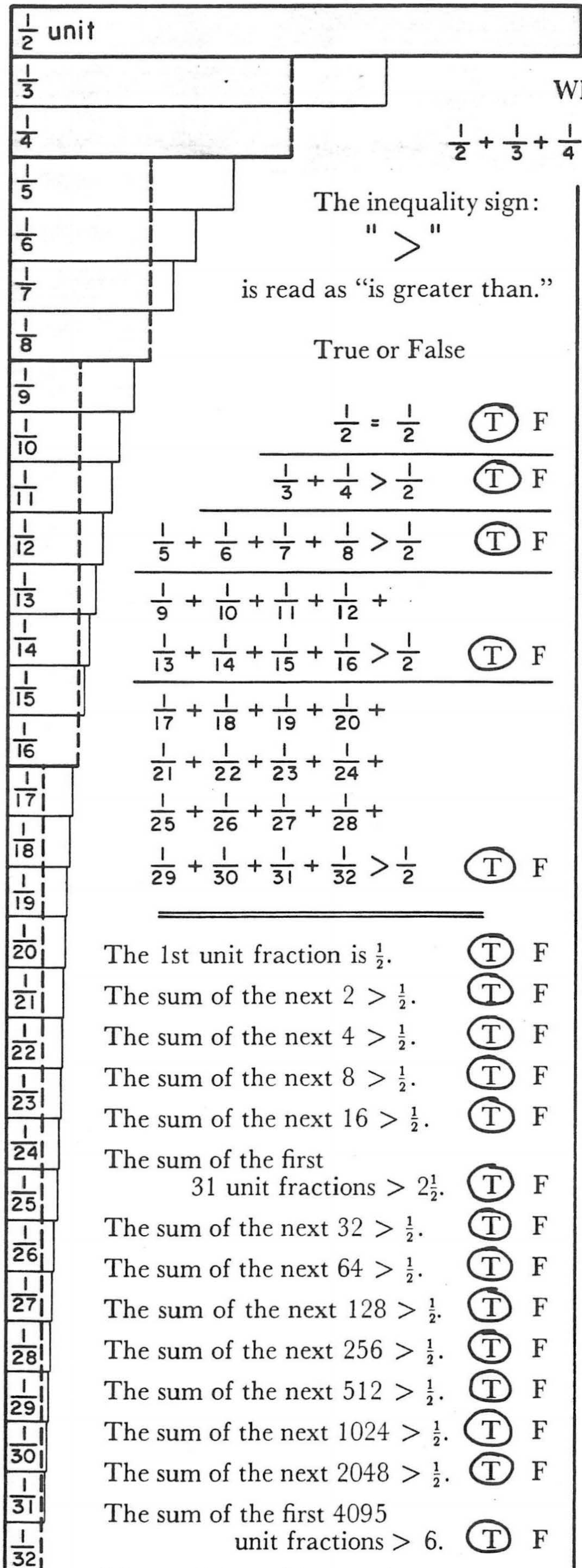
you should look for the smallest whole number n such that

$$a \times n > b \times 1.$$

Try some examples:

$$\frac{4}{9} = \frac{1}{3} + \dots\dots \qquad \frac{3}{8} = \frac{1}{3} + \dots\dots$$

$$\frac{3}{11} = \frac{1}{4} + \dots\dots \qquad \frac{5}{9} = \frac{1}{2} + \dots\dots$$



What is the sum of all unit fractions?

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\ldots \text{etc.}$$

The inequality sign:

"$>$"

is read as "is greater than."

True or False

$\frac{1}{2}=\frac{1}{2}$ — T F

$\frac{1}{3}+\frac{1}{4}>\frac{1}{2}$ — T F

$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{2}$ — T F

$\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}>\frac{1}{2}$ — T F

$\frac{1}{17}+\frac{1}{18}+\frac{1}{19}+\frac{1}{20}+\frac{1}{21}+\frac{1}{22}+\frac{1}{23}+\frac{1}{24}+\frac{1}{25}+\frac{1}{26}+\frac{1}{27}+\frac{1}{28}+\frac{1}{29}+\frac{1}{30}+\frac{1}{31}+\frac{1}{32}>\frac{1}{2}$ — T F

The 1st unit fraction is $\frac{1}{2}$. — T F

The sum of the next 2 $> \frac{1}{2}$. — T F

The sum of the next 4 $> \frac{1}{2}$. — T F

The sum of the next 8 $> \frac{1}{2}$. — T F

The sum of the next 16 $> \frac{1}{2}$. — T F

The sum of the first 31 unit fractions $> 2\frac{1}{2}$. — T F

The sum of the next 32 $> \frac{1}{2}$. — T F

The sum of the next 64 $> \frac{1}{2}$. — T F

The sum of the next 128 $> \frac{1}{2}$. — T F

The sum of the next 256 $> \frac{1}{2}$. — T F

The sum of the next 512 $> \frac{1}{2}$. — T F

The sum of the next 1024 $> \frac{1}{2}$. — T F

The sum of the next 2048 $> \frac{1}{2}$. — T F

The sum of the first 4095 unit fractions > 6. — T F

We could keep this up as long as we wish. — T F

The sum of the first three unit fractions is more than 1 unit.

Is the sum of all unit fractions less than $\frac{1}{4}$ and greater than $\frac{1}{33}$ more or less than another whole unit?

Guess. Don't try to carry out the addition because you will quickly encounter very large common denominators. Start the computation, but stop as soon as you see the difficulty.

Actually, you can answer the question without any difficult computations at all.

Study the questions on the left using the diagram to help you.

Are you convinced that the sum of the first 31 unit fractions is more than $2\frac{1}{2}$ units?

If the final statement in the left-hand column is true, what do you have to say about the question we started with:

What is the sum of all unit fractions?

The sum of the first (2^1-1) unit fractions $= \frac{1}{2}$.

The sum of the first (2^2-1) unit fractions $>$ 1.

The sum of the first (2^3-1) unit fractions $>$ $1\frac{1}{2}$.

The sum of the first (2^4-1) unit fractions $>$ 2.

The sum of the first (2^5-1) unit fractions $>$ $2\frac{1}{2}$.

The sum of the first $(2^{50}-1)$ unit fractions $>$ 25.

The sum of the first $(2^{1000}-1)$ unit fractions $>$ 500.

For each $n>0$,
the sum of the first (2^n-1) unit fractions $>\frac{n}{2}$.

Can you say that the sum of all unit fractions is larger than any number you can think of?

Do you feel that we have proved that it must be true?

 What is the "sum" of all unit fractions?

Tasks: *Students estimate the sums of groups of consecutive unit fractions and the "sum" of all unit fractions.*

Purpose: To help students discover that by combining consecutive unit fractions we can write a sum that is larger than any number we choose.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: The problem posed and discussed on pupil page 42 leads to a most surprising result, one for which the student has little preparation. If we think about the expression:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots,$$

we eventually realize that as we add more terms, we get closer and closer to 1. This limiting number, 1, can be thought of as the sum of the entire series.

Now consider the expression:

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\ldots.$$

We might suppose that here, too, as we add more and more terms of the series, we approach a limiting number that can be thought of as the sum of the entire series.

Quite the opposite is true! If we add enough terms, we can get a sum larger than any number we wish. To see why this is so, we can group the terms in a special way:

$$\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}\right)+\ldots.$$

The numbers of terms in the enclosed groups are consecutive powers of 2: 2, 4, 8, etc. The sum of the terms in each group is greater than ½. Therefore, if we add enough terms, we can get a sum that is as large as we please.

FPS-42

(Note: Don't tackle this page unless you have completed the work on Practice Sheet 40.)

Draw a loop around each fraction that *is greater than* (>) the unit fraction shown.

VI $\frac{3}{7}$ $\frac{4}{25}$ $\frac{2}{3}$ $\frac{3}{11}$ $\frac{4}{9}$ $\frac{2}{11}$ $\frac{2}{15}$ $> \frac{1}{7}$

$\frac{3}{8}$ $\frac{3}{14}$ $\frac{4}{33}$ $\frac{2}{19}$ $\frac{2}{5}$ $\frac{3}{10}$ $\frac{2}{13}$ $\frac{3}{16}$

$\frac{4}{13}$ $\frac{3}{22}$ $\frac{2}{100}$ $\frac{3}{17}$ $\frac{4}{29}$ $\frac{2}{9}$ $\frac{5}{8}$ $\frac{3}{40}$

$\frac{2}{7}$ $\frac{5}{33}$ $\frac{3}{25}$ $\frac{5}{9}$ $\frac{2}{17}$ $\frac{3}{32}$ $\frac{3}{28}$ $\frac{3}{13}$

VII $\frac{3}{7}$ $\frac{4}{25}$ $\frac{2}{3}$ $\frac{3}{11}$ $\frac{4}{9}$ $\frac{2}{11}$ $\frac{2}{15}$ $> \frac{1}{8}$

$\frac{3}{8}$ $\frac{3}{14}$ $\frac{4}{33}$ $\frac{2}{19}$ $\frac{2}{5}$ $\frac{3}{10}$ $\frac{2}{13}$ $\frac{3}{16}$

$\frac{4}{13}$ $\frac{3}{22}$ $\frac{2}{100}$ $\frac{3}{17}$ $\frac{4}{29}$ $\frac{2}{9}$ $\frac{5}{8}$ $\frac{3}{40}$

$\frac{2}{7}$ $\frac{5}{33}$ $\frac{3}{25}$ $\frac{5}{9}$ $\frac{2}{17}$ $\frac{3}{32}$ $\frac{3}{28}$ $\frac{3}{13}$

VIII $\frac{3}{7}$ $\frac{4}{25}$ $\frac{2}{3}$ $\frac{3}{11}$ $\frac{4}{9}$ $\frac{2}{11}$ $\frac{2}{15}$ $> \frac{1}{9}$

$\frac{3}{8}$ $\frac{3}{14}$ $\frac{4}{33}$ $\frac{2}{19}$ $\frac{2}{5}$ $\frac{3}{10}$ $\frac{2}{13}$ $\frac{3}{16}$

$\frac{4}{13}$ $\frac{3}{22}$ $\frac{2}{100}$ $\frac{3}{17}$ $\frac{4}{29}$ $\frac{2}{9}$ $\frac{5}{8}$ $\frac{3}{40}$

$\frac{2}{7}$ $\frac{5}{33}$ $\frac{3}{25}$ $\frac{5}{9}$ $\frac{2}{17}$ $\frac{3}{32}$ $\frac{3}{28}$ $\frac{3}{13}$

IX $\frac{3}{7}$ $\frac{4}{25}$ $\frac{2}{3}$ $\frac{3}{11}$ $\frac{4}{9}$ $\frac{2}{11}$ $\frac{2}{15}$ $> \frac{1}{10}$

$\frac{3}{8}$ $\frac{3}{14}$ $\frac{4}{33}$ $\frac{2}{19}$ $\frac{2}{5}$ $\frac{3}{10}$ $\frac{2}{13}$ $\frac{3}{16}$

$\frac{4}{13}$ $\frac{3}{22}$ $\frac{2}{100}$ $\frac{3}{17}$ $\frac{4}{29}$ $\frac{2}{9}$ $\frac{5}{8}$ $\frac{3}{40}$

$\frac{2}{7}$ $\frac{5}{33}$ $\frac{3}{25}$ $\frac{5}{9}$ $\frac{2}{17}$ $\frac{3}{32}$ $\frac{3}{28}$ $\frac{3}{13}$

X $\frac{3}{7}$ $\frac{4}{25}$ $\frac{2}{3}$ $\frac{3}{11}$ $\frac{4}{9}$ $\frac{2}{11}$ $\frac{2}{15}$ $> \frac{1}{11}$

$\frac{3}{8}$ $\frac{3}{14}$ $\frac{4}{33}$ $\frac{2}{19}$ $\frac{2}{5}$ $\frac{3}{10}$ $\frac{2}{13}$ $\frac{3}{16}$

$\frac{4}{13}$ $\frac{3}{22}$ $\frac{2}{100}$ $\frac{3}{17}$ $\frac{4}{29}$ $\frac{2}{9}$ $\frac{5}{8}$ $\frac{3}{40}$

$\frac{2}{7}$ $\frac{5}{33}$ $\frac{3}{25}$ $\frac{5}{9}$ $\frac{2}{17}$ $\frac{3}{32}$ $\frac{3}{28}$ $\frac{3}{13}$

Complete each sentence, filling in the blanks with whole numbers or unit fractions.

1. A quart is $\frac{1}{2}$ of a half-gallon.
2. A dime is $\frac{1}{10}$ of a dollar.
3. A foot is $\frac{1}{3}$ of a yard.
4. A pint is $\frac{1}{2}$ of a quart.
5. A nickel is $\frac{1}{10}$ of a half-dollar.
6. There are 16 half-pints in a gallon.
7. There are 10 dimes in a dollar.
8. An ounce is $\frac{1}{16}$ of a pound.
9. A quarter of an hour is $\frac{1}{2}$ of a half-hour.
10. A week is 7 days.
11. A day is $\frac{1}{7}$ of a week.
12. There are 5,280 feet in a mile.
13. A foot is $\frac{1}{5280}$ of a mile.
14. A yard is $\frac{1}{1,760}$ of a mile.
15. A rod ($16\frac{1}{2}$ ft.) is $\frac{1}{320}$ of a mile.
16. A square foot is $\frac{1}{9}$ of a square yard.
17. A square inch is $\frac{1}{144}$ of a square foot.
18. A cubic foot is $\frac{1}{27}$ of a cubic yard.
19. There is $\frac{1}{16}$ of a pound in one ounce.

As you were exploring this problem out of the past, did you realize that by the fourth or fifth page of this chapter, you knew more about unit fractions than most of the well educated people in the world?

We do not say this primarily as a compliment to you. Rather, we wish to illustrate something about the subject of mathematics:

Select a problem and work at it. Before you know it, you are learning something that very few people know about.

But don't think that you have learned all there is to know about unit fractions! You've only touched the subject lightly.

As the authors were working on this chapter, many problems arose that we did not raise with you. Some of them might be interesting for you to work at. We shall mention one.

Which *common* fractions (smaller than 1) can be expressed as the sum of two unit fractions?

No fraction larger than $\frac{5}{6}$ can be expressed as the sum of two unit fractions. Why? Because the sum of the two largest is $\frac{5}{6}$.

But $\frac{4}{5}$, which is smaller than $\frac{5}{6}$, cannot be expressed as the sum of two unit fractions. We experiment:

$$\frac{4}{5} = \frac{1}{2} + \frac{3}{10} \qquad \frac{4}{5} = \frac{1}{3} + \frac{7}{15}$$

If we try $\frac{1}{2}$ as the first fraction, the second must be $\frac{3}{10}$. But this isn't a unit fraction. If we try $\frac{1}{3}$ as the first, the second must be $\frac{7}{15}$ — not a unit fraction. Besides, it's larger than $\frac{1}{3}$.

So, we have nothing more to do but report that more than two unit fractions will be required.

Here are some other fractions, less than $\frac{5}{6}$, that call for more than two unit fractions:

$$\frac{3}{7} = \frac{1}{4} + \frac{1}{7} + \frac{1}{28} \qquad \frac{7}{9} = \frac{1}{2} + \frac{1}{6} + \frac{1}{9}$$

$$\frac{5}{7} = \frac{1}{2} + \frac{1}{7} + \frac{1}{14} \qquad \frac{5}{11} = \frac{1}{3} + \frac{1}{11} + \frac{1}{33}$$

(We show one of *several* possible combinations using three unit fractions.)

What is unusual here? Are all numerators and denominators prime numbers? No, 9 is not a prime.

Peter's rule (page 37) showed that every unit fraction can be expressed as the sum of two unit fractions.

We can also show that every fraction with 2 as a numerator can be so expressed. (Of course, the denominator would be odd — otherwise we would have reduced the fraction.) The method is this:

$$\frac{2}{7} = \frac{1}{4} + \frac{1}{28} \qquad \frac{2}{9} = \frac{1}{5} + \frac{1}{45} \qquad \frac{2}{11} = \frac{1}{6} + \frac{1}{66}$$

$$2 \times 4 = 7 + 1 \qquad 2 \times 5 = 9 + 1 \qquad 2 \times 6 = 11 + 1$$

$$7 \times 4 = 28 \qquad 9 \times 5 = 45 \qquad 11 \times 6 = 66$$

You try some:

$$\frac{2}{13} = \frac{1}{7} + \frac{1}{91} \qquad \frac{2}{19} = \frac{1}{10} + \frac{1}{190}$$

Let's try other even numerators:

$$\frac{4}{11} = \frac{1}{3} + \frac{1}{33} \qquad \frac{4}{13} = \frac{1}{?} + \frac{1}{}$$

The first is encouraging — the second is a counterexample.

Perhaps you would like to explore this problem further. It won't be easy.

43 Expressing common fractions as sums of unit fractions

Tasks: Students attempt to determine some common fractions which can be expressed as the sum of two unit fractions.

Purpose: To help students think about a procedure for exploring a mathematical problem.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: The first few paragraphs on this pupil page deserve some discussion.

If any students have selected problems (as we suggested earlier) and have spent several hours on them, they will undoubtedly know more about these problems than almost anyone who doesn't specialize in mathematics. This is a rather remarkable fact. Mathematics is a fundamental part of our culture, yet surprisingly few people ever have a glimpse of the subject, except as it appears between the covers of school textbooks.

When one explores beyond the straight and narrow path prescribed by the syllabus, he often encounters the need for proof, the need to find arguments to support his guesses or hunches. It is the rare student who can be persuaded to question seriously the assertions in a textbook. But when he makes surmises of his own, he feels very keenly the need to defend his position. Every successful defense against his own and others' attacks or doubts represents an attempt at a proof. The attempt may later be found inadequate or some of the assertions may turn out to be wrong. Gradually the student will learn about standards for proofs required of mathematicians, and he will be better able to judge arguments in support of his assertions.

He will learn the role of the *counterexample,* a single example that can challenge an assertion and demolish it. In testing hunches, the counterexample is often useful in pointing the way toward more reliable guesses.

At the close of pupil page 43, we present a technique that works with any fraction with numerator 2, and try to apply it to fractions with numerator 4. The technique works with $4/11$ but $4/13$ is a counterexample. So we have only started the search for a way to know quickly whether a given fraction can be expressed as the sum of two unit fractions.

Practice sheet 43 may help initiate the search. What do all the examples requiring more than two unit fractions have in common? What do the rest have in common?

FPS-43

Can you find a way of expressing the fractions on this page as the sum of the number of unit fractions indicated?

This is a lot to do, so you may wish to organize some teamwork.

$\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$

$\frac{1}{3} = \frac{1}{4} + \frac{1}{12}$

$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$

$\frac{1}{4} = \frac{1}{5} + \frac{1}{20}$

$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$

$\frac{1}{5} = \frac{1}{6} + \frac{1}{30}$

$\frac{2}{5} = \frac{1}{3} + \frac{1}{15}$

$\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$

$\frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$

$\frac{1}{6} = \frac{1}{7} + \frac{1}{42}$

$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$

$\frac{1}{7} = \frac{1}{8} + \frac{1}{56}$

$\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$

$\frac{3}{7} = \frac{1}{4} + \frac{1}{7} + \frac{1}{28}$

$\frac{4}{7} = \frac{1}{2} + \frac{1}{14}$

$\frac{5}{7} = \frac{1}{2} + \frac{1}{7} + \frac{1}{14}$

$\frac{6}{7} = \frac{1}{2} + \frac{1}{3} + \frac{1}{42}$

$\frac{1}{8} = \frac{1}{9} + \frac{1}{72}$

$\frac{3}{8} = \frac{1}{4} + \frac{1}{8}$

$\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$

$\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$\frac{1}{9} = \frac{1}{10} + \frac{1}{90}$

$\frac{2}{9} = \frac{1}{5} + \frac{1}{45}$

$\frac{4}{9} = \frac{1}{3} + \frac{1}{9}$

$\frac{5}{9} = \frac{1}{2} + \frac{1}{18}$

$\frac{7}{9} = \frac{1}{2} + \frac{1}{6} + \frac{1}{9}$

$\frac{8}{9} = \frac{1}{2} + \frac{1}{3} + \frac{1}{18}$

$\frac{1}{10} = \frac{1}{11} + \frac{1}{110}$

$\frac{3}{10} = \frac{1}{5} + \frac{1}{10}$

$\frac{7}{10} = \frac{1}{2} + \frac{1}{5}$

$\frac{9}{10} = \frac{1}{2} + \frac{1}{3} + \frac{1}{15}$

$\frac{1}{11} = \frac{1}{12} + \frac{1}{132}$

$\frac{2}{11} = \frac{1}{6} + \frac{1}{66}$

$\frac{3}{11} = \frac{1}{4} + \frac{1}{44}$

$\frac{4}{11} = \frac{1}{3} + \frac{1}{33}$

$\frac{5}{11} = \frac{1}{3} + \frac{1}{11} + \frac{1}{33}$

$\frac{6}{11} = \frac{1}{2} + \frac{1}{22}$

$\frac{7}{11} = \frac{1}{2} + \frac{1}{11} + \frac{1}{22}$

$\frac{8}{11} = \frac{1}{2} + \frac{1}{6} + \frac{1}{22} + \frac{1}{66}$

$\frac{9}{11} = \frac{1}{2} + \frac{1}{4} + \frac{1}{22} + \frac{1}{44}$

$\frac{10}{11} = \frac{1}{2} + \frac{1}{3} + \frac{1}{22} + \frac{1}{33}$

Here are three more to express as the sum of two unit fractions:

$\frac{13}{42} = \frac{1}{6} + \frac{1}{7}$

$\frac{20}{51} = \frac{1}{3} + \frac{1}{17}$

$\frac{22}{63} = \frac{1}{3} + \frac{1}{63}$

Chapter SEVEN . . . SYSTEMS FOR

LOCATING THINGS IN SPACE

If someone knows your address, he can drop a properly addressed card into almost any mail box in the world and it will eventually reach you — provided that it has enough postage on it.

This is possible because of an address system that may require (1) the country you live in, (2) the state, (3) the city, (4) the street, (5) the house number on that street and, perhaps, (6) an apartment number.

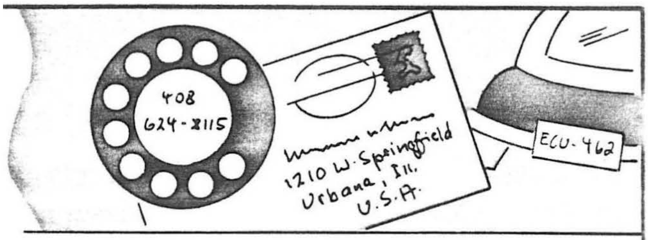
Some people can be located by their telephone number: a 3-digit area code, a 3-digit exchange, and a 4-digit local number — such as 408-624-8115.

The owner of a car can be located through his license number:

Calif. ECU-462

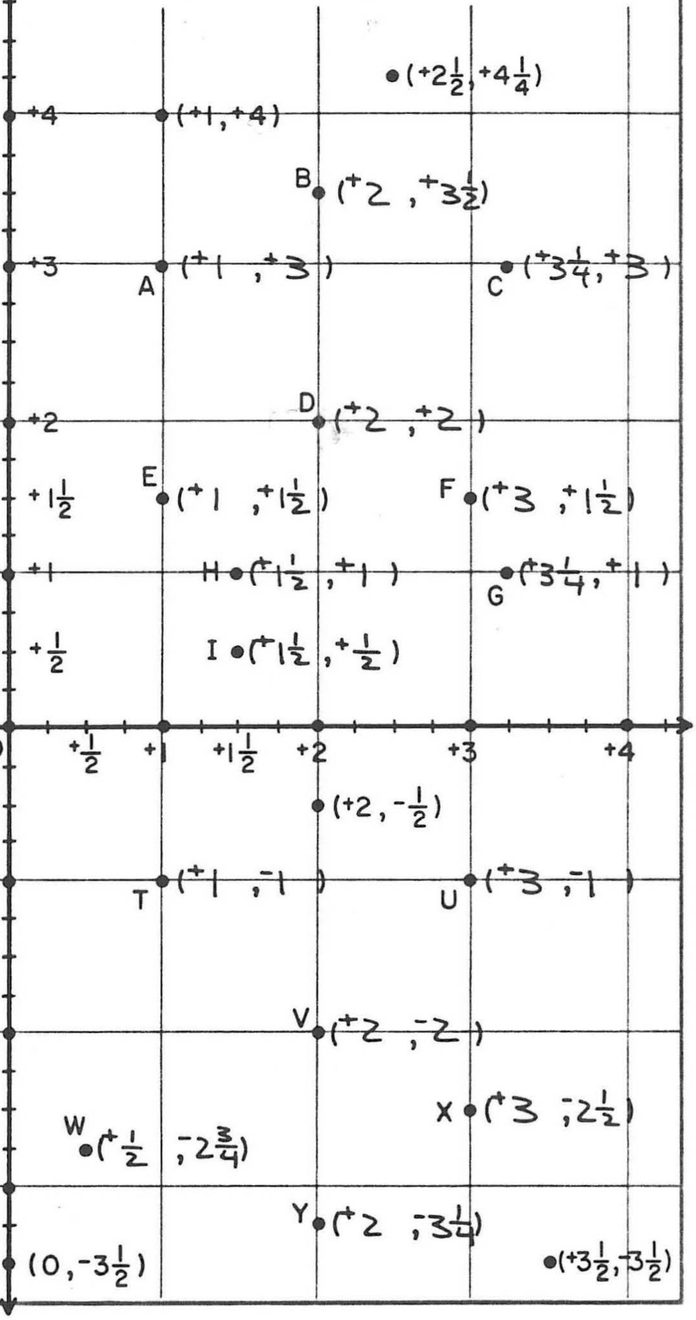
During school hours, you have a school address that's recorded in the office, noting which room you will be in and at what time.

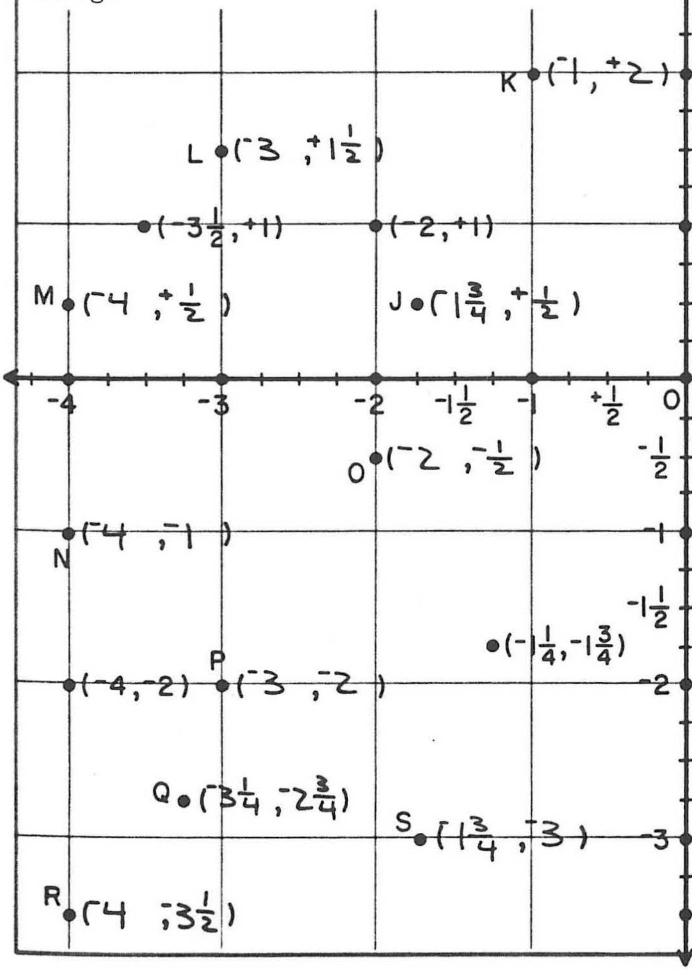
Let's consider other systems for locating things.



On this page we have used a system that gives an address for every point inside the blue borders. Our addressing system requires only two "signed" numbers.

Study those points we have labeled with their addresses. When you think that you have figured out our system, use it to find the addresses of the other points (the ones which have letters next to them). Write the addresses next to the points.





Tasks: *Students study the nine points which are labeled with their addresses to determine the system and then use the system to find the addresses of the 25 letter-labeled points.*

Purpose: To introduce pupils to the idea of a coordinate system for assigning addresses to points on a flat surface.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: Suppose that some students have never encountered positive and negative numbers nor been introduced to coordinate systems.

What can they notice about the nine points already labeled with addresses?

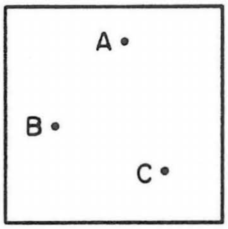
Most students, if given a few minutes to study these nine points and addresses, will be able to find addresses for the other points. They may speak of "numbers with plus and minus signs in front of them" and "the number written on the left, inside the parentheses," etc. We are not at all concerned with whether or not they use the standard terminology. Rather, we are interested in whether or not the students can study the clues and attach address-labels to the points. Can they grasp the system for giving an address for every point on the page?

The role of "+" and "–" is usually more apparent than the role of position of the coordinates within the parentheses. However, an addressing system would be impractical if two points were to have the same address. Not only must every point have an address, but no address can be used for two different points. Thus, for example, the address ($^+2$, $^+3$) is different from the address ($^+3$, $^+2$).

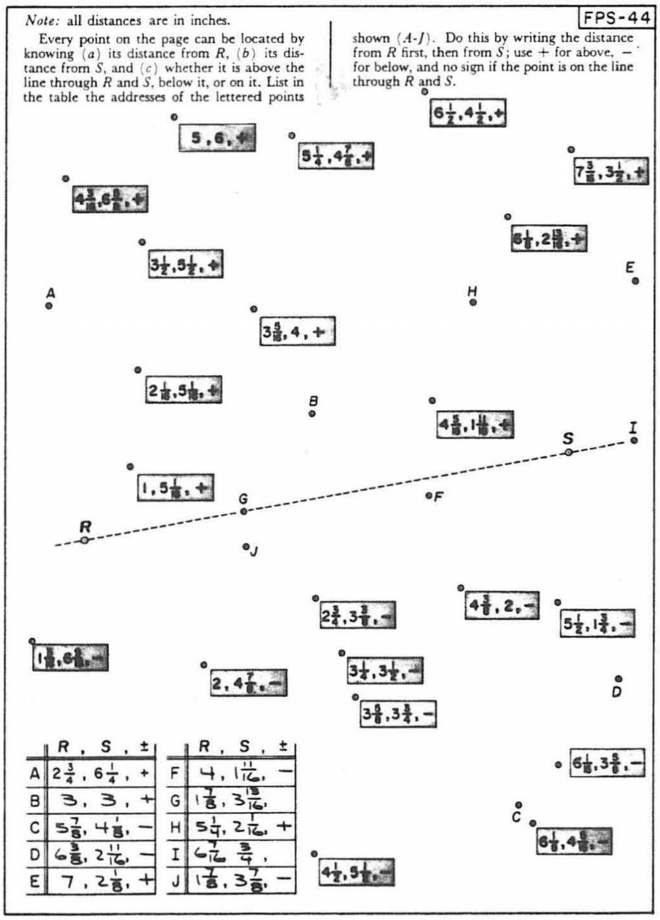
Is the system used on pupil page 44 the only possible system for giving every point on the page its own address? Perhaps this is a question some students with more background could work at while others catch up.

Of course, there are many systems. Consider practice sheet 44. Each point has a three-part address—two distance measurements followed by a sign for "direction." Again, the order of the distance measurements is important. The first measurement (on the left) gives the distance from point *R*, the second is the distance from point *S*; "+" indicates above the line, "–" below the line. The absence of a direction sign indicates that the point is on the line. Quite simple.

Consider another system. Suppose that we mark three points on the page, as shown in the diagram:



We find an address for a given point by noting, first, its distance from *A*, second, its distance from *B*, and, third, its distance from *C*. Would every point on the page have an address? (Yes.) Would each address belong to only one point? (Yes.) Does this system work if *A*, *B*, and *C* are on the same straight line? (No.) Why? (Two different points can have the same address.) Encourage students to invent other systems.



	R , S , ±		R , S , ±
A	$2\frac{3}{4}$, $6\frac{1}{4}$, +	F	4, $1\frac{11}{16}$, –
B	3, 3, +	G	$1\frac{7}{8}$, $3\frac{13}{16}$,
C	$5\frac{7}{8}$, $4\frac{1}{8}$, –	H	$5\frac{1}{4}$, $2\frac{1}{16}$, +
D	$6\frac{3}{8}$, $2\frac{11}{16}$, –	I	$6\frac{7}{16}$, $\frac{3}{4}$,
E	7, $2\frac{1}{8}$, +	J	$1\frac{7}{8}$, $3\frac{7}{8}$, –

An Alternate System of Addresses

(+90°, 6.4") or (−270°, 6.4")

(+120°, 7") or (−240°, 7")

A

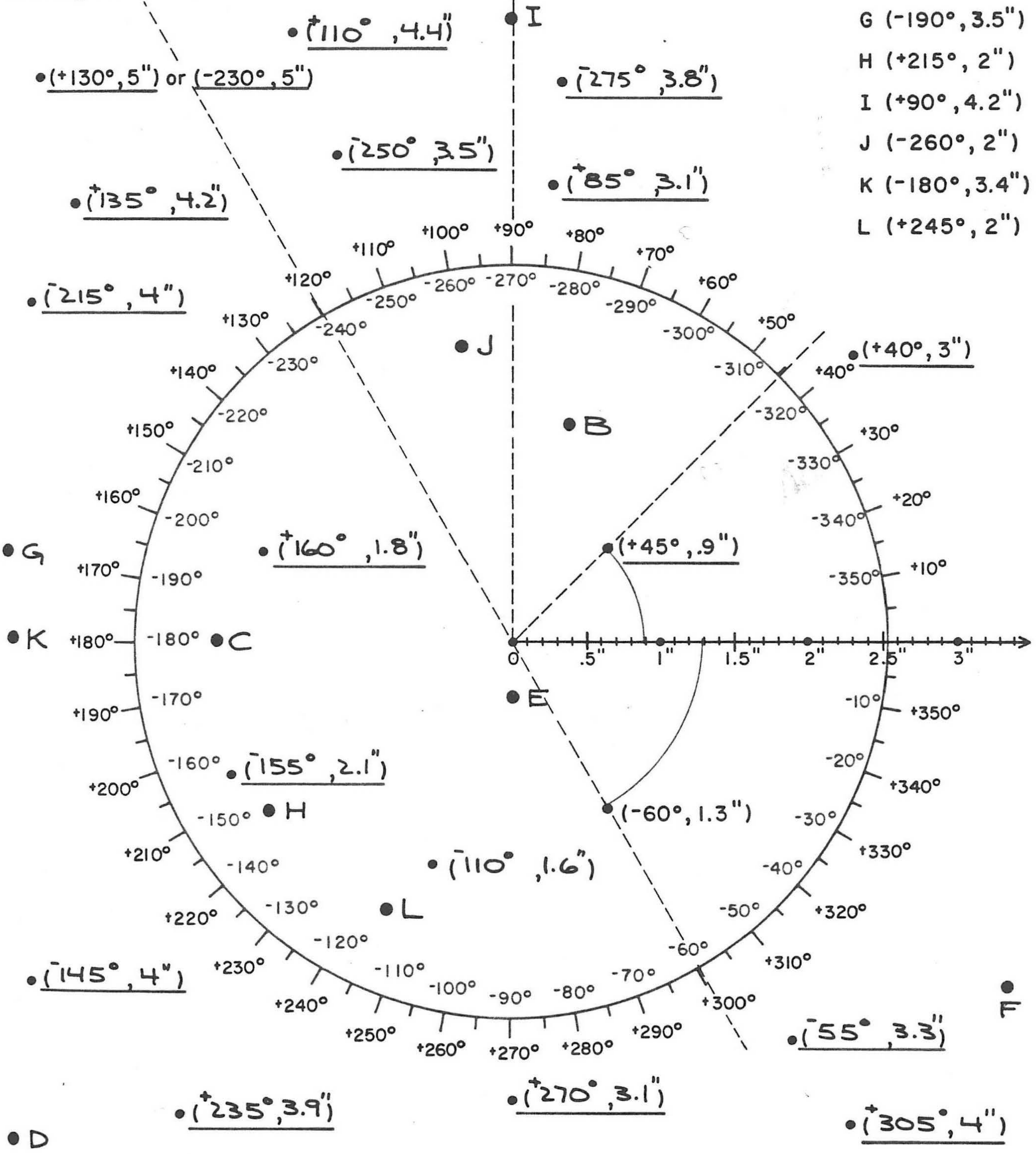
Study the points labeled with addresses. From these clues, can you figure out the system well enough to find addresses for all the points?

Each point has two addresses, but no address fits more than one point.

I. Write an address next to each point (dot), and be sure to follow the color scheme in choosing the address.

II. Please mark with dots the points that have the following addresses and label them with the appropriate letter.

A (−300°, 7")
B (+75°, 1.5")
C (+180°, 2")
D (−135°, 4.8")
E (−90°, .4")
F (+325°, 4.1")
G (−190°, 3.5")
H (+215°, 2")
I (+90°, 4.2")
J (−260°, 2")
K (−180°, 3.4")
L (+245°, 2")



Tasks: Students study the few address-labeled points to determine the system. In I they write addresses for the unlabeled points on the page. In II they mark points for given addresses and label each point with its letter name. (Points A *and* D *are already marked.)*

Purpose: To extend the student's understanding of the idea of a coordinate system for assigning addresses to points on a flat surface.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: On pupil page 45 we consider a very different system of assigning addresses to points.

It depends (a) on a rule or scale to find a distance from the point O, and (b) on a built-in 360° protractor with two scales, one scaled counterclockwise and the other scaled clockwise.

Again, let the students consider the three points for which two addresses already are given, and the three points for which only one address is given. How does the system work? It provides two addresses for every point. Does any address fit more than one point?

In some ways, it's the simplest system yet. Could it be of any practical use? Is there any advantage to having both scales? Wouldn't either one be sufficient? In fact, if there were only one scale, the address-labels would not require + and − signs.

The fact is that this is precisely the system used to locate points in air navigation. (We shall explore this use starting on practice sheet 46.)

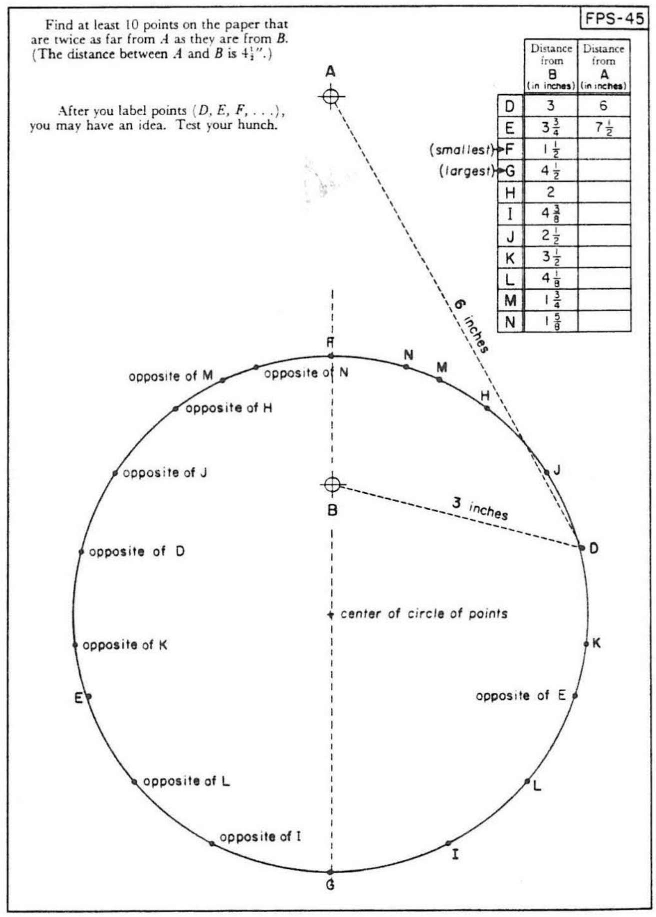
We actually use this system when we say, "The park is about eight miles northwest of town," or, "We walked about four miles a little to the east of due south." When the system is based on a compass, it will probably have much in common with the plan followed on this pupil page.

Practice sheet 45 is a variation of these "addressing" activities. The difference here is that our description of the location of points does not lead to a single point; rather, it leads to a set of points.

We are given points A and B, and we wish to locate points meeting the requirement that they be twice as far from A as they are from B.

Almost no one will have any notion about where these points lie before he goes to work. Many students may feel that they will be scattered randomly all over the page. Others may think that there are only one or two such points.

Most students will begin by using a ruler to try locating the points. After they have found a few, suggest that using a pair of dividers (or a compass) might speed up their work. If they set the dividers at 3 inches and draw a circle with center at B, they will have all points 3 inches from point B. Reset the dividers at 6 inches and draw a circle with center at A, thus locating all points 6 inches from point A. These circles will intersect in two points, each of which meets the requirement of being twice as far from A as from B.


FPS-45

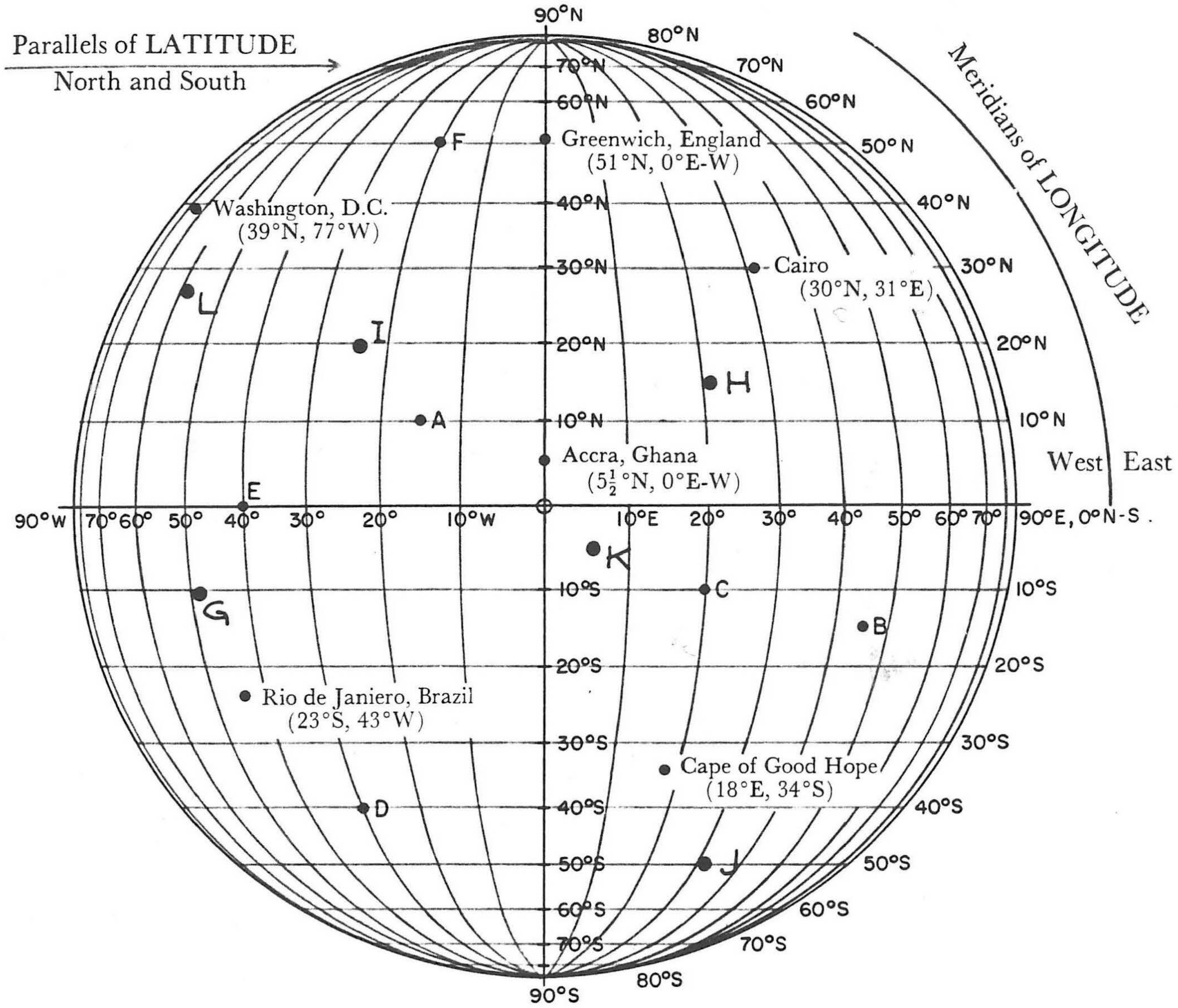
Find at least 10 points on the paper that are twice as far from A as they are from B. (The distance between A and B is $4\frac{1}{2}$".)

After you label points ($D, E, F, \ldots$), you may have an idea. Test your hunch.

	Distance from B (in inches)	Distance from A (in inches)
D	3	6
E	$3\frac{3}{4}$	$7\frac{1}{2}$
(smallest) F	$1\frac{1}{2}$	
(largest) G	$4\frac{1}{2}$	
H	2	
I	$4\frac{3}{8}$	
J	$2\frac{1}{2}$	
K	$3\frac{1}{2}$	
L	$4\frac{1}{8}$	
M	$1\frac{3}{4}$	
N	$1\frac{5}{8}$	

Locating Points on a Sphere such as the Earth.

There are, of course, many systems we might choose to give every point a geographical address. However, the one in most common use since Ptolemy in the second century A.D. is based on imagining two sequences of circles.



Step I. Locate the North Pole, the South Pole, and Greenwich, England. Imagine a circle drawn from the North Pole through Greenwich, on to the South Pole, and on around the world back to the North Pole. We call this *the Prime Meridian.*

Step II. On the semicircle from pole to pole that passes through Greenwich, locate a point halfway between the poles. We have indicated it with an O on our sketch.

Step III. At this point O, imagine a circle drawn perpendicular to the Prime Meridian and around the globe. This circle is called *the Equator*.

Step IV. Draw imaginary circles around the globe that are parallel to the Equator. We call them *parallels of latitude*.

Step V. Draw imaginary circles around the globe that pass through both poles. We call them *meridians of longitude*.

Tasks: Students study the latitude-longitude system for assigning addresses to points on a sphere.

Purpose: To extend student understanding of the idea of a coordinate system for assigning addresses to points on a flat or a spherical surface.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

The Lesson: Pupil page 46 presents still another system for assigning addresses to points, this time on a globe rather than on a flat surface. It is, in effect, a combination of the ideas explored on pupil pages 44 and 45.

It resembles the perpendicular base lines of pupil page 44. One base line (the Equator) runs due east and west from a zero point. The difference is that a traveler going due east from zero and another going due west from the same point would eventually meet each other.

The other base line crosses the Equator at the zero point and goes due north and south. The zero is selected on the Equator so that a due north route would pass through Greenwich, England. This locates the zero point in the Atlantic Ocean off the coast of Africa.

Now the parallels of latitude have much in common with the horizontal lines on pupil page 44—all points on a parallel of latitude are at the same distance from the Equator. Of course, on a globe these parallels turn out to be circles.

Meridians of longitude have much in common with the vertical lines on pupil page 44—each is perpendicular to the Equator as the vertical lines on pupil page 44 are perpendicular to the horizontal base line. But on a globe they all meet at the North Pole and at the South Pole.

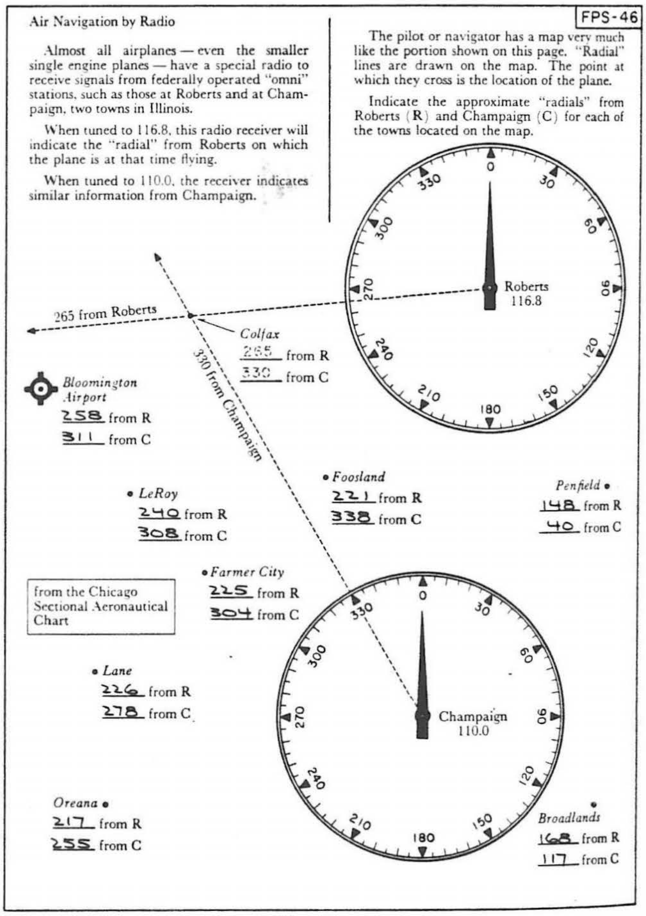
Parallels of latitude and meridians of longitude are labeled with a system similar to that used on the circular scales on pupil page 45—degrees of the circle.

The Equator is divided into 360°, 180° to the west of the zero point and 180° to the east of that point. Meridians of longitude are labeled by the points at which they cross the Equator.

The base line through Greenwich, England, is divided into 90° between the Equator and each pole, so that 90° north on any meridian is the North Pole and 90° south on any meridian is the South Pole.

This provides the basis for an addressing system considered further on pupil page 47.

The air navigation system introduced on practice sheet 46 is very similar to that developed on pupil page 45. A vertical position indicates north, and 0° is located at that point on the scale. The compass points are then labeled on a 360° scale in a clockwise direction. Thus, any point can be located by (a) specifying an *omni station,* (b) a *radial* from that point, and (c) a *distance* from that point. (This page is a scale copy of the actual map published by the U.S. Government and available at any airport: Chicago Sectional Map, 30¢ each.)



Step VI. Divide the Equator and the Prime Meridian each into 360 parts of the same length. We call each part *a degree.*

Step VII. Number the parallels of latitude by the number of degrees from the Equator. The North Pole is 90° north, and the South Pole is 90° south.

Step VIII. Number the meridians of longitude from the Prime Meridian, to the west and to the east. Half-way around the Equator from the Prime Meridian is 180° east and 180° west.

If we use 25,000 miles as the circumference of the earth then a degree of latitude must be about 69 miles (to the nearest mile).

Give the geographical address of the points labeled with letters on the sketch on the previous page.

A. (10°N, 15°W)	B. (15°S, 45°E)
C. (10°S, 20°E)	D. (40°S, 30°W)
E. (0°N-S, 40°W)	F. (50°N, 20°W)

In the sketch on the previous page, locate and label the following points:

G. (10°S, 50°W)	H. (15°N, 20°E)
I. (20°N, 25°W)	J. (50°S, 30°E)
K. (5°S, 5°E)	L. (27½°N, 60°W)

A bear started at a certain point, walked 5 miles south, then 5 miles east, and then 5 miles north. He had walked 15 miles and had returned to his starting point.

What was the color of the bear? WHITE

HE MUST HAVE STARTED AT THE NORTH POLE.

This is obviously a kind of trick question, but think about it for a few minutes.

A Refinement of this System

The latitude-longitude system would be sufficient if the world were perfectly spherical and perfectly smooth. It is neither.

To help locate points better, we can add another bit of information to their addresses.

We shall locate points in terms of their distance (usually in feet) above or below sea level — midway between low tide and high tide, called *Mean Sea Level* (MSL). We shall write 100 ft. above sea level as "$^{+}$100 ft." and 100 feet below sea level as "$^{-}$100 feet."

The lowest point in the United States has this address:

(35°N, 117°W, -282 ft.)

It is DEATH VALLEY IN SO. CALIF.

The highest point in the United States has this address:

(73°N, 150°W, +20,320 ft.)

It is MT. McKINLEY IN ALASKA

There is a large city in the United States that is below sea level. It's address is:

(29°N, 90°W, -5 ft.)

It is NEW ORLEANS

The deepest known point on the floor of the sea has this address:

(12°N, 140°E, -36,198 ft.)

It is in the Mariana Trench which is about 1,400 miles east of the PHILIPPINE Islands.

The labeling center of this addressing system would have the address:

(0° N-S, 0° E-W, 0 ft.)

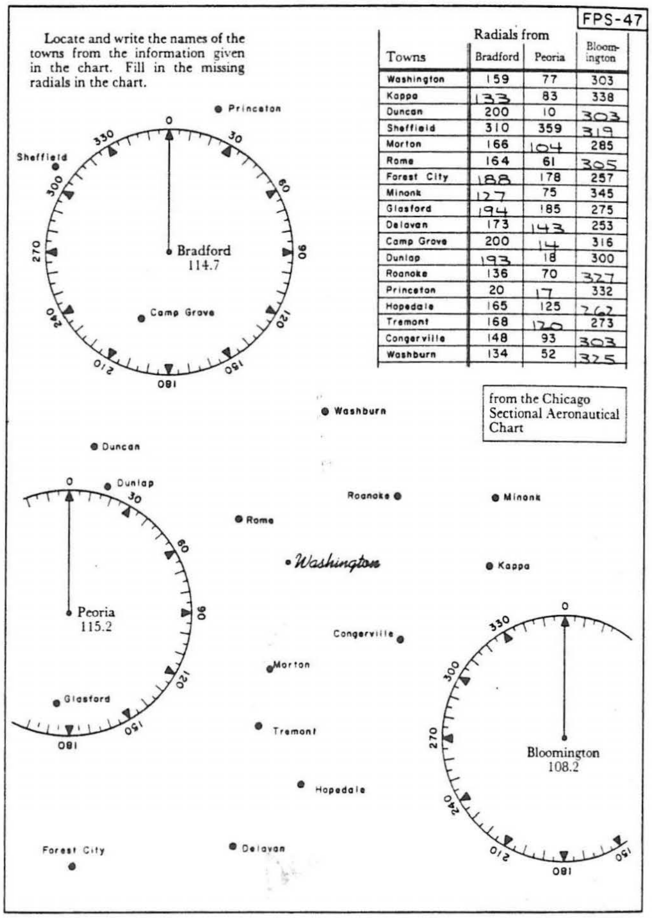
and is located about 380 miles south of Accra, Ghana. $69 \times 5\frac{1}{2} = 379.5$

Tasks: *Students estimate the number of miles in a degree of latitude. They then determine the latitude-longitude address-label for indicated points on the representation of the globe on pupil page 46. They solve a trick problem. Finally, they determine locations on the earth described in terms of latitude-longitude and distance above or below sea level.*

Purpose: To extend student understanding of the idea of a coordinate system for assigning addresses to points on a flat or a spherical surface. To help students consider a third dimension in an addressing system.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

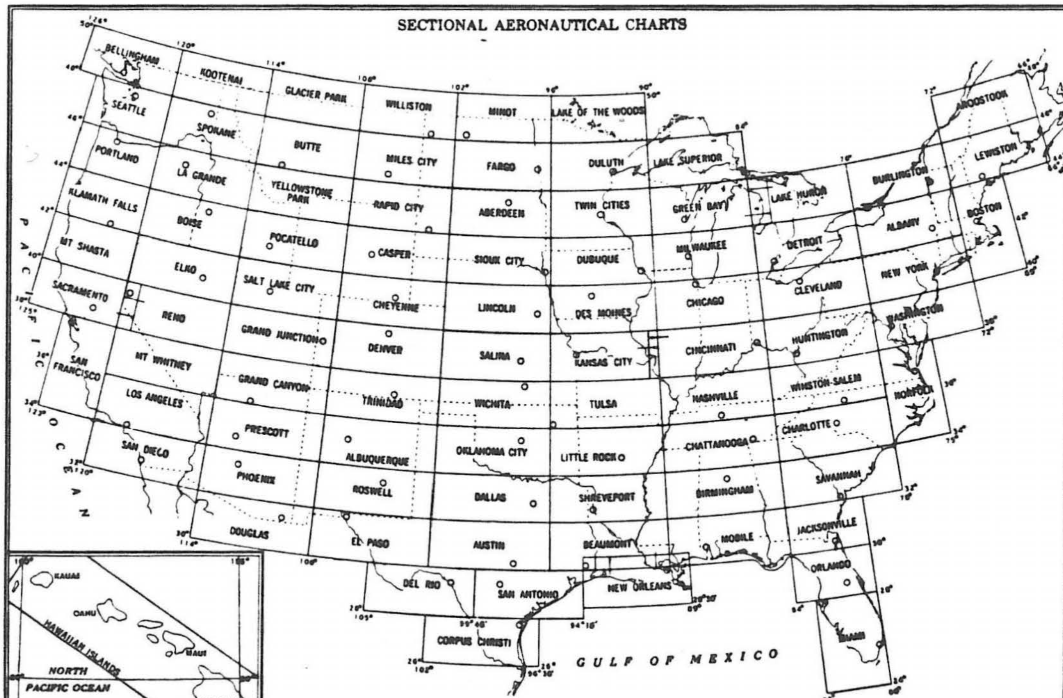
The Lesson: Sectional aeronautical charts will fascinate many students. They are available for 30¢ at airports or from the Director, Coast and Geodetic Survey, U.S. Department of Commerce, Washington Science Center, Rockville, Maryland 20852. Most of them are about two feet by four feet. Much specific information is given on the face of the map and much general information is printed on the back. The map below will indicate the name of the sectional chart for your own region.


FPS-47

Locate and write the names of the towns from the information given in the chart. Fill in the missing radials in the chart.

Towns	Radials from Bradford	Radials from Peoria	Radials from Bloomington
Washington	159	77	303
Kappa	133	83	338
Duncan	200	10	303
Sheffield	310	359	319
Morton	166	104	285
Rome	164	61	305
Forest City	188	178	257
Minonk	127	75	345
Glasford	194	185	275
Delavan	173	143	253
Camp Grove	200	14	316
Dunlap	193	18	300
Roanoke	136	70	327
Princeton	20	17	332
Hopedale	165	125	262
Tremont	168	120	273
Congerville	148	93	303
Washburn	134	52	325

from the Chicago Sectional Aeronautical Chart

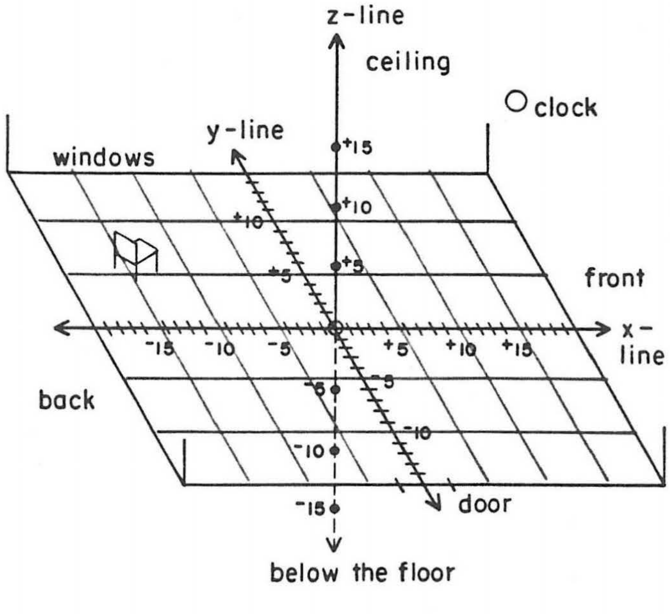


Let's consider a system that will enable us to find an address for every point in a classroom. We shall construct the system so it can be extended to every point in the building. This system will give every point its very own address.

Consider a room that is 30 ft. by 40 ft. It has windows on the left and a door on the right as you face the front. We shall start our job by locating the center of the floor.

Next we draw 3 imaginary lines and mark them off in feet. One is through the center of the floor and parallel to the window wall — our "*x*-line." The second is through the center of the floor and perpendicular to the *x*-line — our "*y*-line." The third is through the center and perpendicular to both the *x*-line and *y*-line. This is our "*z*-line." The *z*-line, in our imagination, goes up through the ceiling and down through the floor.

Here's a sketch:



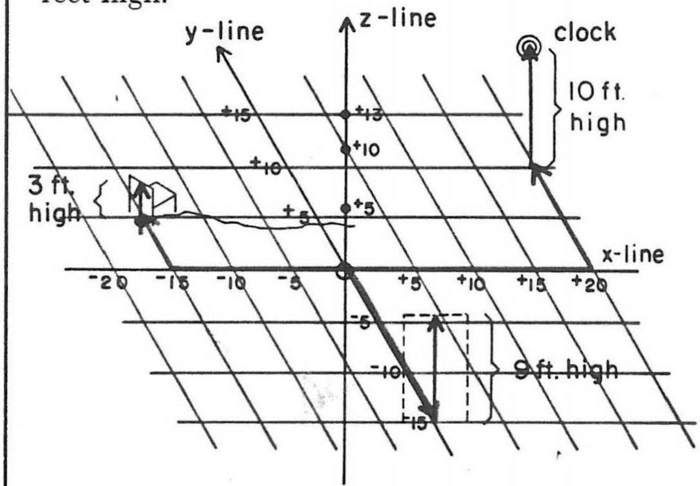
Our addressing system, similar to that on page 44, uses the positive number sign to indicate direction from our center toward the front (*x*-line), toward the windows (*y*-line), and toward the ceiling (*z*-line). The negative number sign indicates direction toward the back (*x*-line), toward the wall with the door (*y*-line), and below the floor (*z*-line).

Our addresses will all have 3 parts. The first tells us how far to move toward the front or toward the back of the room.

The second tells us how far to move and in what direction to move on a line parallel to the *y*-line.

The third tells us how far to move and whether up or down on a line parallel to the *z*-line.

Let's try our system on three points — the center of the clock face, 10 feet high; the midpoint of the top of the door, 9 feet high; and the midpoint of the top of the chair back, 3 feet high.



(distances below are in feet)

	x-line	*y*-line	*z*-line
Center of clock ____	($^+20$,	$^+10$,	$^+10$)
Top of door ______	(0,	$^-15$,	$^+9$)
Top of chair back ___	($^-15$,	$^+5$,	$^+3$)
Center of floor _____	(0,	0,	0)

What can you say about the point whose address is ($^+7$, $^+20$, $^-8$)? IT IS OUT OF DOORS AND BELOW THE FLOOR.

If the ceiling is 13 ft. high, give the address of the point on the ceiling above O on the floor.

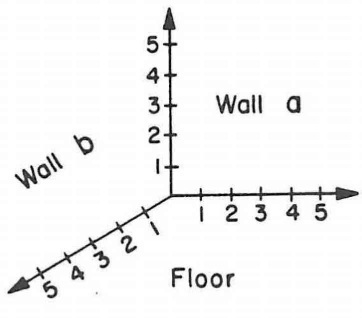
(0, 0, $^+13$)

48-49 Locating points in a room

Tasks: *Students study a system for assigning addresses to points and determine the address-labels for points in a three-dimensional space.*

Purpose: To extend student understanding of a coordinate system of address-labeling in a three-dimensional space.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.



The Lesson: We anticipated a third dimension in address-labeling systems by introducing distance above or below sea level on pupil page 47.

Does the system described here give an address to every point in the room and beyond? Would any point have two addresses? Would any address fit more than one point?

Notice that in all the work thus far we have freely used "+" and "–" signs as a device in helping design a simple system. At no time have we made any attempt to talk about operating with directed numbers.

After the students have become familiar with these things as parts of an address, we shall (in the next chapter) regard them as numbers. We hope that in this way we have provided motivation before talking about positive and negative numbers.

Some students might enjoy acting out the activity described on pupil pages 48 and 49. They must, of course, decide upon the location of the x-line and y-line. The z-line will then be imagined as passing through the intersection of the x- and y-lines.

Must the x- and z-lines pass through the center of the room? No! They could be arbitrarily fixed anywhere. Must they be perpendicular to each other? No. But if they are not, then measurements had better be made along lines parallel to the appropriate axis.

Perhaps someone might prefer planning a system that starts in a corner—so that every point in the room can be address-labeled without the explicit use of direction signs. The x-line would start at a corner and run along the line where wall a meets the floor. The y-line would run out from the corner where wall b meets the floor. The z-line would start from the corner and run up along the line where wall a and wall b meet.

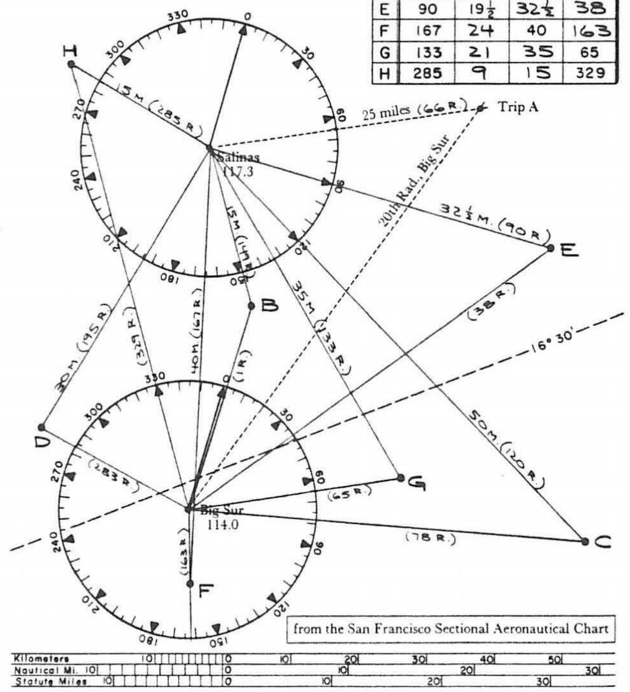
Practice sheet 48 has a broken line labeled 16°30′E. This is included to pique the curiosity of some students. This line tells pilots that the North Magnetic Pole is 16½° east of the True North Pole. So, if his compass points to north, the pilot knows that True North is 16½° to the west, or 360° – 16½°, which is 343½°. This is not a subject suitable for general discussion in all groups.

FPS-48

The Salinas, California Omni transmitter is at the Salinas Airport. Often, a pilot will leave Salinas on a selected radial — perhaps 66. His speed is 100 statute miles per hour. Fifteen minutes later, he has gone 25 miles. Then he checks his course by finding out what radial he is flying on from Big Sur Omni at that time. (We have indicated his position.) Each line in the chart gives information about the beginning of a trip.

Fill in the missing information:

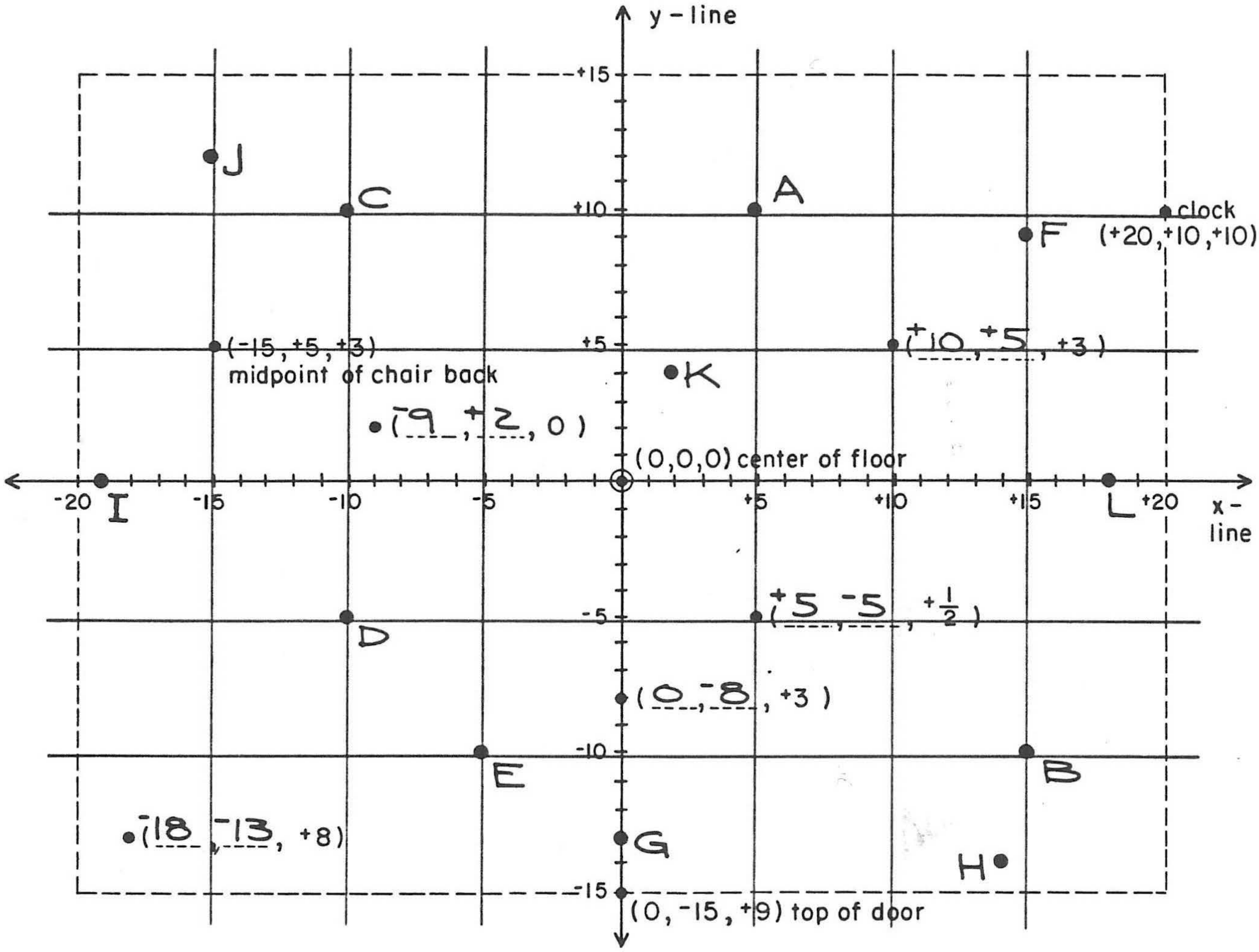
Trip	Radial from Salinas	Min. from Salinas	Statute miles from Salinas	Radial from Big Sur
A	66	15	25	20
B	149	9	15	1
C	120	30	50	78
D	195	18	30	283
E	90	19½	32½	38
F	167	24	40	163
G	133	21	35	65
H	285	9	15	329



from the San Francisco Sectional Aeronautical Chart

It's hard to make a sketch in two dimensions (length and width of a piece of paper) of a room that has three dimensions — length, width, and height.

It's easy, however, to make a floor plan and give addresses to all points on it (page 44). Then the complete address can indicate the height from the floor.



Please complete the address-labels that now indicate only the number of feet above the floor.

Can you locate the following points? When you do, label them with the letters given below.

A (+5, +10, +7)	B (+15, -10, +7½)
C (-10, +10, 0)	D (-10, -5, +3)
E (-5, -10, +11)	F (+15, +9, +9)
G (0, -13, 0)	H (+14, -14, +8)
I (-19, 0, +8)	J (-15, +12, 0)
K (+2, +4, 0)	L (+18, 0, 0)

Assuming that the unit used above is one foot, use any method you like to find the distance to the nearest foot between the pairs of points whose address-labels are given below (be careful with the last three examples):

(+15, 0, 0) and (-8, 0, 0) 23 ft.

(-3, +5, +8) and (-3, +9, +8) 4 ft.

(-7, -6, +5) and (-7, -6, 0) 5 ft.

(+3, 0, +7) and (0, +4, +7) 5 ft.

(-5, 0, +12) and (0, 0, 0) 13 ft.

(-15, -10, +8) and (-9, -10, 0) 10 ft.

(-3, 0, +12) and (0, -4, 0) 13 ft.

Tasks: Students study a three-dimensional system for assigning addresses to points and determine the address-labels for points in a three-dimensional space.

Purpose: To extend student understanding of a coordinate system of address-labeling in a three-dimensional space.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations.

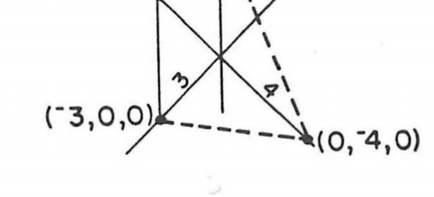
The Lesson: At this point, we abandon the (inevitably awkward) attempt to display a three-dimensional map on a two-dimensional piece of paper. Instead, we present only a map of the floor and rely only on the address-label of a point to indicate its distance above the floor.

On pupil page 49 finding points and address-labeling them is quite simple. However, finding the distances between pairs of points can lead to rather sophisticated problems.

As long as both points are on the floor or the same distance above the floor, the distance between them can be determined from the drawing.

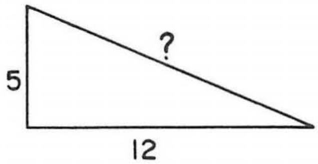
If the two points are on a plane parallel to any two of the axes then the distance can be imagined as on the floor—and determined from the drawing.

However, the last three examples require more imagination. The last one is by far the most difficult example. Here is a three-dimensional view of the problem:



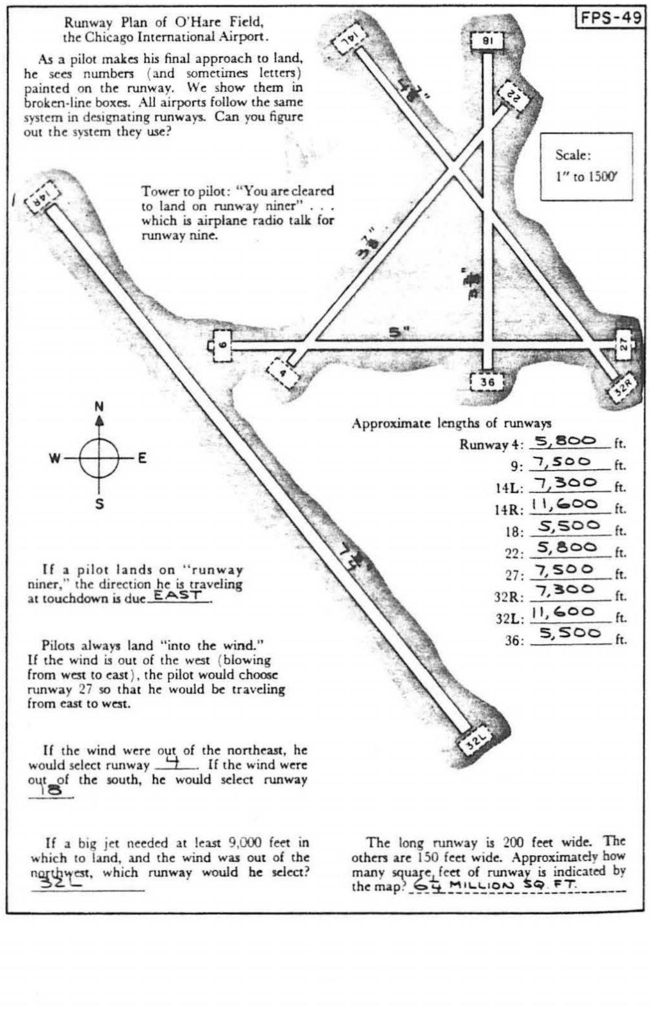
Point ($^-3$,0,0) is on the floor, and point ($^-3$,0,$^+12$) is 12 units above ($^-3$,0,0). Point (0, $^-4$, 0) is also on the floor. We locate these points on our map of the floor and find that they are 5 units apart.

Now we can imagine a right triangle with the distance we want to find as the length of its hypotenuse. We can lay this triangle out on the floor and measure it. The perpendicular sides are 5 units and 12 units:



By measurement, we find that the length of the hypotenuse is 13 units. Or, if we remember the Pythagorean Theorem: $5^2 + 12^2 = 169$, and $\sqrt{169} = 13$.

Practice sheet 49 is a scale drawing of O'Hare Field at Chicago. It presents problems in using scale drawings.


FPS-49

Runway Plan of O'Hare Field, the Chicago International Airport.

As a pilot makes his final approach to land, he sees numbers (and sometimes letters) painted on the runway. We show them in broken-line boxes. All airports follow the same system in designating runways. Can you figure out the system they use?

Tower to pilot: "You are cleared to land on runway niner" . . . which is airplane radio talk for runway nine.

Scale: 1" to 1500′

Approximate lengths of runways

Runway 4: 5,800 ft.
9: 7,500 ft.
14L: 7,300 ft.
14R: 11,600 ft.
18: 5,500 ft.
22: 5,800 ft.
27: 7,500 ft.
32R: 7,300 ft.
32L: 11,600 ft.
36: 5,500 ft.

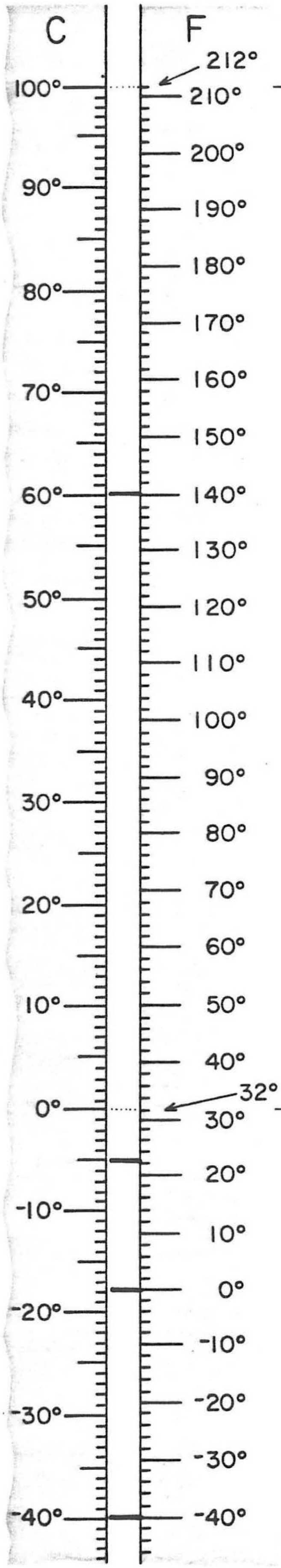
If a pilot lands on "runway niner," the direction he is traveling at touchdown is due EAST.

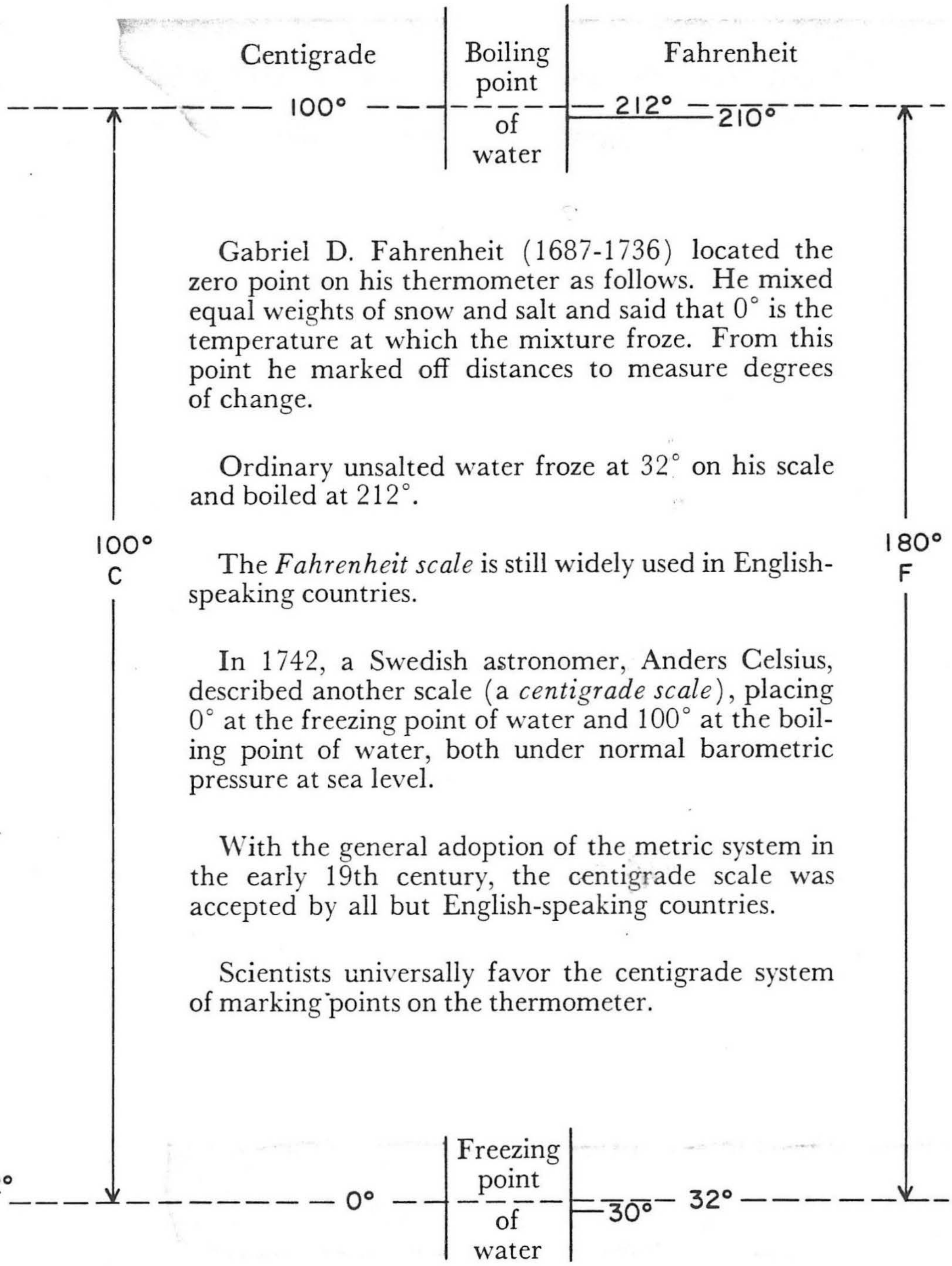
Pilots always land "into the wind." If the wind is out of the west (blowing from west to east), the pilot would choose runway 27 so that he would be traveling from east to west.

If the wind were out of the northeast, he would select runway 4. If the wind were out of the south, he would select runway 18

If a big jet needed at least 9,000 feet in which to land, and the wind was out of the northwest, which runway would he select? 32L

The long runway is 200 feet wide. The others are 150 feet wide. Approximately how many square feet of runway is indicated by the map? 64 MILLION SQ. FT.





Gabriel D. Fahrenheit (1687-1736) located the zero point on his thermometer as follows. He mixed equal weights of snow and salt and said that 0° is the temperature at which the mixture froze. From this point he marked off distances to measure degrees of change.

Ordinary unsalted water froze at 32° on his scale and boiled at 212°.

The *Fahrenheit scale* is still widely used in English-speaking countries.

In 1742, a Swedish astronomer, Anders Celsius, described another scale (a *centigrade scale*), placing 0° at the freezing point of water and 100° at the boiling point of water, both under normal barometric pressure at sea level.

With the general adoption of the metric system in the early 19th century, the centigrade scale was accepted by all but English-speaking countries.

Scientists universally favor the centigrade system of marking points on the thermometer.

You are familiar with the abbreviation for *degree* (°). Our abbreviation for readings *above 0°* is a raised plus sign ($^{+}10°$, $^{+}32°$); for readings *below 0°* we shall use a raised minus sign ($^{-}5°$, $^{-}11°$). Read "$^{+}10°$" as "positive ten degrees" or as "ten degrees above zero"; read "$^{-}5°$" as "negative five degrees" or as "five degrees below zero."

Using the scales on the left, please complete the following sentences.

A reading of $^{+}60°$ C is about the same as a reading of $^{+}140°$ F.

A reading of $^{-}18°$ C is about the same as a reading of 0° F.

A reading of $^{-}5°$ C is about the same as a reading of $^{+}22°$ F.

A reading of $^{-}40°$ C is about the same as a reading of $^{-}40°$ F.

50 Fahrenheit and centigrade thermometer scales

Tasks: Students compare centigrade and Fahrenheit temperature scales.

Purpose: To help students understand an application of directed numbers in a familiar setting.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations; measurement.

The Lesson: Pupil page 50 focuses on the use of directed-number notation for labeling points on a scale. Since these scales run in opposite directions from a zero point, we merely agree that "+" signals that the point we are referring to is "above 0°" and that "–" signals that the point is "below 0°." The label for the point tells the amount and direction of the temperature change from 0° to that point.

The bits of history emphasize the rather arbitrary ways in which such common scales of measurement were invented. Why did Fahrenheit decide to mix equal weights of snow and salt and freeze the mixture to determine a zero point for his scale? Perhaps he wished the zero point to be at a very low temperature so that "below zero" conditions would not often come up naturally in experiments. And why did he divide his scale above this point in such a way that water boils at a reading of 212°?

Celsius' scale is more logical. He chose the two points at which water changes its form (solid to liquid and liquid to gas) as his 0° and 100° points. But these changes depend in part on the barometric pressure, which in turn depends on things such as weather conditions and altitude.

You might wish to discuss converting from Fahrenheit to centigrade, or vice versa. The ratio can be worked out by noticing the difference between the freezing point and the boiling point of water:

Centigrade: 0° to 100°—100° difference
Fahrenheit: 32° to 212°—180° difference

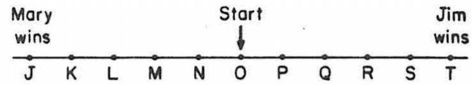
Thus each centigrade degree is $^{180}/_{100}$ or 1.8 of a Fahrenheit degree. Each Fahrenheit degree is $^{100}/_{180}$ or $^{5}/_{9}$ of a centigrade degree. Since the zero points are 32° apart, we can say that, for any temperature of F° (on the Fahrenheit scale) and C° (on the centigrade scale),

$$C = (F - 32) \times \frac{5}{9}$$

and

$$F = C \times 1.8 + 32$$

Practice sheets 50 and 51 introduce a game which uses directed-number notation both for changes in position on a line and for positions on the line. This double use of notation may cause some confusion at the outset, but it reflects common usage. One could play the same game on practice sheet 50 if the labels were letters instead of directed numbers:



In Game I, the score for the first turn is ⁻1. This is a jump from *O* to *N*. So the score is *N*. The next turn-score is ⁻2. The marker jumps from *N* to *L*. The game is over when the total-score is *J* or *T*.

The important advantage in using directed-numbers rather than letters is that a total-score at the end of a turn can be determined by finding the sum of the preceding total-score and the turn-score. This is so because of the nature of addition of directed numbers, because we have assigned 0 to the starting point, and because every other point gets the directed number which is the measure of a jump to it from the starting point.

FPS-50

The Game of Jumps

Here is the playing board:

Mary				
	X	⁺1	⁻2	⁺1
	Y	0	⁺1	⁻1
	Z	⁻1	⁺1	0
		A	B	C
		Jim		

Jim and Mary are playing.

Each turn, Jim picks a letter on his side of the board (*A*, *B*, or *C*). He writes it down. Mary cannot look at Jim's choice.

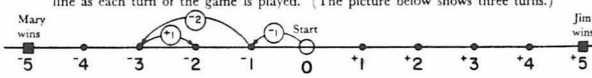
Then Mary picks a letter on her side of the board (*X*, *Y*, or *Z*).

The score for the turn is found where the column picked by Jim and the row picked by Mary cross.

Example: Jim chooses *A* and Mary chooses *Z*. The turn-score is ⁻1, a jump of 1 to the left.

Keeping Score

Put a marker on the starting point, 0. Mary wins if the marker reaches the point ⁻5. Jim wins if the marker reaches the point ⁺5. Jump the marker on the number line as each turn of the game is played. (The picture below shows three turns.)



The score at the end of the example is ⁻1. This is because the turn-score was ⁻1, and a jump of 1 to the left from 0 brings the marker to the point ⁻1.

On the next turn, Jim chooses *B*, and Mary chooses *X*. (*BX*: ⁻2) A turn-score of ⁻2 brings the marker from ⁻1 to ⁻3.

So now the total-score is ⁻3. Who is ahead? MARY

On the next turn, Jim chooses *A*, and Mary chooses *X* again. (*AX*: ⁺1) The turn-score is ⁺1.

Now the total-score is ⁻2.

That helped Jim, but Mary is still ahead. (Fill in the turn-scores and the total-scores for Game I in the table at the left.)

Game I

Jim	Mary	turn-score	total-score
			0
A	Z	⁻1	⁻1
B	X	⁻2	⁻3
A	X	⁺1	⁻2
C	Z	0	⁻2
C	Y	⁻1	⁻3
B	Y	⁺1	⁻2
B	X	⁻2	⁻4
C	Y	⁻1	⁻5

Study the complete records for Games I, II, and III; then fill in the missing total-scores for each of Games IV and V.

II. $0 \xrightarrow{AX} {}^{+}1 \xrightarrow{AZ} 0 \xrightarrow{BZ} {}^{+}1 \xrightarrow{BX} {}^{-}1 \xrightarrow{AZ} {}^{-}2 \xrightarrow{CY} {}^{-}3 \xrightarrow{BX} {}^{-}5$ end

III. $0 \xrightarrow{CX} {}^{+}1 \xrightarrow{BZ} {}^{+}2 \xrightarrow{AX} {}^{+}3 \xrightarrow{AZ} {}^{+}2 \xrightarrow{BY} {}^{+}3 \xrightarrow{BZ} {}^{+}4 \xrightarrow{AX} {}^{+}5$ end

IV. $0 \xrightarrow{CZ} 0 \xrightarrow{BY} {}^{+}1 \xrightarrow{BX} {}^{-}1 \xrightarrow{BX} {}^{-}3 \xrightarrow{AZ} {}^{-}4 \xrightarrow{AX} {}^{-}3 \xrightarrow{BX} {}^{-}5$

V. $0 \xrightarrow{BX} {}^{-}2 \xrightarrow{AX} {}^{-}1 \xrightarrow{CX} 0 \xrightarrow{BY} {}^{+}1 \xrightarrow{BZ} {}^{+}2 \xrightarrow{CZ} {}^{+}2 \xrightarrow{CX} {}^{+}3$ (cont.)

$\xrightarrow{BZ} {}^{+}4 \xrightarrow{BX} {}^{+}2 \xrightarrow{AX} {}^{+}3 \xrightarrow{BZ} {}^{+}4 \xrightarrow{AY} {}^{+}4 \xrightarrow{AZ} {}^{+}3 \xrightarrow{BZ} {}^{+}4 \xrightarrow{BY} {}^{+}5$ end

$^{-}5^\circ$ as a point and $^{-}5^\circ$ as a change

It's five degrees below zero: $^{-}5^\circ$. That is a point or a position on a temperature scale.

The temperature has fallen five degrees: $^{-}5^\circ$. That describes a change — giving the direction ($^{-}$) and the amount (5°).

For the remainder of this page, we shall use a $^{-}5^\circ$ to mean a drop of 5°, and a $^{+}5^\circ$ to mean a rise of 5°.

If the temperature goes up 10° and then up another 7°, it has gone up a total of 17°. We shall abbreviate this statement by writing:

$$^{+}10^\circ + {}^{+}7^\circ = {}^{+}17^\circ$$

Read it as "positive 10° plus positive 7° equals positive 17°."

If the temperature goes up 3° and then falls 8°, we shall indicate that by:

$$^{+}3^\circ + {}^{-}8^\circ = \underline{{}^{-}5^\circ}$$

Read this as "positive 3° plus negative 8° equals negative 5°."

Here are a few examples you can use to check whether you understand this way of showing the total change in temperature that results when a first change is followed by a second change.

1. $^{+}3^\circ + {}^{+}7^\circ = \underline{{}^{+}10}^\circ$
2. $^{-}3^\circ + {}^{+}7^\circ = \underline{{}^{+}4}^\circ$
3. $^{+}3^\circ + {}^{-}7^\circ = \underline{{}^{-}4}^\circ$
4. $^{-}3^\circ + {}^{-}7^\circ = \underline{{}^{-}10}^\circ$
5. $^{+}18^\circ + {}^{+}14^\circ = \underline{{}^{+}32}^\circ$
6. $^{+}18^\circ + {}^{-}14^\circ = \underline{{}^{+}4}^\circ$
7. $^{-}18^\circ + {}^{+}14^\circ = \underline{{}^{-}4}^\circ$
8. $^{-}18^\circ + {}^{-}14^\circ = \underline{{}^{-}32}^\circ$

9. $0^\circ + {}^{+}9^\circ = \underline{{}^{+}9}^\circ$
10. $0^\circ + {}^{-}9^\circ = \underline{{}^{-}9}^\circ$
11. $^{+}17^\circ + 0^\circ = \underline{{}^{+}17}^\circ$
12. $^{-}17^\circ + 0^\circ = \underline{{}^{-}17}^\circ$
13. $^{+}3.5^\circ + {}^{+}.8^\circ = \underline{{}^{+}4.3}^\circ$
14. $^{-}3.5^\circ + {}^{+}.8^\circ = \underline{{}^{-}2.7}^\circ$
15. $^{-}3.5^\circ + {}^{-}.8^\circ = \underline{{}^{-}4.3}^\circ$
16. $^{+}3.5^\circ + {}^{-}.8^\circ = \underline{{}^{+}2.7}^\circ$

We are using an *addition sign* as a short way of saying:

followed by a change of

We shall use a *subtraction sign* as a short way of saying:

followed by the opposite of a change of

The *opposite of a rise of* 5° ($^{+}5^\circ$) is *a fall of* 5° ($^{-}5^\circ$).

$$^{+}7^\circ - {}^{+}5^\circ = {}^{+}2^\circ$$

The *opposite of a fall of* 8° ($^{-}8^\circ$) is *a rise of* 8° ($^{+}8^\circ$).

$$^{+}3^\circ - {}^{-}8^\circ = {}^{+}11^\circ$$

Here are a few examples to check your understanding. (Let's agree that we shall consider the opposite of a change of 0° to be 0°.)

17. $^{+}4^\circ - {}^{+}5^\circ = \underline{{}^{-}1}^\circ$
18. $^{-}4^\circ - {}^{+}5^\circ = \underline{{}^{-}9}^\circ$
19. $^{+}4^\circ - {}^{-}5^\circ = \underline{{}^{+}9}^\circ$
20. $^{-}4^\circ - {}^{-}5^\circ = \underline{{}^{+}1}^\circ$
21. $0^\circ - {}^{+}16^\circ = \underline{{}^{-}16}^\circ$
22. $0^\circ - {}^{-}16^\circ = \underline{{}^{+}16}^\circ$
23. $^{+}16^\circ - 0^\circ = \underline{{}^{+}16}^\circ$
24. $^{-}16^\circ - 0^\circ = \underline{{}^{-}16}^\circ$

Suppose that a temperature went through three successive changes — each being a change of $^{-}5^\circ$. We shall use a *multiplication sign* to show successive changes of the same size and direction. Thus:

$$^{-}5^\circ \times 3 = {}^{-}15^\circ \quad \text{or} \quad 3 \times {}^{-}5^\circ = {}^{-}15^\circ$$

Read this as "negative 5° multiplied by 3 equals negative 15°, or 3 times negative 5° equals negative 15°."

Please do the following examples.

25. $^{+}7^\circ \times 3 = {}^{+}21^\circ$
26. $^{-}7^\circ \times 3 = {}^{-}21^\circ$
27. $3 \times {}^{+}7^\circ = \underline{{}^{+}21}^\circ$
28. $3 \times {}^{-}7^\circ = \underline{{}^{-}21}^\circ$

29. $^{+}9^\circ \times 5 = \underline{{}^{+}45}^\circ$
30. $^{-}9^\circ \times 5 = \underline{{}^{-}45}^\circ$
31. $9 \times {}^{-}5^\circ = \underline{{}^{-}45}^\circ$
32. $9 \times {}^{+}5^\circ = \underline{{}^{+}45}^\circ$
33. $10 \times {}^{-}8^\circ = \underline{{}^{-}80}^\circ$
34. $10 \times {}^{+}8^\circ = \underline{{}^{+}80}^\circ$
35. $^{+}10^\circ \times 8 = \underline{{}^{+}80}^\circ$
36. $^{-}10^\circ \times 8 = \underline{{}^{-}80}^\circ$

Because this arithmetic we use to compute total changes will keep coming up, let's take a further look at it.

51 Doing some computations with changes

Tasks: Students study the use of the positive and negative signs in describing positions on a scale and in describing changes of position. They also complete a number of examples of computations with changes.

Purpose: To help students understand the application of directed numbers in a familiar setting. In particular, the student learns new meanings for the addition sign, the subtraction sign, and the multiplication sign.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: So far we have used such notations as ${}^{+}5$ and ${}^{-}7$ as labels for points on a scale. But there is another use for such notations.

Let's consider their use as a way of describing a change—both the magnitude and the direction of a change.

"Ed drove 5 miles north and then 7 miles south." From that information alone, we can say that Ed could have ended up at the same point simply by driving 2 miles south in the first place. To express this idea we might write: $5N+7S=2S$. Thus we have a notion of "addition" as finding the "net change" when one change is followed by another.

In using ${}^{+}$ and ${}^{-}$ to indicate the direction of a change, it is important to remember that our assignment of an interpretation to either sign is purely arbitrary.

If we decided that ${}^{+}$ will indicate north and ${}^{-}$ will indicate south then we could replace:

$$5N+7S=2S$$

with: $${}^{+}5+{}^{-}7={}^{-}2$$

However, if we switch these assignments of direction then we would replace:

$$5N+7S=2S$$

with: $${}^{-}5+{}^{+}7={}^{+}2$$

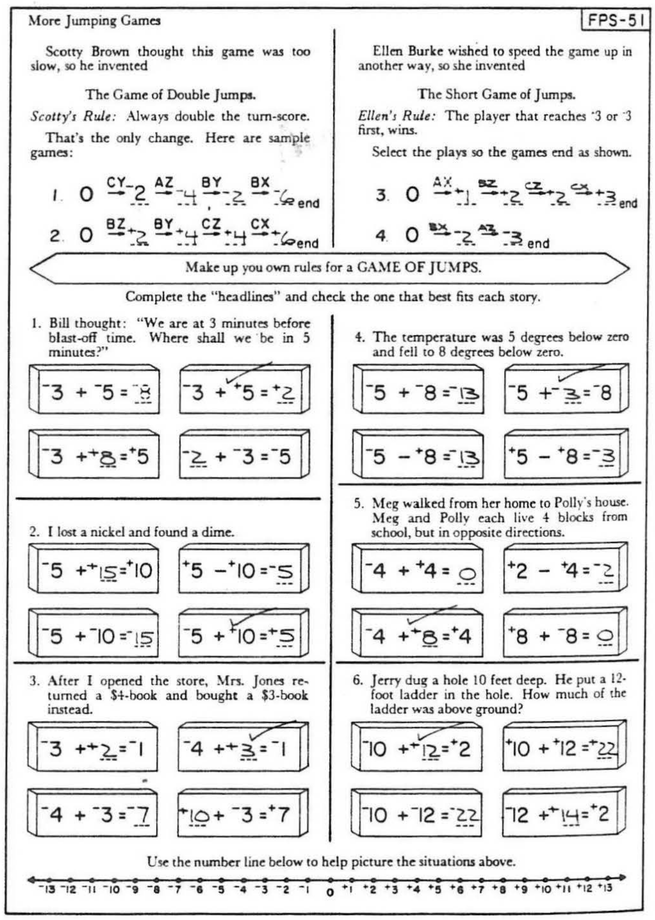
Because we customarily associate directions and qualities such as *up, to the right, profit, north, east,* and *good* with ${}^{+}$ and their opposites with ${}^{-}$, we can easily forget that these assignments are entirely arbitrary from the standpoint of mathematics.

Any statement that uses positive and negative numbers to express the net result of a pair of successive changes must, therefore, be true if the direction signs are changed:

$${}^{+}2+{}^{+}8={}^{+}10 \text{ and } {}^{-}2+{}^{-}8={}^{-}10,$$
$${}^{-}2+{}^{+}8={}^{+}6 \text{ and } {}^{+}2+{}^{-}8={}^{-}6,$$
$${}^{+}2-{}^{+}8={}^{-}6 \text{ and } {}^{-}2-{}^{-}8={}^{+}6,$$
$${}^{+}2-{}^{-}8={}^{+}10 \text{ and } {}^{-}2-{}^{+}8={}^{-}10, \text{ etc.}$$

Multiplication by an undirected number is introduced briefly as a kind of repetitive addition: ${}^{-}5+{}^{-}5+{}^{-}5={}^{-}15$ or ${}^{-}5\times 3={}^{-}15$ (negative 5 multiplied by 3 is negative 15). We shall, of course, return to this idea when we have freed it from its single "temperature" application.

Other applications are suggested on practice sheet 51. Students select the mathematical sentence or "headline" that most adequately reflects the idea in a given statement.



More Jumping Games

FPS-51

Scotty Brown thought this game was too slow, so he invented

The Game of Double Jumps.

Scotty's Rule: Always double the turn-score.

That's the only change. Here are sample games:

1. $0 \xrightarrow{CY} {}^{-}2 \xrightarrow{AZ} {}^{-}4 \xrightarrow{BY} {}^{-}2 \xrightarrow{BX} {}^{-}6$ end
2. $0 \xrightarrow{BZ} {}^{+}2 \xrightarrow{BY} {}^{+}4 \xrightarrow{CZ} {}^{+}4 \xrightarrow{CX} {}^{+}6$ end

Ellen Burke wished to speed the game up in another way, so she invented

The Short Game of Jumps.

Ellen's Rule: The player that reaches ${}^{+}3$ or ${}^{-}3$ first, wins.

Select the plays so the games end as shown.

3. $0 \xrightarrow{AX} {}^{+}1 \xrightarrow{BZ} {}^{+}2 \xrightarrow{CZ} {}^{+}2 \xrightarrow{CX} {}^{+}3$ end
4. $0 \xrightarrow{BX} {}^{-}2 \xrightarrow{AZ} {}^{-}3$ end

Make up you own rules for a GAME OF JUMPS.

Complete the "headlines" and check the one that best fits each story.

1. Bill thought: "We are at 3 minutes before blast-off time. Where shall we be in 5 minutes?"

$${}^{-}3+{}^{-}5={}^{-}8 \qquad {}^{-}3+{}^{+}5={}^{+}2$$
$${}^{-}3+{}^{+}8={}^{+}5 \qquad {}^{-}2+{}^{-}3={}^{-}5$$

2. I lost a nickel and found a dime.

$${}^{-}5+{}^{+}15={}^{+}10 \qquad {}^{+}5-{}^{+}10={}^{-}5$$
$${}^{-}5+{}^{-}10={}^{-}15 \qquad {}^{-}5+{}^{+}10={}^{+}5$$

3. After I opened the store, Mrs. Jones returned a \$4-book and bought a \$3-book instead.

$${}^{-}3+{}^{+}2={}^{-}1 \qquad {}^{-}4+{}^{+}3={}^{-}1$$
$${}^{-}4+{}^{-}3={}^{-}7 \qquad {}^{+}10+{}^{-}3={}^{+}7$$

4. The temperature was 5 degrees below zero and fell to 8 degrees below zero.

$${}^{-}5+{}^{-}8={}^{-}13 \qquad {}^{-}5+{}^{-}3={}^{-}8$$
$${}^{-}5-{}^{+}8={}^{-}13 \qquad {}^{+}5-{}^{+}8={}^{-}3$$

5. Meg walked from her home to Polly's house. Meg and Polly each live 4 blocks from school, but in opposite directions.

$${}^{-}4+{}^{+}4=0 \qquad {}^{+}2-{}^{+}4={}^{-}2$$
$${}^{-}4+{}^{+}8={}^{+}4 \qquad {}^{+}8+{}^{-}8=0$$

6. Jerry dug a hole 10 feet deep. He put a 12-foot ladder in the hole. How much of the ladder was above ground?

$${}^{-}10+{}^{+}12={}^{+}2 \qquad {}^{+}10+{}^{+}12={}^{+}22$$
$${}^{-}10+{}^{-}12={}^{-}22 \qquad {}^{-}12+{}^{+}14={}^{+}2$$

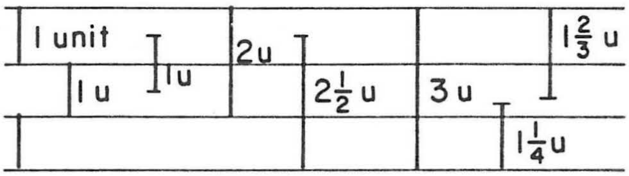
Use the number line below to help picture the situations above.

A General Method

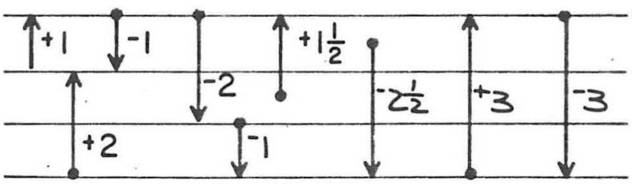
We often need to describe changes and whole sequences of changes — changes in temperature, changes in position, changes in a bank account, changes in weight, changes in scores, changes in height, changes in time, etc.

Let's develop a system we can adapt to meet any needs of this kind.

First, we shall select a convenient way to indicate a unit of change. Let the unit be the distance between two successive green lines.

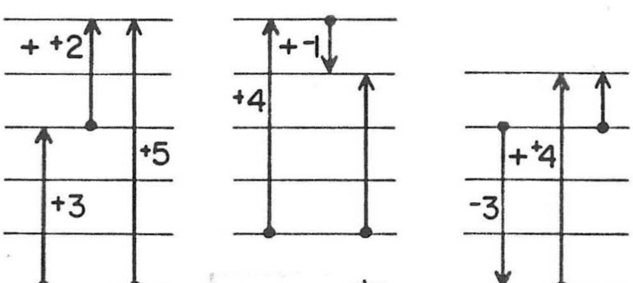


Next, we shall say that a unit of change in the up direction is a *positive* unit of change, and in the opposite direction, a *negative* unit of change. *(We left four changes for you to label. Please do so.)*



ADDITION

We shall think of one change followed by another as *adding* the second change, and the sum of the changes as the *net change*. Here's a sketch of the idea:

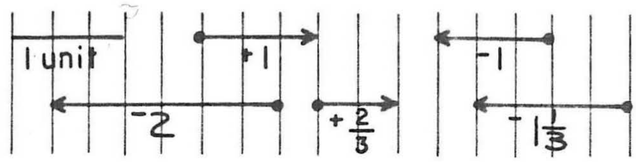


$^{+}4 + {}^{-}1 = {}^{+}3$

$^{+}3 + {}^{+}2 = {}^{+}5$ $\qquad$ $-3 + {}^{+}4 = {}^{+}1$

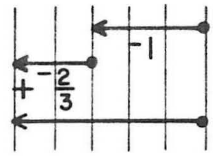
$$\underbrace{\text{positive 3}}_{\text{first change}}\ \underbrace{\text{plus positive 2}}_{\text{followed by second change}} = \underbrace{\text{positive 5}}_{\text{net change}}$$

We can, of course, change our unit and sketch. Let us say that our unit is the horizontal distance as shown, and a positive change is shown as moving from left to right. *(Please label the arrows.)*

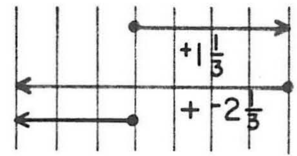


A change followed by another.

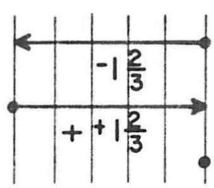
(Please complete the sentences and label the arrows.)



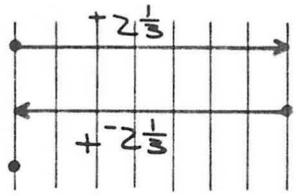
$^{-}1 + {}^{-}\frac{2}{3} = {}^{-}1\frac{2}{3}$



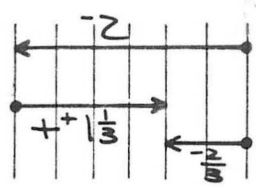
$^{+}1\frac{1}{3} + {}^{-}2\frac{1}{3} = {}^{-}1$



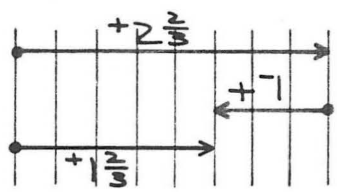
$^{-}1\frac{2}{3} + {}^{+}1\frac{2}{3} = 0$



$^{+}2\frac{1}{3} + {}^{-}2\frac{1}{3} = 0$

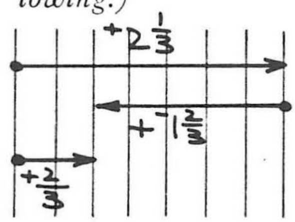


$^{-}2 + {}^{+}1\frac{1}{3} = {}^{-}\frac{2}{3}$

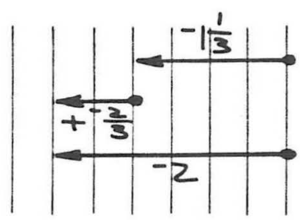


$^{+}2\frac{2}{3} + {}^{-}1 = {}^{+}1\frac{2}{3}$

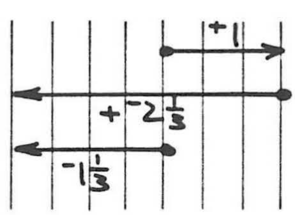
(Please draw labeled sketches for the following.)



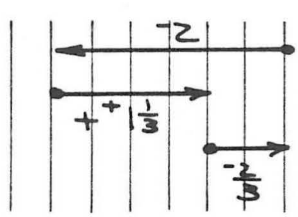
$^{+}2\frac{1}{3} + {}^{-}1\frac{2}{3} = {}^{+}\frac{2}{3}$



$^{-}1\frac{1}{3} + {}^{-}\frac{2}{3} = {}^{-}2$



$^{+}1 + {}^{-}2\frac{1}{3} = {}^{-}1\frac{1}{3}$



$^{-}2 + {}^{+}1\frac{1}{3} = {}^{-}\frac{2}{3}$

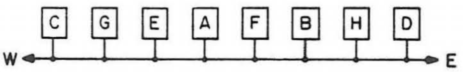
52 Adding changes

Tasks: *Students complete the labeling of representations of changes, and complete corresponding computation sentences.*

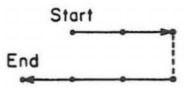
Purpose: To help students understand a general method for adding directed numbers.

Unifying Ideas: Structure; addition and subtraction; functions and relations.

The Lesson: Consider a road with mileposts:



Suppose we know that Mr. Brown drove 2 miles east and 3 miles west. We could represent that combination of two trips in some such fashion as this:



The same end point would have been reached if, in the beginning, he had driven 1 mile to the west. If we use $^+$ to indicate *to the east* and $^-$ to indicate *to the west* then we can express the physical fact by writing: $^+2+{}^-3={}^-1$. In other words, we have made a statement about his "combined" trips that we are confident is true despite the fact that we have no idea where Mr. Brown started his trips nor where they ended. But we can say that if he started at *B*, he ended at *F*; if he started at *G*, he ended at *C*; etc.

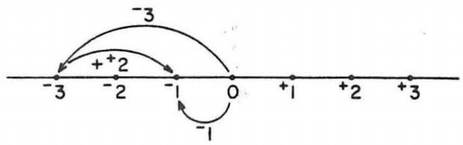
Suppose Mr. Jones started at *B* and ended at *A*. Even if we know he made the trip in two parts, we cannot tell what they were. The net change in his position is that accomplished by a trip of 2 miles to the west, or $^-2$ miles, but he could have taken many different combinations of trips: $^-2={}^+4+{}^-6$, $^-2={}^-1+{}^-1$, $^-2={}^-100+{}^+98$, etc. In any case, the net change is $^-2$ since he ended up 2 miles west of his starting point.

On pupil page 52 we use arrows to represent trips. Units of magnitude are indicated; $^+$ signals a change of position to the right, and $^-$ signals a change to the left. Addition is defined as finding the net change of a pair of successive changes in position—one change followed by another. A label such as "$+{}^+2$" signifies that the first change is to be followed by a second change of $^+2$.

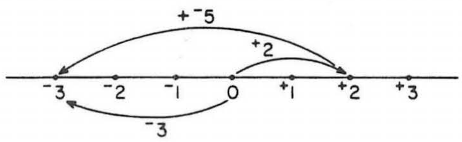
Practice sheet 52 explores what would happen to *Cross Number Puzzles* in which the numbers are directed numbers and the operation is addition. Will the check box always be both the sum of the sums for the two rows and the sum of the sums for the two columns? Of course it will! No matter what order or grouping is used on a succession of given changes, the net change will be the same. Addition of directed numbers is commutative and associative.

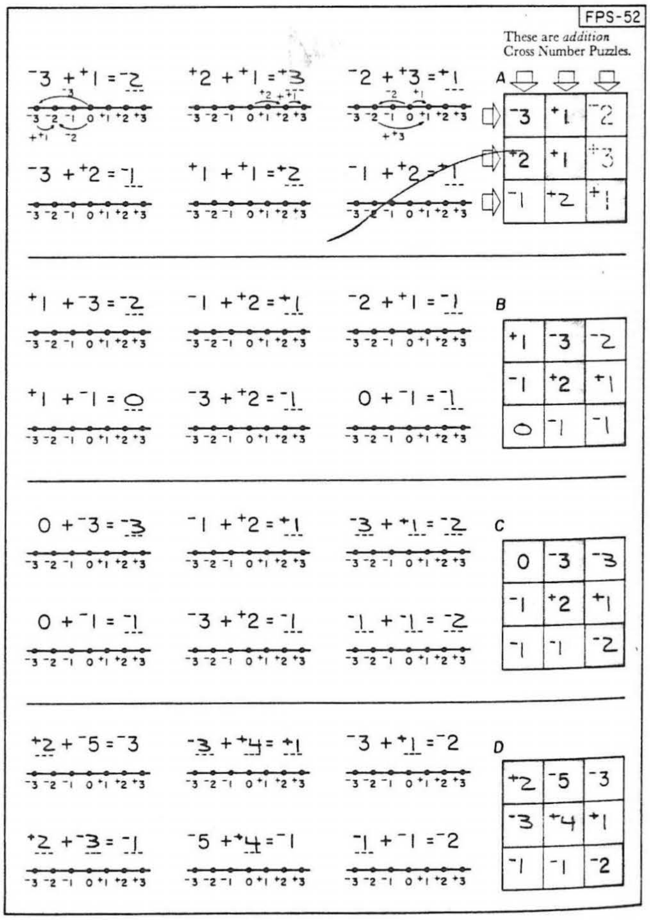
Most of the examples of addition in the *Cross Number Puzzles* can be shown on the number line chunks on the practice sheet.

$^-3+{}^+2={}^-1$



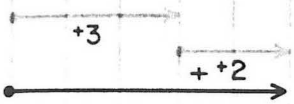
$^+2+{}^-5=$ ----

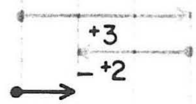




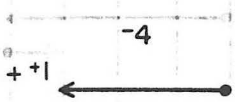
SUBTRACTION

We shall think of one change followed by the *opposite* of a second change as *subtracting* the second change. So, to subtract a second change, just add its opposite. Here are sketches to suggest the idea. *(Please complete the sentences.)*

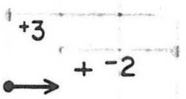


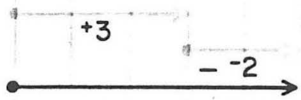


$^{+}3 + {}^{+}2 = {}^{+}5$ $\qquad$ $^{+}3 - {}^{+}2 = {}^{+}1$

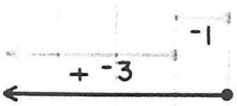


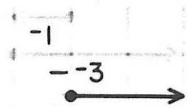
$^{-}4 + {}^{+}1 = {}^{-}3$ $\qquad$ $^{-}4 - {}^{+}1 = {}^{-}5$





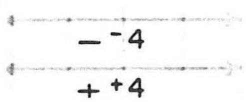
$^{+}3 + {}^{-}2 = {}^{+}1$ $\qquad$ $^{+}3 - {}^{-}2 = {}^{+}5$

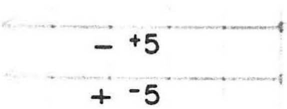




$^{-}1 + {}^{-}3 = {}^{-}4$ $\qquad$ $^{-}1 - {}^{-}3 = {}^{+}2$

We might summarize this way:





In general, for all numbers *a* and *n*,

$$^{+}a - {}^{-}n = {}^{+}a + {}^{+}n$$
$$^{-}a - {}^{-}n = {}^{-}a + {}^{+}n$$
$$^{+}a - {}^{+}n = {}^{+}a + {}^{-}n$$
$$^{-}a - {}^{+}n = {}^{-}a + {}^{-}n.$$

Any problem in subtracting positive and negative numbers can be solved by doing a related problem in addition.

Yesterday, between 6 A.M. and noon, the temperature rose 3 degrees. Between noon and 6 P.M., it fell 9 degrees.

$$^{+}3 + {}^{-}9 = {}^{-}6$$

It was 6 degrees COLDER (warmer or colder) at 6 P.M. yesterday than at 6 A.M.

Today, the rise in the morning was the same, but in the afternoon its change was just the opposite of the day before.

$$^{+}3 - {}^{-}9 = {}^{+}12$$

It was 12 degrees WARMER (warmer or colder) at 6 P.M. today than at 6 A.M.

I had \$15 more in savings than in bills. The postman brought me a bill for \$7.

$$^{+}15 + {}^{-}7 = {}^{+}8$$

So, I now have \$8.00 more in savings than in bills.

On a different date, I again had \$15 more in savings than in bills. When the postman came, he explained that he should have delivered a bill of \$7 to someone else, so he took the bill away.

$$^{+}15 - {}^{-}7 = {}^{+}22$$

After he took away the bill, I had \$22.00 more in savings than in bills.

At still another time, I found I had \$10 more in bills than I had in savings.

When the postman came, he told me of another mistake. He had by mistake brought me a \$2-check the day before, so I gave him back the \$2-check.

$$^{-}10 - {}^{+}2 = {}^{-}12$$

After he took the check back, I had \$12.00 more in bills than in savings.

Tasks: Students complete computation sentences leading to a first change followed by a second change (addition) and to a first change followed by the opposite of a second change (subtraction).

Purpose: To help students understand a general system for subtracting changes.

Unifying Ideas: Structure; addition and subtraction; functions and relations.

The Lesson: We define *subtracting a second change* from a first as *adding the opposite of the second change* to the first.

Note that, in this definition, subtraction of directed numbers bears the same relation to addition of directed numbers as subtraction of undirected numbers does to addition of undirected numbers. Subtracting 5 undoes the effect of adding 5:

$$(9 + 5) - 5 = 9$$

Similarly, subtracting ⁺5 undoes the adding of ⁺5:

$$({}^{+}9 + {}^{+}5) - {}^{+}5 = ({}^{+}9 + {}^{+}5) + {}^{-}5 = {}^{+}9$$

And subtracting ⁻5 undoes the effect of adding ⁻5:

$$({}^{+}9 + {}^{-}5) - {}^{-}5 = ({}^{+}9 + {}^{-}5) + {}^{+}5 = {}^{+}9$$

In short, for both undirected numbers and directed numbers, subtracting is the *inverse* of adding.

Make a list of changes and their opposites:

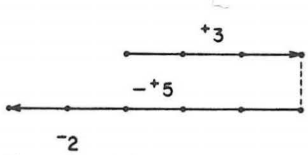
Changes	Their Opposites
⁺3	⁻3
⁻3	⁺3
⁺11	⁻11

With this understanding and the definition of subtraction, we can write the following:

$$ {}^{+}3 - {}^{+}5 = {}^{+}3 + {}^{-}5 = {}^{-}2$$
$$ {}^{-}7 - {}^{+}6 = {}^{-}7 + {}^{-}6 = {}^{-}13$$
$$ {}^{+}6 - {}^{+}9 = {}^{+}6 + {}^{-}9 = {}^{-}3$$
$$ {}^{-}10 - {}^{-}3 = {}^{-}10 + {}^{+}3 = {}^{-}7$$

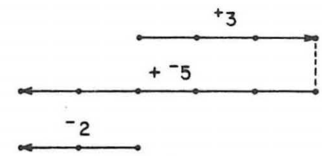
We sketch this notion on pupil page 53 by indicating the second change with a subtraction sign:

$${}^{+}3 - {}^{+}5 = {}^{-}2$$



The label "− ⁺5" signifies that the first change is to be followed by the opposite of a change of ⁺5. That is, the first change is to be followed by a change of ⁻5. For contrast, we illustrate an addition problem by indicating the second change with an addition sign:

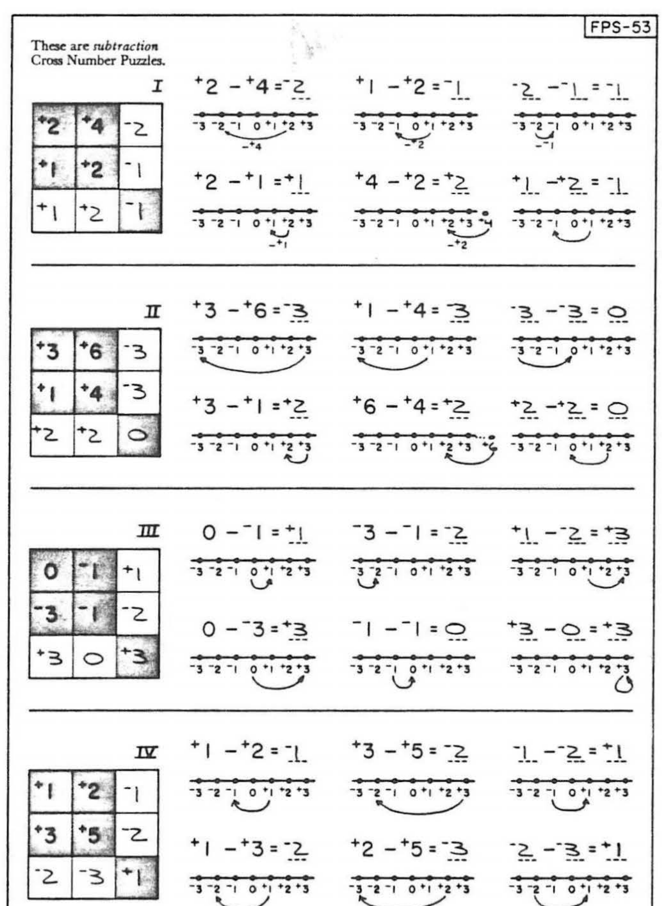
$${}^{+}3 + {}^{-}5 = {}^{-}2$$



The label "+ ⁻5" signifies that the first change is to be followed by a change of ⁻5.

In the right-hand column, two situations exemplify where addition and subtraction of positive and negative numbers might arise. Drawing sketches and inventing such situations will help students understand what they do—and why.

Number line chunks are provided on practice sheet 53, but students should be allowed to write down answers they are sure of without having to draw sketches.



MULTIPLICATION

You know how to make sense out of the following statements in addition and subtraction of positive and negative numbers.

$$^{-}5 + {}^{+}3 = {}^{-}2$$

$$^{+}2 - {}^{-}4 = {}^{+}6$$

In the addition example, just think of the net change produced by a change of $^{-}5$ followed by a change of $^{+}3$. The net change is $^{-}2$.

In the subtraction example, think of the net change produced by a change of $^{+}2$ followed by the opposite of a change of $^{-}4$. The net change is $^{+}6$.

Now, how shall we make sense out of examples of multiplication of positive and negative numbers?

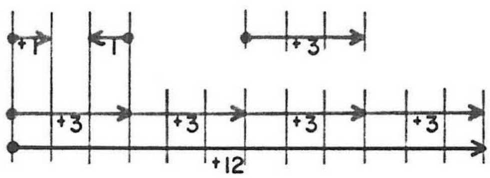
$$^{+}3 \times {}^{+}2 = ?$$

$$^{-}4 \times {}^{-}5 = ?$$

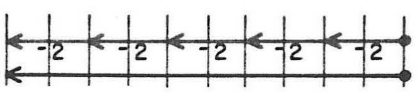
Let's build from what we already know. We can easily make sense out of one kind of multiplication example.

$^{+}3 \times 4 = ?$ $\qquad$ $5 \times {}^{-}2 = ?$

A few sketches will suggest the idea:



$^{+}3 \times 4 = {}^{+}12$ or $4 \times {}^{+}3 = {}^{+}12$



$^{-}2 \times 5 = {}^{-}10$ or $5 \times {}^{-}2 = {}^{-}10$

You may draw sketches if you like, to help you complete the following:

$^{+}3 \times 8 = {}^{+}24$ or $8 \times {}^{+}3 = {}^{+}24$

$^{-}7 \times 5 = {}^{-}35$ or $5 \times {}^{-}7 = {}^{-}35$

$^{+}12 \times \frac{1}{2} = {}^{+}6$ or $\frac{1}{2} \times {}^{+}12 = {}^{+}6$

$^{-}9 \times \frac{2}{3} = {}^{-}6$ or $\frac{2}{3} \times {}^{-}9 = {}^{-}6$

$^{-}81 \times \frac{1}{3} = {}^{-}27$ or $\frac{1}{3} \times {}^{-}81 = {}^{-}27$

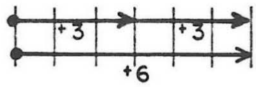
$^{+}36 \times \frac{1}{4} = {}^{+}9$ or $\frac{1}{4} \times {}^{+}36 = {}^{+}9$

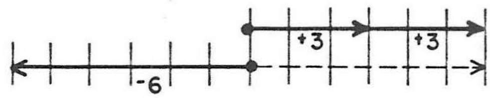
Let's introduce a new idea. Not only do we wish to multiply a change by a number, but we also wish to signal a change of direction of the result. We might use a small *CD* as a signal to change direction, and a small *NC* as a signal for no change in direction.

Examples.

$^{+}3 \times {}^{NC}2 = {}^{+}6$

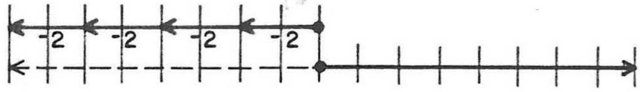
no change





$^{+}3 \times {}^{CD}2 = {}^{-}6$

change direction



$^{-}2 \times {}^{CD}4 = {}^{+}8$

"Negative two multiplied by four . . . and change direction of the result."

Please complete the following sentences. (Draw sketches if you need them to help you.)

$^{-}4 \times {}^{CD}3 = {}^{+}12$ or ${}^{CD}3 \times {}^{-}4 = {}^{+}12$

$^{-}3 \times {}^{CD}4 = {}^{+}12$ or ${}^{CD}4 \times {}^{-}3 = {}^{+}12$

$^{+}9 \times {}^{NC}8 = {}^{+}72$ or ${}^{NC}8 \times {}^{+}9 = {}^{+}72$

$^{+}8 \times {}^{NC}9 = {}^{+}72$ or ${}^{NC}9 \times {}^{+}8 = {}^{+}72$

$^{+}7 \times {}^{CD}4 = {}^{-}28$ or ${}^{CD}4 \times {}^{+}7 = {}^{-}28$

$^{+}4 \times {}^{CD}7 = {}^{-}28$ or ${}^{CD}7 \times {}^{+}4 = {}^{-}28$

54-55 Multiplication of positive and negative numbers

Tasks: Students do computations involving multiplication.

Purpose: To introduce an interpretation which leads to multiplication of directed numbers.

Unifying Ideas: Numeration; multiplication and division; functions and relations.

The Lesson: We are back again to a familiar problem that grows out of the fundamental difference between addition and multiplication as those ideas are applied to real world situations. It is easy to find illustrations for adding quantities, but difficult to find interpretations for multiplying quantities.

We can add changes in temperature (directed quantities), but it makes no sense to multiply them:

$$^{+}3^\circ + {}^{+}7^\circ = 10^\circ$$

$$^{+}3^\circ \times {}^{+}7^\circ = \underline{\ ?\ }$$

We have no interpretation for a rise of 3° "multiplied by" a rise of 7°. But we can easily interpret:

$$3 \times {}^{+}7^\circ \quad \text{and} \quad {}^{+}3^\circ \times 7$$

as 3 times a rise of 7° and as a rise of 3° multiplied by 7. Each expression stands for a rise of 21°. So, when multiplication and division are involved, we can lay the most intuitively sound basis of understanding by talking about multiplying a change by a number—(that is, an undirected number)—just as we multiply any quantity by a number.

On pupil page 54 we introduce temporarily the abbreviations *CD* for "change of direction" and *NC* for "no change of direction." This helps us make the transition from multiplication by undirected numbers to multiplication by directed numbers. We shall replace our *ad hoc* shorthand on the next page.

Practice sheet 55 will be easy for most students. It will provide worthwhile practice for those students who are still uncertain about addition and subtraction of positive and negative numbers. Note that exercises *K-O* require students to reverse the usual computational order. Instead of going from an indicated sum or difference to a result, they start with a result and find equivalent indicated sums and differences.

FPS-54

$^{-}6 \quad ^{-}5 \quad ^{-}4 \quad ^{-}3 \quad ^{-}2 \quad ^{-}1 \quad 0 \quad ^{+}1 \quad ^{+}2 \quad ^{+}3 \quad ^{+}4 \quad ^{+}5 \quad ^{+}6$

A	B	C	D	E
$^{+}3 + {}^{+}1 = {}^{+}4$	$^{+}3 + {}^{+}2 = {}^{+}5$	$^{+}4 + {}^{+}2 = {}^{+}6$	$^{+}5 - {}^{+}3 = {}^{+}2$	$^{+}4 - {}^{+}2 = {}^{+}2$
$^{+}2 + {}^{+}1 = {}^{+}3$	$^{+}3 + {}^{+}1 = {}^{+}4$	$^{+}3 + {}^{+}1 = {}^{+}4$	$^{+}4 - {}^{+}3 = {}^{+}1$	$^{+}3 - {}^{+}2 = {}^{+}1$
$^{+}1 + {}^{+}1 = {}^{+}2$	$^{+}3 + 0 = {}^{+}3$	$^{+}2 + 0 = {}^{+}2$	$^{+}3 - {}^{+}3 = 0$	$^{+}2 - {}^{+}2 = 0$
$0 + {}^{+}1 = {}^{+}1$	$^{+}3 + {}^{-}1 = {}^{+}2$	$^{+}1 + {}^{-}1 = 0$	$^{+}2 - {}^{+}3 = {}^{-}1$	$^{+}1 - {}^{+}2 = {}^{-}1$
$^{-}1 + {}^{+}1 = 0$	$^{+}3 + {}^{-}2 = {}^{+}1$	$0 + {}^{-}2 = {}^{-}2$	$^{+}1 - {}^{+}3 = {}^{-}2$	$0 - {}^{+}2 = {}^{-}2$
$^{-}2 + {}^{+}1 = {}^{-}1$	$^{+}3 + {}^{-}3 = 0$	$^{-}1 + {}^{-}3 = {}^{-}4$	$0 - {}^{+}3 = {}^{-}3$	$^{-}1 - {}^{+}2 = {}^{-}3$
$^{-}3 + {}^{+}1 = {}^{-}2$	$^{+}3 + {}^{-}4 = {}^{-}1$	$^{-}2 + {}^{-}4 = {}^{-}6$	$^{-}1 - {}^{+}3 = {}^{-}4$	$^{-}2 - {}^{+}2 = {}^{-}4$
$^{-}4 + {}^{+}1 = {}^{-}3$	$^{+}3 + {}^{-}5 = {}^{-}2$	$^{-}3 + {}^{-}5 = {}^{-}8$	$^{-}2 - {}^{+}3 = {}^{-}5$	$^{-}3 - {}^{+}2 = {}^{-}5$

F	G	H	I	J
$^{+}2 - {}^{+}4 = {}^{-}2$	$^{-}3 - {}^{+}2 = {}^{-}5$	$^{-}3 + {}^{+}3 = 0$	$^{+}3 + {}^{+}2 = {}^{+}5$	$^{+}2 + {}^{+}3 = {}^{+}5$
$^{+}2 - {}^{+}3 = {}^{-}1$	$^{-}3 - {}^{+}1 = {}^{-}4$	$^{-}2 + {}^{+}2 = 0$	$^{-}3 + {}^{+}2 = {}^{-}1$	$^{+}2 + {}^{-}3 = {}^{-}1$
$^{+}2 - {}^{+}2 = 0$	$^{-}3 - 0 = {}^{-}3$	$^{-}1 + {}^{+}1 = 0$	$^{+}3 + {}^{-}2 = {}^{+}1$	$^{-}2 + {}^{+}3 = {}^{+}1$
$^{+}2 - {}^{+}1 = {}^{+}1$	$^{-}3 - {}^{-}1 = {}^{-}2$	$0 + 0 = 0$	$^{-}3 + {}^{-}2 = {}^{-}5$	$^{-}2 + {}^{-}3 = {}^{-}5$
$^{+}2 - 0 = {}^{+}2$	$^{-}3 - {}^{-}2 = {}^{-}1$	$^{+}1 + {}^{-}1 = 0$	$^{+}3 - {}^{+}2 = {}^{+}1$	$^{+}2 - {}^{+}3 = {}^{-}1$
$^{+}2 - {}^{-}1 = {}^{+}3$	$^{-}3 - {}^{-}3 = 0$	$^{+}2 + {}^{-}2 = 0$	$^{-}3 - {}^{+}2 = {}^{-}5$	$^{-}2 - {}^{+}3 = {}^{-}5$
$^{+}2 - {}^{-}2 = {}^{+}4$	$^{-}3 - {}^{-}4 = {}^{+}1$	$^{+}3 + {}^{-}3 = 0$	$^{+}3 - {}^{-}3 = {}^{+}6$	$^{+}2 - {}^{-}3 = {}^{+}5$
$^{+}2 - {}^{-}3 = {}^{+}5$	$^{-}3 - {}^{-}5 = {}^{+}2$	$^{+}4 + {}^{-}4 = 0$	$^{-}3 - {}^{-}2 = {}^{-}1$	$^{-}2 - {}^{-}3 = {}^{+}1$
$^{+}2 - {}^{-}4 = {}^{+}6$	$^{-}3 - {}^{-}6 = {}^{+}3$	$^{+}5 + {}^{-}5 = 0$	$^{+}2 + {}^{-}3 = {}^{-}1$	$^{+}3 + {}^{+}2 = {}^{+}5$

K. Expressions for $^{-}2$	L. Expressions for $^{-}1$	M. Expressions for 0	N. Expressions for $^{+}1$	O. Expressions for $^{+}2$
$^{-}3 + {}^{+}1$	$^{+}2 - {}^{+}3$	$^{+}3 - {}^{+}3$	$0 + {}^{+}1$	$^{+}1 + {}^{+}1$
$^{+}3 + {}^{-}5$	$^{-}2 + {}^{+}1$	$^{+}2 - {}^{+}2$	$^{+}3 + {}^{-}2$	$^{+}3 + {}^{-}1$
$0 + {}^{-}2$	$^{+}3 + {}^{-}4$	$^{-}3 - {}^{-}3$	$^{+}4 - {}^{+}3$	$^{+}2 + 0$
$^{+}1 - {}^{+}3$	$^{+}1 - {}^{+}2$		$^{+}3 - {}^{+}2$	$^{+}5 - {}^{+}3$
$0 - {}^{+}2$	$^{-}3 - {}^{-}2$		$^{+}2 - {}^{+}1$	$^{+}4 - {}^{+}2$
$^{+}2 - {}^{+}4$	$^{-}3 + {}^{+}2$		$^{-}3 - {}^{-}4$	$^{+}2 - 0$
$^{-}3 - {}^{-}1$	$^{+}2 + {}^{-}3$		$^{-}2 + {}^{+}3$	$^{-}3 - {}^{-}5$
			$^{-}2 - {}^{-}3$	

Simplifying the Shorthand

Do we need all of the following signs?

$$+ \qquad - \qquad NC \qquad CD$$

Mathematicians use the $^{+}$ instead of the ^{NC} and the $^{-}$ instead of the ^{CD}. Look through the twelve examples at the end of the previous page. Mathematicians would write them like this:

$${}^{-}4 \times {}^{-}3 = {}^{-}3 \times {}^{-}4 = {}^{-}3 \times {}^{-}4 = {}^{-}4 \times {}^{-}3 = {}^{+}12$$

$${}^{+}9 \times {}^{+}8 = {}^{+}8 \times {}^{+}9 = {}^{+}8 \times {}^{+}9 = {}^{+}9 \times {}^{+}8 = {}^{+}72$$

$${}^{+}7 \times {}^{-}4 = {}^{-}4 \times {}^{+}7 = {}^{+}4 \times {}^{-}7 = {}^{-}7 \times {}^{+}4 = {}^{-}28$$

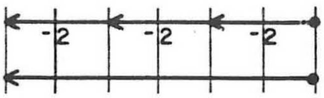
We might have lost a little information because we can no longer tell which is the multiplier.

$${}^{-}2 \times {}^{NC}3 = {}^{NC}3 \times {}^{-}2 = {}^{+}3 \times {}^{-}2 = {}^{-}6$$

In the first two expressions, we are taking a change of $^{-}2$ three times and the result, $^{-}6$, is not changed in direction.

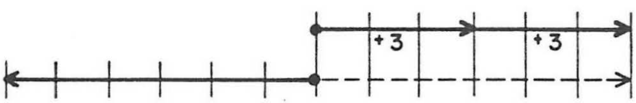
The third expression could mean just what the first two do, but it could also mean a change of $^{+}3$ taken twice and the result changed in direction.

Let's illustrate both ideas with sketches:



$${}^{-}2 \times {}^{NC}3 = {}^{-}6$$

and



$${}^{+}3 \times {}^{-}2 = {}^{-}6$$

The results are alike. So we can indeed use our *NC* and *CD* ideas to make sense out of the mathematician's way of doing multiplication with positive and negative numbers.

This situation is similar to the one in which we try to make sense out of the statement that $3 \times 2 = 6$. This could be thought of as saying that

$$\$3 \times 2 = \$6 \quad \text{or} \quad 3 \times \$2 = \$6.$$

Both ways of looking at "$3 \times 2 = 6$" make sense.

More Simplification

Let's agree that we shall sometimes omit the $^{+}$ in our computations with directed numbers. We shall act just as if it had been written. So, for positive numbers like $^{+}3$ and $^{+}7$, we shall simply write 3 and 7. We shall write negative numbers as we have always done.

Does this cause any confusion in multiplication examples?

$${}^{+}12 \times {}^{-}4 = 12 \times {}^{-}4 = {}^{-}48$$

$${}^{+}9 \times {}^{+}7 = 9 \times 7 = 63$$

$${}^{-}6 \times {}^{+}8 = {}^{-}6 \times 8 = {}^{-}48$$

$${}^{-}10 \times {}^{-}10 = {}^{-}10 \times {}^{-}10 = 100$$

$${}^{+}8 \times {}^{+}5 = 8 \times 5 = 40$$

$${}^{-}8 \times {}^{-}5 = {}^{-}8 \times {}^{-}5 = 40$$

Does our agreement cause any confusion in addition and subtraction examples?

$3 + {}^{-}4 = {}^{-}1$	${}^{-}7 + {}^{-}3 = {}^{-}10$
$5 - 7 = {}^{-}2$	$9 + 8 = 17$
$5 - {}^{-}7 = 12$	${}^{-}6 - {}^{-}10 = 4$
${}^{-}5 \times 7 = {}^{-}35$	${}^{-}6 \times {}^{-}10 = 60$
$3 \times {}^{-}12 = {}^{-}36$	$5 \times 13 = 65$
$23 - {}^{-}18 = 41$	$23 - 18 = 5$
$18 - 23 = {}^{-}5$	$18 - {}^{-}23 = 41$
${}^{-}24 \times \frac{1}{3} = {}^{-}8$	${}^{-}24 \times {}^{-}\frac{1}{3} = 8$

Now let's think about division.

Tasks: Students complete sentences involving multiplication, addition, and subtraction of directed numbers.

Purpose: To introduce multiplication of directed numbers, and to present the usual convention for omitting the $^{+}$ in the notation for positive numbers.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

The Lesson: On pupil page 55 we tell students that the operation called *multiplication of directed numbers* is faithfully reflected by the *NC* and *CD* interpretation of multiplication on pupil page 54. Moreover, the interpretation leads us to discover that multiplication of directed numbers is commutative. For example, how could we tell in advance that

$${}^{+}3 \times {}^{-}4 = {}^{-}4 \times {}^{+}3 \quad ?$$

Just think of:

${}^{+}3 \overset{CD}{\times} 4$ as replacing: ${}^{+}3 \times {}^{-}4$

and:

${}^{-}4 \overset{NC}{\times} 3$ as replacing: ${}^{-}4 \times {}^{+}3$

Then:

${}^{+}3 \overset{CD}{\times} 4 =$ the opposite of $({}^{+}3 + {}^{+}3 + {}^{+}3 + {}^{+}3) = {}^{-}12$

and:

${}^{-}4 \overset{NC}{\times} 3 = {}^{-}4 + {}^{-}4 + {}^{-}4 = {}^{-}12$.

If you wish, have students present the definition of multiplication of directed numbers as for subtraction on pupil page 53:

For all numbers a and n,

$$
\begin{aligned}
{}^{+}a \times {}^{+}n &= {}^{+}(an),\\
{}^{+}a \times {}^{-}n &= {}^{-}(an),\\
{}^{-}a \times {}^{+}n &= {}^{-}(an),\\
\text{and} \quad {}^{-}a \times {}^{-}n &= {}^{+}(an).
\end{aligned}
$$

The simplification of notation afforded by omitting the $^{+}$ causes no confusion in computation, because the system of nonnegative directed numbers (positive numbers and 0) has precisely the same additive and multiplicative *structures* as the system of undirected numbers. Any computation without the raised plus signs can be thought of as an example in each system:

$$
\begin{array}{ccccccc}
5 + 3 = 8 & \quad & 6 - 2 = 4 & \quad & 7 \times 3 = 21 \\
\updownarrow\ \updownarrow\ \updownarrow\ \ \updownarrow & & \updownarrow\ \updownarrow\ \updownarrow\ \ \updownarrow & & \updownarrow\ \updownarrow\ \updownarrow\ \ \updownarrow \\
{}^{+}5 + {}^{+}3 = {}^{+}8 & & {}^{+}6 - {}^{+}2 = {}^{+}4 & & {}^{+}7 \times {}^{+}3 = {}^{+}21
\end{array}
$$

This is a logical consequence of the definitions of addition and multiplication for both systems.

However, the similarity of structures does not mean that we should claim that the directed number $^{+}7$ is the same as the undirected number 7. We like to think of these numbers as measures of quantities in the real world, and, in this sense, a directed quantity is not the same thing as an undirected one.

Some students may have learned the terms *commutativity*, *associativity*, and *distributivity* from earlier work with undirected numbers. They may be interested in the fact that addition and multiplication in the system of directed numbers are also commutative and associative, and that multiplication distributes over addition. They may convince themselves of this by checking a few examples:

$$
\begin{aligned}
{}^{+}9 + {}^{-}2 &= {}^{+}7 = {}^{-}2 + {}^{+}9\\
({}^{-}3 + {}^{+}4) + {}^{-}8 &= {}^{-}7 = {}^{-}3 + ({}^{+}4 + {}^{-}8)\\
{}^{+}4 \times {}^{-}2 &= {}^{-}8 = {}^{-}2 \times {}^{+}4\\
({}^{-}2 \times {}^{-}3) \times {}^{+}4 &= {}^{+}24 = {}^{-}2 \times ({}^{-}3 \times {}^{+}4)\\
({}^{+}2 + {}^{-}5) \times {}^{-}3 &= {}^{+}9 = ({}^{+}2 \times {}^{-}3) + ({}^{-}5 \times {}^{-}3)
\end{aligned}
$$

FPS-55

Directional signs can be used for many ideas. For example, if the $^{+}$ means forward then the $^{-}$ will mean backward. Here are more examples. Please complete them.

	1.	2.	3.	4.	5.	6.
If $^{+}$ means . . .	UP	PROFIT	EAST	ABOVE	FOUND	NORTH
then $^{-}$ means . . .	DOWN	LOSS	WEST	BELOW	LOST	SOUTH

Write appropriate "headlines" or number sentences for the following stories.

7. ${}^{-}8 + {}^{+}17 = {}^{+}9$ Bill started down the basement stairs. After going down 8 stairs, he changed his mind. Then he went up 17 stairs.

8. ${}^{+}32 - {}^{+}5 = {}^{+}27$ Mr. Swanson found he had gone 5 miles in the wrong direction. "Now I'm 32 miles from home." How far would he have had to go if he had not made a mistake?

9. ${}^{-}1\frac{1}{2} + {}^{-}3 = {}^{-}4\frac{1}{2}$ Harry lost $1\frac{1}{2}$ pounds one week and twice as much the next. How had his weight changed?

10. ${}^{+}175 + {}^{-}210 = {}^{-}35$ Bill walked 175 feet into a cave. Then he turned around and walked 210 feet. Where was he?

11. ${}^{-}63 + {}^{+}25 = {}^{-}38$ The skin diver went down 63 feet. After he came up 25 feet, he saw a beautiful piece of coral. How far was the coral beneath the surface?

12. ${}^{+}212 + {}^{-}75 = {}^{+}137$ The water had cooled 75 degrees Fahrenheit after it had boiled. This happened at the seashore last summer.

Write appropriate stories for the following headlines. (MANY POSSIBLE ANSWERS.)

13. ${}^{+}17 + {}^{-}25 = {}^{-}8$ HARRY GAINED 17 POUNDS ONE YEAR AND LOST 25 POUNDS THE NEXT. HOW HAD HIS WEIGHT CHANGED?

14. ${}^{-}8 + {}^{+}29 = {}^{+}21$ SAM LOST 8 POINTS IN THE FIRST ROUND OF HIS GAME WITH JOE. HE FINISHED THE SECOND ROUND 21 POINTS AHEAD OF JOE.

15. ${}^{-}1.25 + {}^{-}3.75 = {}^{-}5$ JACK RECEIVED TWO BILLS IN THE MAIL. ONE BILL AMOUNTED TO \$1.25. THE TOTAL AMOUNT OF BOTH BILLS WAS \$5.00.

16. ${}^{+}2\frac{1}{2} + {}^{-}4 = {}^{-}1\frac{1}{2}$ TOM GAINED SOME WEIGHT DURING HIS FIRST WEEK AT CAMP. THE NEXT WEEK HE WENT ON A DIET AND LOST 4 POUNDS. AT THE END OF THE 2 WEEKS HE FOUND HIS WEIGHT HAD DECREASED BY $1\frac{1}{2}$ POUNDS.

DIVISION

Division with positive and negative numbers is much like division with undirected numbers. *(Please complete the following examples.)*

$8 \div 4 =$ 2 because 4 $\times 2 = 8$

$12 \div 3 =$ 4 because 4 $\times 3 = 12$

$3 \div \frac{1}{2} =$ 6 because 6 $\times \frac{1}{2} = 3$

$21 \div 3 = ($ 7 $\times 3) \div 3 =$ 7

Dividing by an undirected number "undoes" the result of multiplying by it. This is precisely what dividing by a directed number does.

$^{+}10 \div {}^{-}5 =$ $^{-}2$ because $^{-}2$ $\times {}^{-}5 = {}^{+}10$

$^{-}8 \div {}^{-}2 =$ $^{+}4$ because $^{+}4$ $\times {}^{-}2 = {}^{-}8$

$^{+}6 \div {}^{+}3 =$ $^{+}2$ because $^{+}2$ $\times {}^{+}3 = {}^{+}6$

$^{-}10 \div {}^{+}2 =$ $^{-}5$ because $^{-}5$ $\times$ $^{+}2$ $=$ $^{-}10$

$^{-}21 \div {}^{+}3 = ($ $^{-}7$ $\times {}^{+}3) \div {}^{+}3 =$ $^{-}7$

To solve this problem:

$^{+}14 \div {}^{-}7 =$ $^{-}2$

just solve this problem:

$^{-}2$ $\times {}^{-}7 = {}^{+}14$

Solve the following problems.

1. $^{-}30 \div {}^{+}5 =$ $^{-}6$
2. $^{+}12 \div {}^{-}3 =$ $^{-}4$
3. $^{-}18 \div {}^{-}9 =$ $^{+}2$
4. $^{+}20 \div {}^{+}4 =$ $^{+}5$
5. $25 \div {}^{-}5 =$ $^{-}5$
6. $^{-}30 \div 6 =$ $^{-}5$
7. $^{-}56 \div 8 =$ $^{-}7$
8. $^{-}56 \div {}^{-}8 =$ $^{+}7$
9. $15 \div {}^{-}3 =$ $^{-}5$
10. $15 \div {}^{-}\frac{1}{3} =$ $^{-}45$
11. $0 \div {}^{-}2 =$ 0
12. 0 $\times {}^{-}2 = 0$
13. $^{-}\frac{1}{2} \div {}^{-}\frac{1}{2} =$ $^{+}1$
14. $^{-}\frac{1}{3} \div \frac{1}{3} =$ $^{-}1$
15. $^{-}\frac{3}{2} \div {}^{-}\frac{3}{2} =$ $^{+}1$
16. $\frac{1}{3} \div {}^{-}\frac{1}{3} =$ $^{-}1$

SUMMARY

Complete the following table to show the results of adding, subtracting, multiplying, and dividing with positive and negative numbers.

	if $a < b$	if $a = b$	if $a > b$
$^{+}a + {}^{+}b$	positive	POSITIVE	POSITIVE
$^{+}a + {}^{-}b$	negative	0	P
$^{-}a + {}^{+}b$	P	O	NEGATIVE
$^{-}a + {}^{-}b$	N	N	N
$^{+}a - {}^{+}b$	N	O	positive
$^{+}a - {}^{-}b$	P	positive	P
$^{-}a - {}^{+}b$	N	N	N
$^{-}a - {}^{-}b$	P	O	N
$^{+}a \times {}^{+}b$	positive	P	P
$^{+}a \times {}^{-}b$	N	N	N
$^{-}a \times {}^{+}b$	N	N	N
$^{-}a \times {}^{-}b$	P	P	P
$^{+}a \div {}^{+}b$	P	P	P
$^{+}a \div {}^{-}b$	N	N	N
$^{-}a \div {}^{+}b$	N	N	N
$^{-}a \div {}^{-}b$	P	P	P

56 Division of positive and negative numbers - summary

Tasks: Students examine division of directed numbers and summarize the "sign laws" for all four operations on positive and negative numbers.

Purpose: To help students understand that dividing by a directed number is the inverse of multiplying by it. To help students generalize their study of positive and negative numbers.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: We define division as the operation which "undoes" or "inverts" multiplication. Consider the operation multiplying by $^-2$. This operation takes each directed number to its product with $^-2$:

multiplying by $^-2$

$^+8 \to {}^-16 \quad 0 \to 0 \quad {}^-3 \to {}^+6 \quad {}^-8 \to {}^+16$

$^+2 \to {}^-4 \quad {}^-1 \to {}^+2 \quad {}^-9 \to {}^+18$

$^+\frac{3}{4} \to {}^-1\frac{1}{2} \quad {}^-2 \to {}^+4 \quad {}^+5 \to {}^-10 \quad {}^-10 \to {}^+20$

There is also an operation which "reverses" the arrows." That is, it takes $^-16$ to $^+8$, 0 to 0, $^+6$ to $^-3$, etc. This operation is *dividing by* $^-2$:

dividing by $^-2$

$^+8 \leftarrow {}^-16 \quad 0 \leftarrow 0 \quad {}^-3 \leftarrow {}^+6 \quad {}^-8 \leftarrow {}^+16$

$^+2 \leftarrow {}^-4 \quad {}^-1 \leftarrow {}^+2 \quad {}^-9 \leftarrow {}^+18$

$^+\frac{3}{4} \leftarrow {}^-1\frac{1}{2} \quad {}^-2 \leftarrow {}^+4 \quad {}^+5 \leftarrow {}^-10 \quad {}^-10 \leftarrow {}^+20$

Anyone who learns the operation multiplying by $^-2$ automatically learns the inverse operation, dividing by $^-2$. To compute $^-10 \div {}^-2$, all one need do is find the number whose product by $^-2$ is $^-10$. The answer is $^+5$. In the language of pupil page 56,

$$^-10 \div {}^-2 = \underline{{}^+5} \text{ because } \underline{{}^+5} \times {}^-2 = {}^-10.$$

Consider the operation multiplying by 0, which takes every directed number to 0:

$$^+4 \times 0 = 0 \quad {}^-3 \times 0 = 0 \quad {}^+8 \times 0 = 0 \quad \text{etc.}$$

If there were an inverse to this operation, we could apply it to any directed number, say $^+6$, and come out with exactly one directed number as the answer. But, since $^+6$ is not the result of applying multiplication by 0 to any directed number, we do not get an answer. For this reason we say that multiplication by 0 does not have an inverse.

$$^+6 \div 0 = \underline{?} \text{ because } \underline{?} \times 0 = {}^+6$$

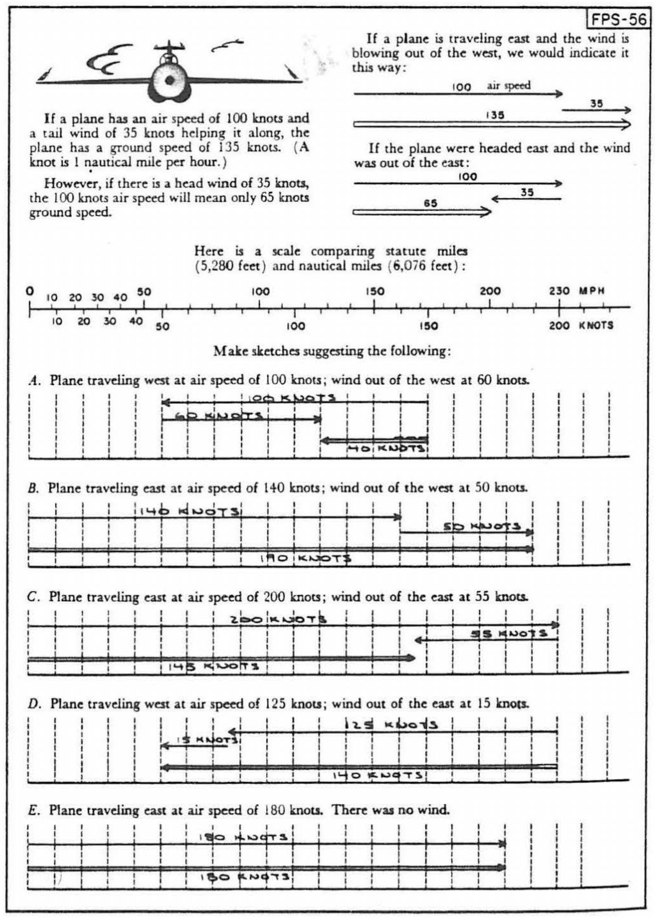
The table on page 56 summarizes work with positive and negative numbers. The student finds results by testing instances. For example, in the third row ($^-a + {}^+b$) he might proceed as follows:

if $a < b$	if $a = b$	if $a > b$
$^-5 + {}^+7 = {}^+2$	$^-5 + {}^+5 = 0$	$^-5 + {}^+4 = {}^-1$
$^-8 + {}^+9 = {}^+1$	$^-8 + {}^+8 = 0$	$^-8 + {}^+2 = {}^-6$
positive	zero	negative

If the student is aware of commutativity, he might recognize that he need not do any testing to find the entries for $^-a + {}^+b$ once he has found the entries for $^+a + {}^-b$. The sign of $^-a + {}^+b$ is the opposite of the sign of $^+a + {}^-b$. Similarly, no testing is needed for the $^-a \times {}^+b$ row once the entries have been found for the $^+a \times {}^-b$ row. The sign of $^-a \times {}^+b$ is the same as the sign of $^+a \times {}^-b$.

Since subtracting a directed number amounts to adding its opposite, the $^+a - {}^+b$ row must be the same as the $^+a + {}^-b$ row.

The summary table deals with operations for positive and negative numbers only. You might wish to extend it with examples such as $^+3 + 0$, $0 + {}^-2$, $^-4 - 0$, $0 - {}^+2$, $^-3 \times 0$, $0 \times {}^+4$, and $0 \div {}^-2$.


FPS-56

If a plane is traveling east and the wind is blowing out of the west, we would indicate it this way:

If a plane has an air speed of 100 knots and a tail wind of 35 knots helping it along, the plane has a ground speed of 135 knots. (A knot is 1 nautical mile per hour.)

However, if there is a head wind of 35 knots, the 100 knots air speed will mean only 65 knots ground speed.

If the plane were headed east and the wind was out of the east:

Here is a scale comparing statute miles (5,280 feet) and nautical miles (6,076 feet):

Make sketches suggesting the following:

A. Plane traveling west at air speed of 100 knots; wind out of the west at 60 knots.

B. Plane traveling east at air speed of 140 knots; wind out of the west at 50 knots.

C. Plane traveling east at air speed of 200 knots; wind out of the east at 55 knots.

D. Plane traveling west at air speed of 125 knots; wind out of the east at 15 knots.

E. Plane traveling east at air speed of 180 knots. There was no wind.

I. In the green-tinted blocks, write the sum of the numbers in the blue-tinted blocks that occur in the same row and column.

II. In the green-tinted blocks, write the product of the numbers in the blue-tinted blocks that occur in the same row and column.

ADDITION

0	+1	+2	+3	+4	+5	+6	+7	+8
-1	0	+1	+2	+3	+4	+5	+6	+7
-2	-1	0	+1	+2	+3	+4	+5	+6
-3	-2	-1	0	+1	+2	+3	+4	+5
-4	-3	-2	-1		+1	+2	+3	+4
-5	-4	-3	-2	-1	0	+1	+2	+3
-6	-5	-4	-3	-2	-1	0	+1	+2
-7	-6	-5	-4	-3	-2	-1	0	+1
-8	-7	-6	-5	-4	-3	-2	-1	0

MULTIPLICATION

-16	-12	-8	-4	+4	+4	+8	+12	+16
-12	-9	-6	-3	+3	+3	+6	+9	+12
-8	-6	-4	-2	+2	+2	+4	+6	+8
-4	-3	-2	-1	+1	+1	+2	+3	+4
-4	-3	-2	-1		+1	+2	+3	+4
+4	+3	+2	+1	-1	-1	-2	-3	-4
+8	+6	+4	+2	-2	-2	-4	-6	-8
+12	+9	+6	+3	-3	-3	-6	-9	-12
+16	+12	+8	+4	-4	-4	-8	-12	-16

III. Pattern hunts.

1)	2)	3)	4)	5)
10 → 17	-8 → -16	7 → 15	7 → -13	6 → -36
-3 → 4	.7 → 1.4	4 → 9	4 → -7	-3 → -9
0 → 7	-90 → -180	1 → 3	1 → -1	3 → -9
-10 → -3	0 → 0	-2 → -3	-2 → 5	-6 → -36
-7 → 0	-.25 → -.5	-5 → -9	-5 → 11	0 → 0
$-9\frac{7}{8} \to -2\frac{7}{8}$	$-\frac{1}{6} \to -\frac{1}{3}$	-8 → -15	-8 → 17	4 → -16
$-5\frac{1}{2} \to 1\frac{1}{2}$	$\frac{3}{8} \to \frac{3}{4}$	-11 → -21	-11 → 23	4 → -16
$a \to a + 7$	$n \to 2n$	$x \to 2x + 1$	$x \to -2x + 1$	$m \to m^2 \times (-1)$

6)	7)	8)	9)	10)
5 → 30	-4 → 20	70 → 30	3 → 15	4 → 25
-5 → 20	-3 → 17	30 → 70	-2.5 → 15	3 → 16
-4 → 12	-2 → 14	60 → 40	4 → 28	2 → 9
4 → 20	-1 → 11	40 → 60	-3.5 → 28	1 → 4
-6 → 30	0 → 8	120 → -20	5 → 45	0 → 1
6 → 42	2 → 2	-20 → 120	-4.5 → 45	-1 → 0
-3 → 6	3 → -1	-50 → 150	7 → 91	-2 → 1
2 → 6	5 → -7	150 → -50	-6.5 → 91	-3 → 4
$y \to y^2 + y$	$c \to -3c + 8$	$p \to 100 - p$	$q \to 2q^2 - q$	$r \to r^2 + 2r + 1$

Tasks: Students fill in the addition and multiplication grids and complete the ten Pattern Hunts.

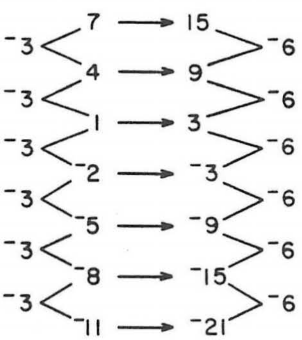
Purpose: To help students review addition and multiplication with signed numbers.

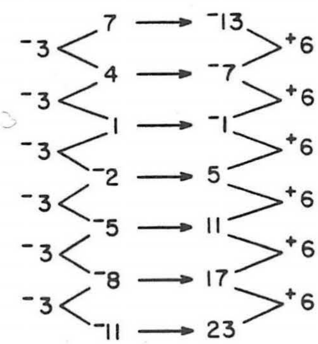
Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Exercises I and II reveal interesting patterns. Each left-to-right row of table I contains an increasing sequence of directed numbers. Each top-to-bottom column contains a decreasing sequence of directed numbers.

Each left-to-right green-tinted row of table II is also a right-to-left green-tinted row of the table. Similarly for the columns. Table II shows quite clearly that the product of a pair of positive numbers is the same as the product of their opposites.

The *Pattern Hunts* provide computational practice—and insight into multiplication of directed numbers. Consider examples 3 and 4 in exercise III:





The force of the pattern in example 3 leads us to feel that $(2 \times {}^-2) + 1$ should be 6 less than 3, or $^-3$; that $(2 \times {}^-5) + 1$ should be 6 less than $^-3$, or $^-9$; etc. Thus, if the pattern is to be maintained then the product of a positive number and a negative number should be negative.

Similarly, if the pattern in example 4 is to be maintained, the product of two negative numbers should be positive.

And, of course, our definition of multiplication is such that the patterns are maintained.

Examples 5, 6, 9, and 10 emphasize that the square of a positive or negative number is positive. They also show that trying to "undo" the operation of squaring does not produce a unique result. Squaring directed numbers is an example of an operation which cannot be undone (that is, does not have an inverse). The process of *extracting the square root* in the system of directed numbers is *always* defined so that only nonnegative numbers are produced. Thus it is correct to write:

$$\sqrt{^+25} = {}^+5 \quad \text{and} \quad \sqrt{^+25} \neq {}^-5$$

The process of extracting the square root can be applied only to nonnegative numbers. Thus the expression "$\sqrt{^-25}$" has no meaning.

A statement to the effect that the square root of $^+25$ is $^+5$ or $^-5$ is misleading because it tempts one to think that the square root of $^+25$ might be $^-5$. The phrase "the square root of $^+25$" and the notation "$\sqrt{^+25}$" stand for one and only one directed number, $^+5$.

FPS-57

Complete the tables.

I

+	$^-6$	$^-4$	$^-2$	0	2	4	6
$^-6$	$^-12$	$^-10$	$^-8$	$^-6$	$^-4$	$^-2$	0
$^-4$	$^-10$	$^-8$	$^-6$	$^-4$	$^-2$	0	2
$^-2$	$^-8$	$^-6$	$^-4$	$^-2$	0	2	4
0	$^-6$	$^-4$	$^-2$	0	2	4	6
2	$^-4$	$^-2$	0	2	4	6	8
4	$^-2$	0	2	4	6	8	10
6	0	2	4	6	8	10	12

II

−	$^-6$	$^-4$	$^-2$	0	2	4	6
$^-6$	0	$^-2$	$^-4$	$^-6$	$^-8$	$^-10$	$^-12$
$^-4$	2	0	$^-2$	$^-4$	$^-6$	$^-8$	$^-10$
$^-2$	4	2	0	$^-2$	$^-4$	$^-6$	$^-8$
0	6	4	2	0	$^-2$	$^-4$	$^-6$
2	8	6	4	2	0	$^-2$	$^-4$
4	10	8	6	4	2	0	$^-2$
6	12	10	8	6	4	2	0

III

×	$^-6$	$^-3$	0	3	6	9
$^-1$	6	3	0	$^-3$	$^-6$	$^-9$
$^-2$	12	6	0	$^-6$	$^-12$	$^-18$
$^-3$	18	9	0	$^-9$	$^-18$	$^-27$
$^-5$	30	15	0	$^-15$	$^-30$	$^-45$
$^-8$	48	24	0	$^-24$	$^-48$	$^-72$
$^-13$	78	39	0	$^-39$	$^-78$	$^-117$
$^-21$	126	63	0	$^-63$	$^-126$	$^-189$

IV

÷	$^-3$	$^-2$	$^-1$	1	2	3	6
$^-6$	2	3	6	$^-6$	$^-3$	$^-2$	$^-1$
$^-12$	4	6	12	$^-12$	$^-6$	$^-4$	$^-2$
$^-18$	6	9	18	$^-18$	$^-9$	$^-6$	$^-3$
24	$^-8$	$^-12$	$^-24$	24	12	8	4
30	$^-10$	$^-15$	$^-30$	30	15	10	5
54	$^-18$	$^-27$	$^-54$	54	27	18	9
36	$^-12$	$^-18$	$^-36$	36	18	12	6

V

× $^-2$	3	4	7	11	18	29
	$^-6$	$^-8$	$^-14$	$^-22$	$^-36$	$^-58$

VI

÷ $^-\frac{1}{2}$	5	6	11	17	28	45
	$^-10$	$^-12$	$^-22$	$^-34$	$^-56$	$^-90$

VII

+ $^-3$	$^-8$	7	$^-4$	11	$^-2$	13
	$^-11$	4	$^-7$	8	$^-5$	10

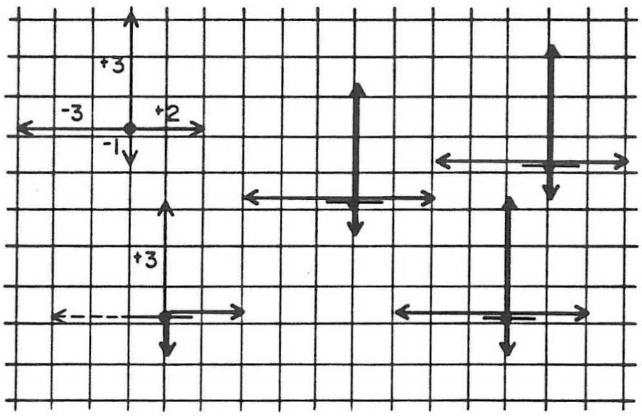
VIII

− $^+3$	$^-8$	7	$^-4$	11	$^-2$	13
	$^-11$	4	$^-7$	8	$^-5$	10

Horizontal and Vertical Changes

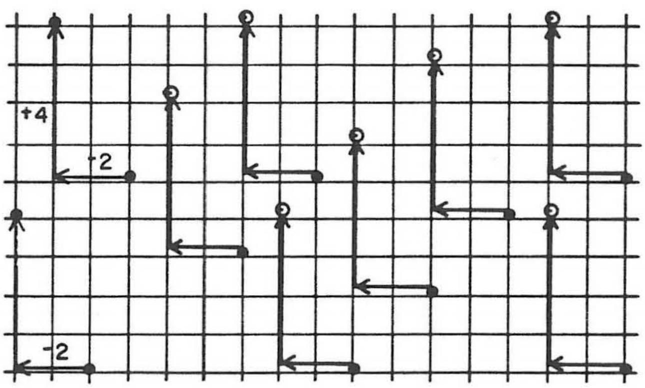
I. We wish to indicate four changes of position for each of the five points shown below. You can get the rules without any explanation.

Horizontal: $^{-}3$, $^{+}2$ Vertical: $^{+}3$, $^{-}1$



II. We wish to indicate a horizontal change followed by a vertical change. Do you get our rules?

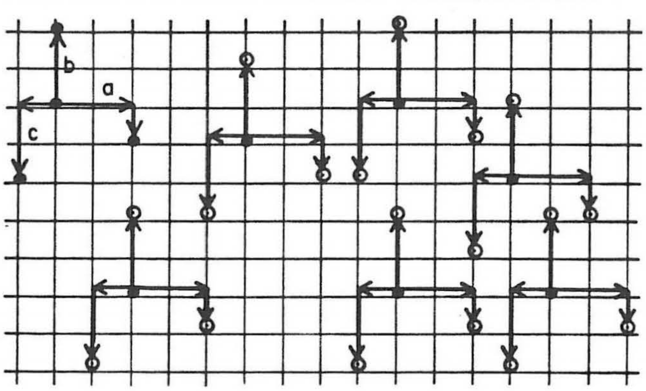
Horizontal: $^{-}2$ followed by Vertical: $^{+}4$



III. Change *a*: (H: $^{+}2$, V: $^{-}1$)

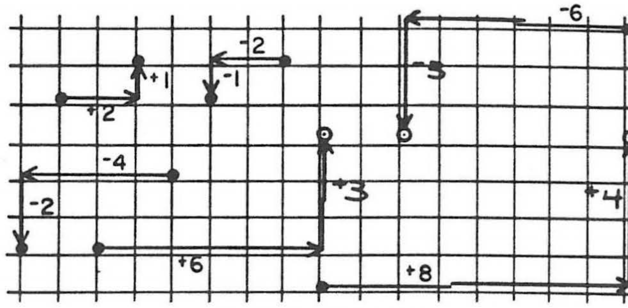
Change *b*: (H: 0, V: $^{+}2$)

Change *c*: (H: $^{-}1$, V: $^{-}2$)



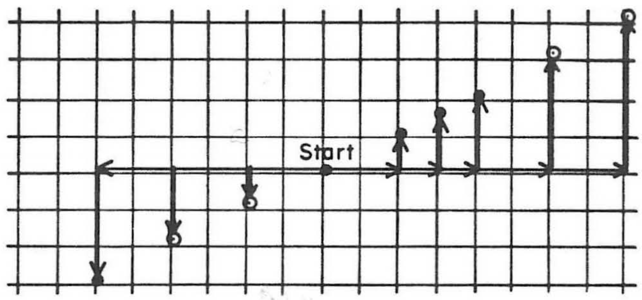
IV. A horizontal change followed by a vertical change that is half the horizontal change.

$$V = \frac{1}{2}H$$



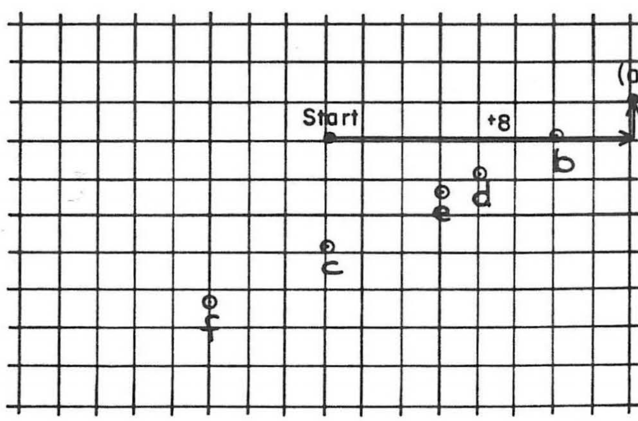
V. Pairs of horizontal followed by vertical changes, all starting from the same point. ($^{+}2$, $^{+}1$), ($^{+}3$, $^{+}1.5$), etc. *(Find at least four more new positions.)*

$$V = \frac{1}{2}H$$



VI. Each vertical change is $^{+}3$ less than half the corresponding horizontal change — all from the same point.

$$V = \frac{1}{2}H - {}^{+}3$$



$V = \frac{1}{2}H - {}^{+}3$

	a	b	c	d	e	f
If H is	+8	+6	0	+4	+3	-3
then V is	+1	0	-3	-1	$-1\frac{1}{2}$	$-4\frac{1}{2}$

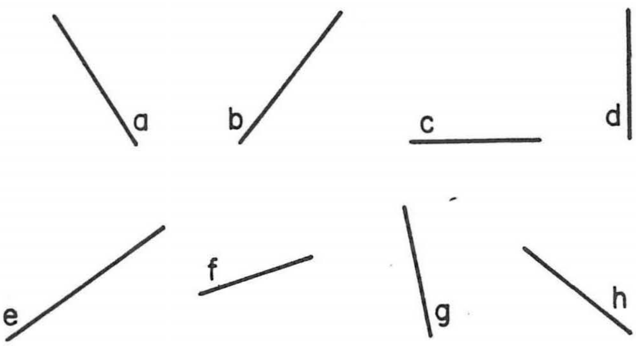
58-61 Horizontal and vertical changes-graphs of equations

Tasks: Students graph new positions of points, given horizontal and vertical components of the changes. The horizontal and vertical components are given in words and by formula.

Purpose: To help students learn to describe and compare the slopes of lines on a graph.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; geometry; measurement.

The Lesson: By agreeing to think in a left-to-right sense, we can make comments about the lines drawn below.



Line a slopes down; it has a steeper slope down than h, but not as steep a slope down as g. Line b slopes up; it has a steeper slope up than e and f. Line c has no slope up or down. Line d is straight up and down; and, since we can't think of it in any left-to-right sense, we will say that it has no slope.

We can make other comparisons. Line a slopes down at about the same angle as b slopes up; the same is true of h and e. Line f slopes up more gradually than g slopes down.

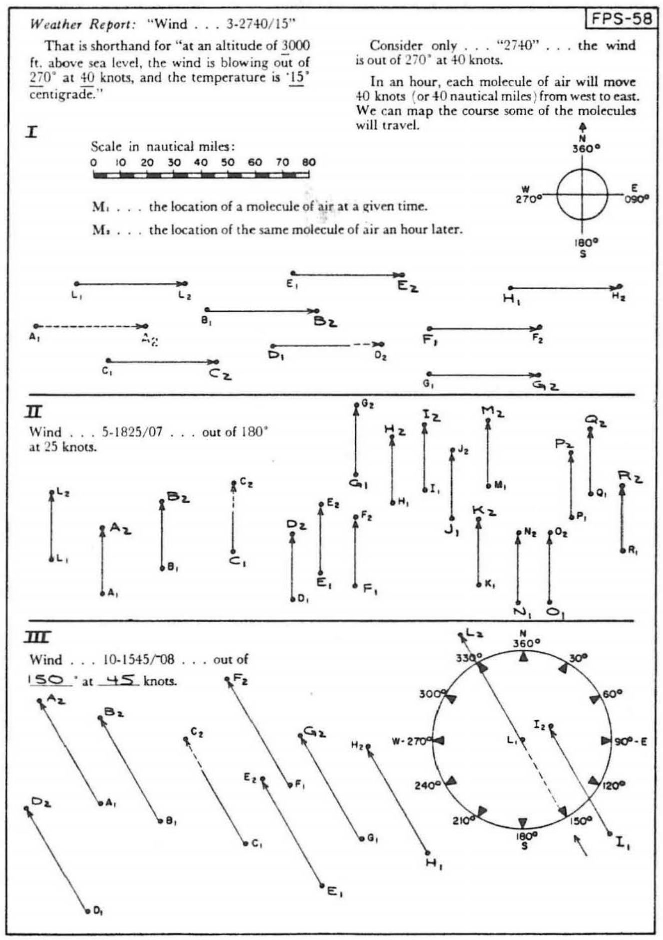
If such lines are drawn on graph paper, we can describe and compare the slopes of lines more precisely. A simple method is to select any two convenient points on the line and describe the location of one point relative to the other by telling how one might get from one point of the line to another.

Any two points of the line will do. We can get from one point to the other by moving a certain number of units to the right or left and then a certain number of units up or down.

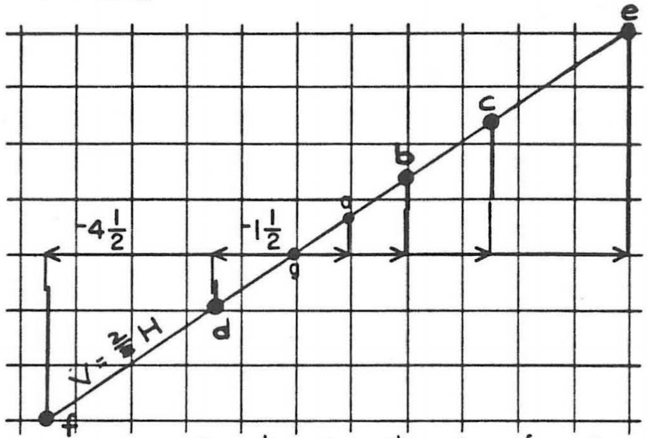
We can describe a line by indicating the relationship between the number of horizontal units (H) and the number of vertical units (V); we can do this by using formulas such as $V=H$ or $V=\frac{1}{2}H$.

Practice sheet 58 introduces another aspect of air navigation—the effect of wind direction and wind velocity on the course a pilot or navigator chooses to follow from one point to another. Practice sheets 59 through 65 develop this material further. The terminology is very close to that used by pilots. The weather reports are given in the same form in which pilots or navigators get them from the U.S. Weather Bureau.

Practice sheet 58 is designed to help students realize that we are thinking about a mass of air moving in a certain direction relative to the surface of the earth.



I. $V = \frac{2}{3}H$

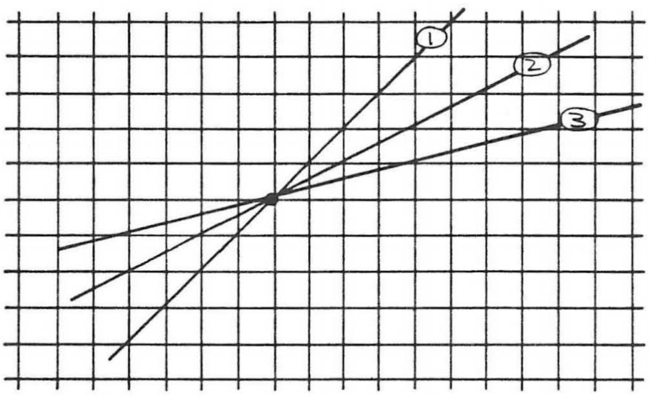


	a	b	c	d	e	f	g
If H is	$^+1$	$^+2$	$^+3\frac{1}{2}$	$^-1\frac{1}{2}$	$^+6$	$^-4\frac{1}{2}$	0
then V is ...	$^+\frac{2}{3}$	$^+1\frac{1}{3}$	$^+2\frac{1}{3}$	$^-1$	$^+4$	$^-3$	0

Draw the line that would include all the points you have located.

II. From a single point, locate many points by following the rules given. You may wish to complete the tables below as you work on the graphs. *(Omit the arrows if possible.)*

$V = H$ $V = \frac{1}{2}H$ $V = \frac{1}{4}H$



$V = H$ ①

H	$^+1$	$^+2$	$^+3$	$^-1$	$^-3$	0	$^-\frac{1}{2}$	$^+.1$	$^-\frac{7}{9}$	$^+19$
V	$^+1$	$^+2$	$^+3$	$^-1$	$^-3$	0	$^-\frac{1}{2}$	$^+.1$	$^-\frac{7}{9}$	$^+19$

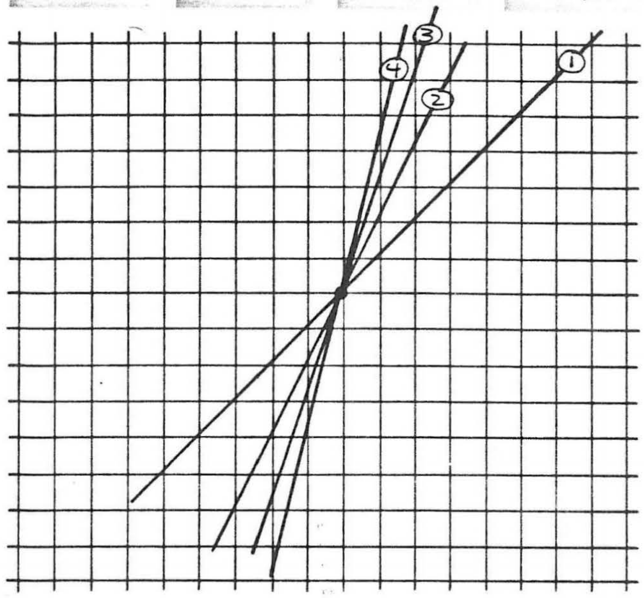
$V = \frac{1}{2}H$ ②

H	$^+1$	$^+2$	$^-3$	$^-4$	0	$^+5$	$^+\frac{1}{2}$	$^+\frac{2}{3}$	$^-.8$	$^-1$
V	$^+\frac{1}{2}$	$^+1$	$^-1\frac{1}{2}$	$^-2$	0	$^+2\frac{1}{2}$	$^+\frac{1}{4}$	$^+\frac{1}{3}$	$^-.4$	$^-\frac{1}{2}$

$V = \frac{1}{4}H$ ③

H	$^+1$	$^-1$	$^+2$	$^-2$	$^+\frac{1}{2}$	$^-\frac{1}{2}$	$^+.8$	$^-.8$	0	$^+\frac{2}{3}$
V	$^+\frac{1}{4}$	$^-\frac{1}{4}$	$^+\frac{1}{2}$	$^-\frac{1}{2}$	$^+\frac{1}{8}$	$^-\frac{1}{8}$	$^+.2$	$^-.2$	0	$^+\frac{1}{6}$

III. $V = H$ $V = 2H$ $V = 3H$ $V = 4H$



$V = H$ ①

H	$^+1$	$^+3$	$^-2$	$^-5$
V	$^+1$	$^+3$	$^-2$	$^-5$

$V = 2H$ ②

H	$^-1$	$^+2$	0	$^-3$
V	$^-2$	$^+4$	0	$^-6$

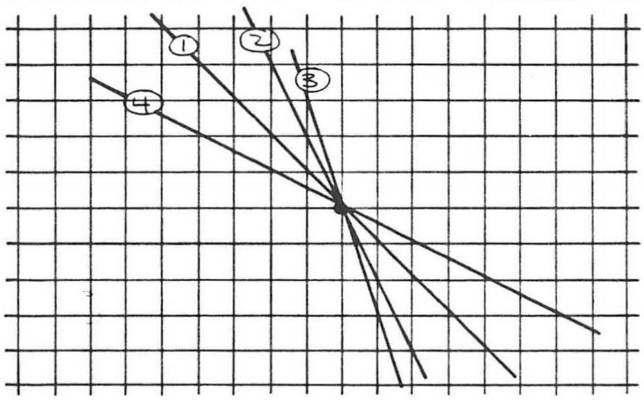
$V = 3H$ ③

H	$^-\frac{1}{2}$	$^-1$	0	$^+2$
V	$^-1\frac{1}{2}$	$^-3$	0	$^+6$

$V = 4H$ ④

H	$^+1$	$^+\frac{1}{2}$	$^-2$	0
V	$^+4$	$^+2$	$^-8$	0

IV. $V = H \times {}^-1$ or $V = {}^-1H$ $V = H \times {}^-2$ or $V = {}^-2H$ $V = H \times {}^-3$ or $V = {}^-3H$ $V = H \times {}^-\frac{1}{2}$ or $V = {}^-\frac{1}{2}H$



$V = {}^-1H$ ①

H	$^+1$	$^-1$	$^+4$	$^-4$
V	$^-1$	$^+1$	$^-4$	$^+4$

$V = {}^-2H$ ②

H	$^-1$	$^-2$	$^+2$	0
V	$^+2$	$^+4$	$^-4$	0

$V = {}^-3H$ ③

H	$^+1$	$^+\frac{1}{2}$	$^-1$	0
V	$^-3$	$^-1\frac{1}{2}$	$^+3$	0

$V = {}^-\frac{1}{2}H$ ④

H	$^-1$	$^-3$	$^+4$	0
V	$^+\frac{1}{2}$	$^+1\frac{1}{2}$	$^-2$	0

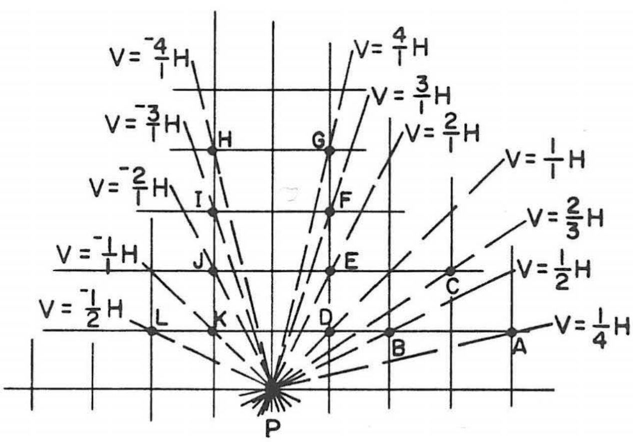
Tasks: Students complete twelve tables based on the formulas given and graph the twelve lines suggested.

Purpose: To help students learn to describe and compare the slopes of lines on a graph.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; geometry; measurement.

The Lesson: As students work through the examples on pupil page 59, they will begin to realize that once they have found a point different from the one given, they can draw the line that includes all the points specified by the given rule.

You may wish to draw a composite diagram on the chalkboard, including all lines considered on pupil page 59. (When identifying a line by its rule or formula, express the multiplier of H as a fraction.)



Indicate the given point clearly. How would you get from point P to point A? (Over 4 to the right and up 1.) From P to B? (Over 2 to the right and up 1.) Etc. Summarize the way to get from P to the first marked intersection on each line:

$V = \frac{1}{4}H$ . . . 4 to right, 1 up
$V = \frac{1}{2}H$ . . . 2 to right, 1 up
$V = \frac{1}{1}H$ . . . 1 to right, 1 up
$V = \frac{2}{1}H$ . . . 1 to right, 2 up
etc.
$V = {}^{-}\frac{2}{1}H$ . . . 1 to left, 2 up
$V = {}^{-}\frac{1}{1}H$ . . . 1 to left, 1 up
$V = {}^{-}\frac{1}{2}H$ . . . 2 to left, 1 up
etc.

Thus, the multiplier (or *coefficient*) of H is a key to the slope of the line. If the coefficient is positive, the line slopes up (from left to right); if it is negative, the line slopes down. The larger the absolute value of the coefficient (corresponding undirected number), the steeper the slope. In locating a second point with respect to a given point, the denominator of the coefficient indicates the number of units to the right (if the coefficient is positive) or left (if negative), and the numerator indicates the number of units up. This is sometimes referred to as the "UP over OVER" system:

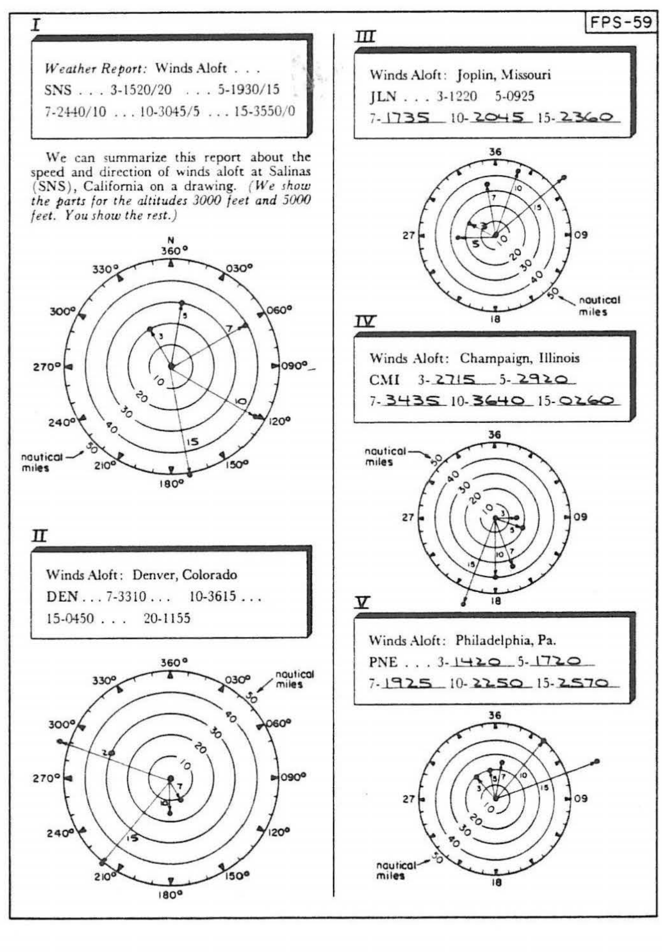
$$\frac{\text{UP}}{\text{OVER}}$$

We can thus use the coefficient of H as a numerical description of the slope of any nonvertical line:

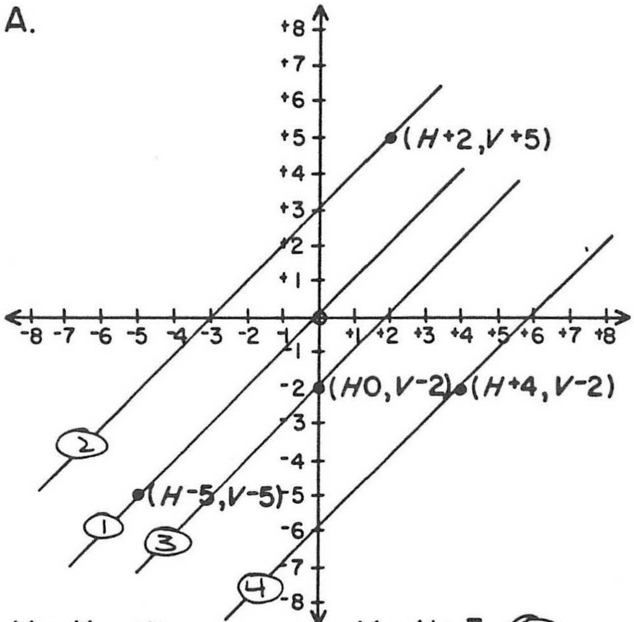
$V = \frac{2}{3}H$ . . . a slope of $\frac{2}{3}$

$V = {}^{-}1H$ . . . a slope of ${}^{-}1$

Practice sheet 59 is an extension of the activity begun on practice sheet 58.



A.



$V = H$ ①

H	+3	-5	+8	-8
V	+3	-5	+8	-8

$V = H + 3$ ②

H	+2	-4	0	-3
V	+5	-1	+3	0

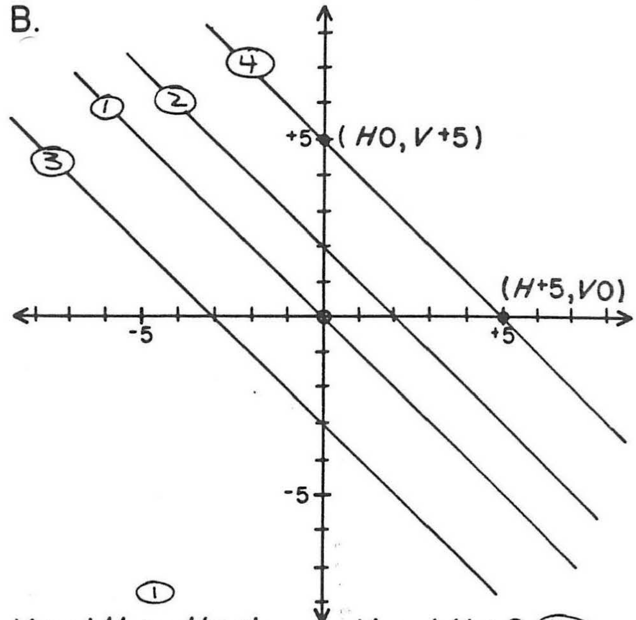
$V = H - 2$ ③

H	+8	-6	0	2
V	+6	-8	-2	0

$V = H - 6$ ④

H	+6	0	+4	-2
V	0	-6	-2	-8

B.



①
$V = -1H$ or $H \times -1$

H	+4	-3	0	-8
V	-4	+3	0	+8

$V = -1H + 2$ ②

H	0	+8	+2	-6
V	+2	-6	0	+8

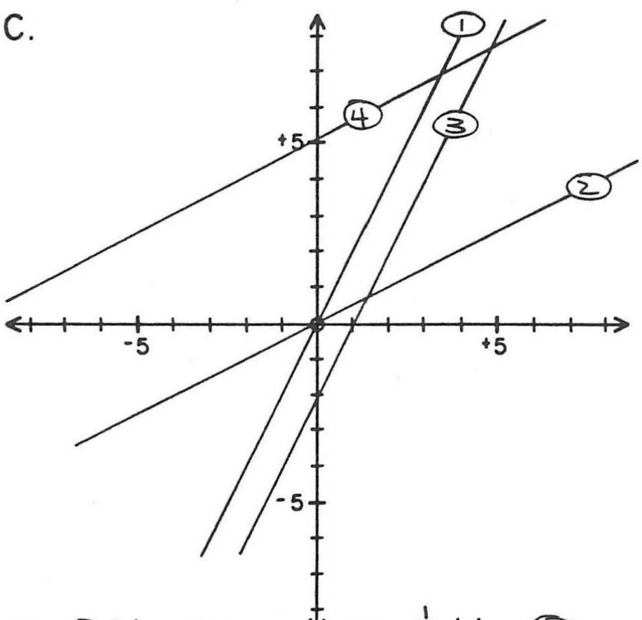
$V = -1H - 3$ ③

H	+1	+3	-7	0
V	-4	-6	+4	-3

$V = -1H + 5$ ④

H	+8	0	+5	-3
V	-3	+5	0	+8

C.



$V = 2H$ ①

H	1	4	-3	0
V	2	8	-6	0

$V = \frac{1}{2}H$ ②

H	0	2	-6	3
V	0	1	-3	$1\frac{1}{2}$

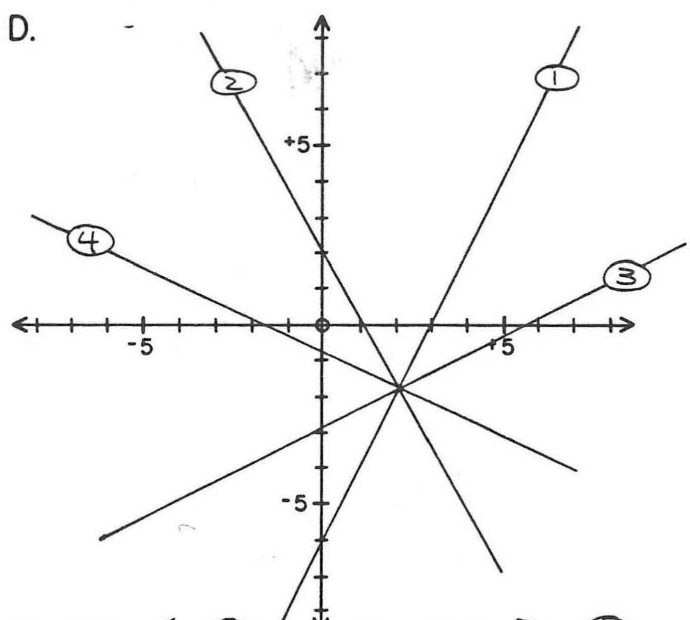
$V = 2H - 2$ ③

H	-2	1	+4	0
V	-6	0	+6	-2

$V = \frac{1}{2}H + 5$ ④

H	3	0	-4	-8
V	$6\frac{1}{2}$	+5	+3	+1

D.



$V = 2H - 6$ ①

H	-1	+3	+7	0
V	-8	0	+8	-6

$V = -2H + 2$ ②

H	+4	-2	0	1
V	-6	+6	+2	0

$V = \frac{1}{2}H - 3$ ③

H	6	0	-2	-6
V	0	-3	-4	-6

$V = -\frac{1}{2}H - 1$ ④

H	0	8	-4	-2
V	-1	-5	+1	0

Tasks: Students complete tables and formulas, and graph the corresponding lines.

Purpose: To help students learn to describe and compare precisely the slopes of lines on a graph.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; geometry; measurement.

The Lesson: Instead of drawing all the grid lines, we use two number lines or *axes*—a horizontal axis and a vertical axis. The point at which the axes cross is called the *origin.*

Two ideas are introduced: (a) some lines do not go through the origin, and (b) we use a system of assigning address-labels to some of the points of the graphs. Each point on an axis has 0 as part of its address. The address of the origin is $(H0, V0)$.

After the graphs on pupil page 60 are completed, what do you notice? The graphs of $V=H$, $V={}^-1H$, $V=\frac{1}{2}H$, and $V=2H$ exhibit nothing new—nothing that couldn't be anticipated from the experiences of the previous page. They all pass through the origin, and their slopes are 1, $^-1$, $\frac{1}{2}$, and 2 respectively.

The four graphs in exercise *A* all have the same slope, but each has been moved up or down with respect to the graph of $V=H$. We might describe this movement up or down by noting the point at which the lines cross the vertical axis once they have been moved:

The graph of $V=H+3$ is obtained by moving the graph of $V=H$ to $(0, {}^+3)$ on the V-axis.

The graph of $V=H-2$ is obtained by moving the graph of $V=H$ to $(0, {}^-2)$ on the V-axis.

The graph of $V=H-6$ is obtained by moving the graph of $V=H$ to $(0, {}^-6)$ on the V-axis.

Studying the graphs in exercise *B*, we notice that the graph of $V={}^-1H$ passes through the origin, and that the graph of $V={}^-1H+5$ is obtained by moving the graph of $V={}^-1H$ to $(0, {}^+5)$ on the V-axis, and that the graph of $V={}^-1H-3$ is obtained by moving $V={}^-1H$ to $(0, {}^-3)$ on the V-axis.

Suppose, however, that the coefficient of H is neither 1 nor $^-1$, as in exercise *C*. Here we notice that the graph of $V=2H-2$ is obtained by moving $V=2H$ to $(0, {}^-2)$ on the V-axis, and that the graph of $V=\frac{1}{2}H+5$ is obtained by moving $V=\frac{1}{2}H$ to $(0, {}^+5)$ on the V-axis. The formula can be written in the form of:

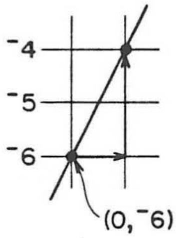
$$V = aH + b$$

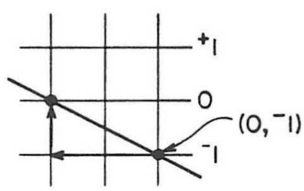
The number a (the coefficient of H) tells the slope of the line, and b gives information about the point at which the line crosses the V-axis.

We could graph the four rules or formulas in exercise *D* by using this procedure:

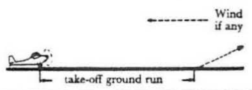
$V = 2H - 6$ . . . passes through $(0, {}^-6)$ on the V-axis; its slope is 2/1, or 1 to the right and up 2.

$V = {}^-\frac{1}{2}H - 1$ . . . passes through $(0, {}^-1)$ on the V-axis; its slope is ${}^-\frac{1}{2}$, or 2 to the left and up 1.





Take-off performance of a light, single-engine aircraft. FPS-60



Altitude in ft.	Weight in lbs.	Air Temp. F°	No. of ft. in ground run at wind velocities shown 0 mph	10 mph	20 mph
At Sea Level	1400	40	790	510	280
	1650	40	1140	770	430
	1400	60	850	560	300
	1650	60	1230	830	470
	1400	80	910	610	320
	1650	80	1330	890	500
2000 ft. above sea level	1400	30	970	650	370
	1550	30	1410	980	580
	1400	50	1030	690	390
	1650	50	1530	1030	600
	1400	70	1090	730	410
	1650	70	1650	1080	620
4000 ft. above sea level	1400	20	1130	770	430
	1650	20	1720	1180	690
	1400	40	1230	830	470
	1650	40	1880	1250	740
	1400	60	1330	890	500
	1650	60	2050	1380	820

(from manufacturer's handbook)

Notice the patterns in the entries given. Can you fill in the missing entries — at least approximations of them?

Study the chart further. What can you learn about problems in flying from the facts given? Remember, pilots always take off into the wind as directly as possible.

As you would expect, when the plane is more heavily loaded, the ground run must be longer. But how do temperature and wind velocity and altitude change the take-off problems? Write down some of your observations.

THE GREATER THE ALTITUDE, THE LONGER THE GROUND RUN. THE HIGHER THE TEMPERATURE THE LONGER THE GROUND RUN. THE GREATER THE WIND VELOCITY THE SHORTER THE GROUND RUN. IN A 20MPH WIND, THE PLANE NEEDS ONLY ABOUT 35% AS LONG A GROUND RUN AS IN NO WIND.

At sea level with a weight of 1400 lbs. and a temperature of 50°, the ground run would be about

820 feet with no wind,

535 feet with a 10 mph wind,

290 feet with a 20 mph wind.

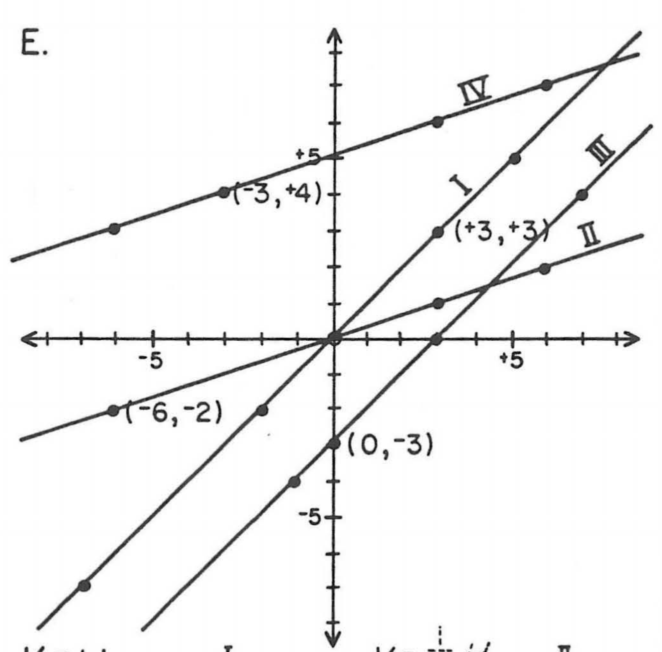
If you knew only the performance information for a weight of 1400 lbs., would it be safe to allow 50% more ground run if the load is actually 1650 lbs.? No (Yes or no)

Explain: IN MOST CASES THAT WOULD NOT BE AS MUCH AS REQUIRED. ADDING 60% WOULD NOT BE ENOUGH IN SOME. ADDING 70% WOULD BE SAFE.

If you knew the performance information at sea level, would you need to allow an additional 100 ft., 150 ft., or 200 ft., of ground run for each additional 1000 feet above sea level? 200 feet.

Explain: WHILE 150 ADDITIONAL FEET IS USUALLY ENOUGH IF THERE IS A WIND, COMPARE A LOAD OF 1650 AT 60° WITH NO WIND—1,230 AT SEA LEVEL, AND 2,050 AT 4,000 FEET.

E.



$V = H$ I

H	+3	-2	+5	-7
V	+3	-2	+5	-7

$V = \frac{1}{3}H$ II

H	+3	+6	-6	0
V	+1	+2	-2	0

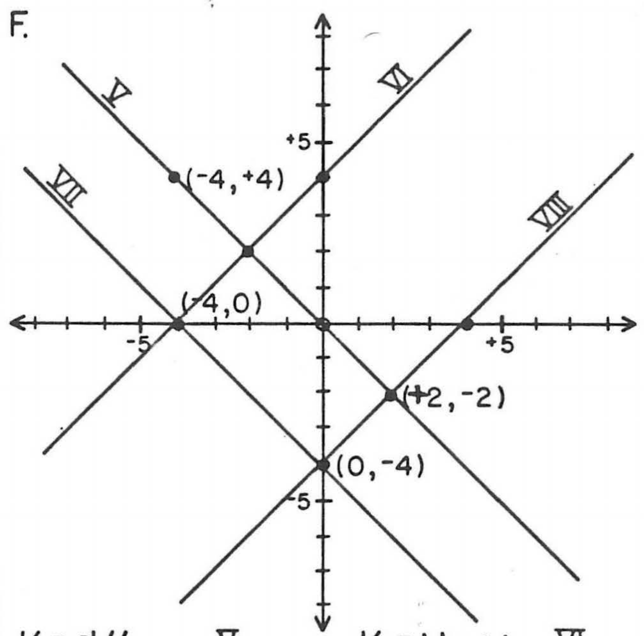
$V = H + {}^{-}3$ III

H	+3	0	+7	-1
V	0	-3	+4	-4

$V = \frac{1}{3}H + 5$ IV

H	+6	-3	+3	-6
V	+7	+4	+6	+3

F.



$V = -1H$ V

H	-4	+2	0	-1 -2
V	+4	-2	0	1 +2

$V = H + 4$ VI

H	-2	-1	0	1	-4
V	+2	3	4	5	0

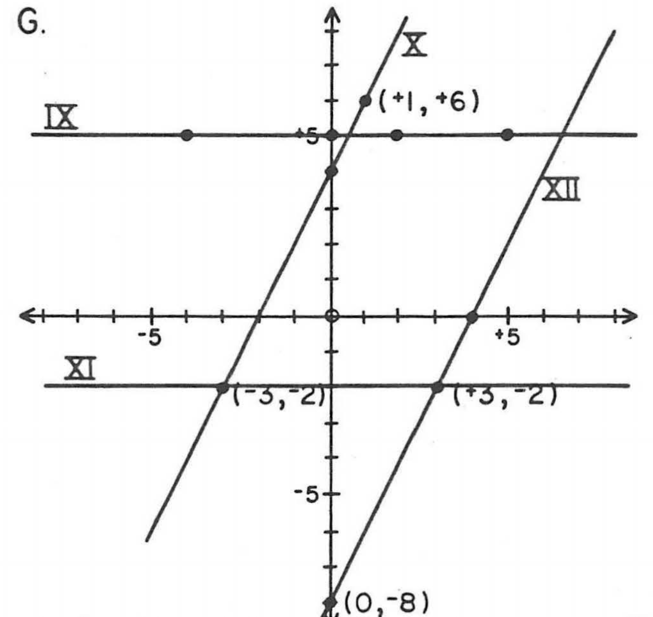
$V = {}^{-}H + {}^{-}4$ VII

H	-4	-3	-2	-1
V	0	-1	-2	-3

$V = H + {}^{-}4$ VIII

H	+2	1	0	-1
V	-2	-3	-4	-5

G.



$V = 0 \times H + 5$ IX

H	+2	0	-4	+5
V	+5	5	+5	5

$V = 2H + 4$ X

H	+1	2	0	3
V	+6	8	4	10

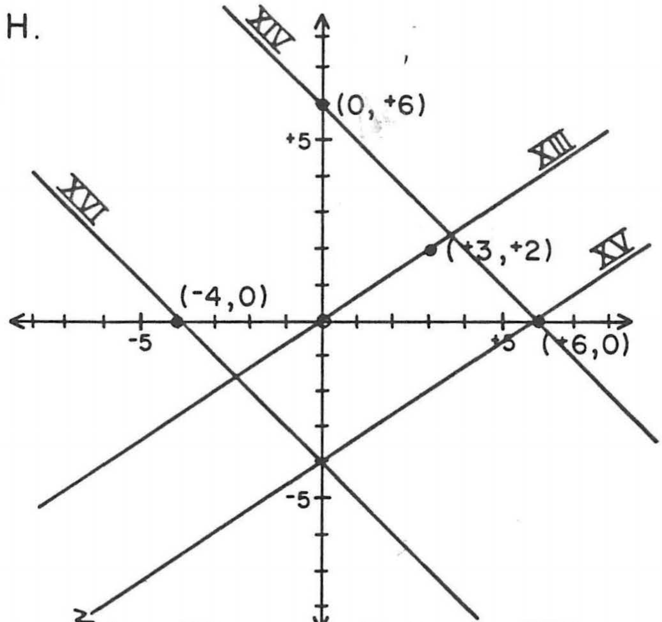
$V = 0 \times H - 2$ XI

H	+3	-3	+2	-2
V	-2	-2	-2	-2

$V = 2H - 8$ XII

H	+3	2	1	0
V	-2	-4	-6	-8

H.



$V = \frac{2}{3}H$ XIII

H	+3	-3	6	-6
V	+2	-2	4	-4

$V = {}^{-}H + 6$ XIV

H	0	1	-1	2
V	+6	5	7	4

$V = \frac{2}{3}H - 4$ XV

H	+6	-6	9	-9
V	0	-8	2	-10

$V = {}^{-}H - 4$ XVI

H	-4	4	-3	3
V	0	-8	-1	-7

Tasks: Students complete tables by using the given graphs and find rules for the graphs.

Purpose: To help students learn to describe and compare precisely the slopes of lines on a graph.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; geometry; measurement.

The Lesson: On pupil page 61 students are given line graphs and asked to complete tables and find rules for the graphs.

One approach is to find certain points by reading the graph and recording the points in the table. Then consider the table as part of a *What's My Rule?* game: Try to find the rule and express it in the form:

$$V = aH + b$$

Such records as:

I

H	$^+3$	$^-2$	$^+5$	$^-7$
V	$^+3$	$^-2$	$^+5$	$^-7$

and

II

H	$^+3$	$^+6$	$^-6$	0
V	$^+1$	$^+2$	$^-2$	0

reveal their rules easily. Obviously, the first indicates that H and V are the same; thus we obtain $V = H$, or $V = 1H + 0$. In the second, V is ⅓ of H; so we have $V = \frac{1}{3}H$, or $V = \frac{1}{3}H + 0$.

Here are the records of III and IV. The last two entries for H and V in table IV depend on individual selections, although we have marked suggested points on the graph:

III

H	$^+3$	0	$^+7$	$^-1$
V	0	$^-3$	$^-1$	$^-4$

and

IV

H	$^+6$	$^-3$	$^+3$	$^-6$
V	$^+7$	$^+4$	$^+6$	$^+3$

While it is rather obvious that the rule for III is $V = H - 3$, IV would be quite difficult as a *What's My Rule?* game.

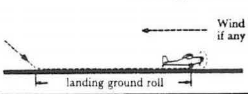
This suggests the need for an alternative approach for IV. Notice that the line crosses the V-axis at $(0, {}^+5)$; so the rule must be of the form $V = aH + 5$. We know from the table that the point $({}^-3, {}^+4)$ is on the line. To get from this point to $(0, {}^+5)$ we could go 3 units to the right and 1 unit up. Thus the line must have a slope of ⅓. We return to $V = aH + 5$, replace a with ⅓, and obtain the rule $V = \frac{1}{3}H + 5$.

Let's look at line XV in this way. It passes through $(0, {}^-4)$ on the V-axis; so the rule is of the form $V = aH - 4$. Another point on the line is $({}^+6, 0)$. We can get from $(0, {}^-4)$ to $({}^+6, 0)$ by going 6 units to the right and 4 units up, so that the slope of line XV is 4/6, or ⅔. Therefore, a rule that would produce this graph is $V = \frac{2}{3}H - 4$.

The work with graphs of formulas on this and preceding pages gives a brief glimpse of mathematics that the student will encounter later on. Mastery of the ideas presented in these pages cannot be expected of all students. But we have introduced methods, terminology, and general ideas that will be useful as we proceed in Level F and as students advance further in mathematics.

Practice sheet 61 is a companion to practice sheet 60. It presents a table that gives information about the landing performance of the same plane referred to on practice sheet 60. Since the entries given in the table do not follow a completely regular pattern, missing entries cannot be exactly predicted. The manufacturer rounds off all entries to the nearest 10; therefore, answers that vary from ours by ±10 ought to be accepted.

Landing performance of a light single-engine aircraft. FPS-61



Altitude in ft.	Weight in lbs.	Air Temp. F°	No. of ft. in ground roll with maximum use of brakes — at wind of 0 mph	10 mph	20 mph
At Sea Level	1400	40	420	340	260
	1650	40	490	400	310
	1400	60	440	360	280
	1650	60	510	420	330
	1400	80	460	350	300
	1650	80	530	430	340
2000 ft. above sea level	1400	30	440	360	280
	1650	30	500	420	330
	1400	50	460	380	290
	1650	50	530	440	340
	1400	70	480	400	300
	1650	70	550	450	350
4000 ft. above sea level	1400	20	460	390	300
	1650	20	540	440	350
	1400	40	480	400	320
	1650	40	560	460	370
	1400	60	500	410	330
	1650	60	580	480	390

(from manufacturer's handbook)

Estimate to find the missing entries. Your results will not be exactly the same as the manufacturer's, but they will be close enough if you follow the pattern suggested by the given information.

The manufacturer has chosen a different range of temperatures for each altitude. Can you think of a reason for this?

TEMPERATURES ARE GENERALLY HIGHER AT SEA LEVEL AND MUCH LOWER AT 4,000 FEET.

Compare the information given in this "landing performance" chart with that in the "take-off performance" chart.

Does the extra 250 lbs. (1650 lbs. rather than 1400 lbs.) have more influence on the "take-off ground run" than on the "landing ground roll"? Explain: IN MANY CASES THE EXTRA LOAD INCREASES THE GROUND RUN BY 50% OR MORE. IT INCREASES THE LANDING ROLL LESS THAN 20%.

Does a wind velocity of 20 mph have more influence on the ground run on take-off than on the ground roll on landing? Explain:

A 20 MPH WIND CUTS THE TAKE-OFF GROUND RUN TO LESS THAN ½ THE RUN WITH NO WIND. A SIMILAR WIND NEVER CUTS ½ OFF THE GROUND ROLL.

With a temperature of 70° F at sea level, the landing roll of the plane with a weight of 1650 lbs. is about

450 ft. with no wind,

355 ft. with 10 mph wind,

290 ft. with 20 mph wind.

With a temperature of 80°F at 4000 ft., the landing roll of the plane with a weight of 1400 lbs. is about

520 ft. with no wind,

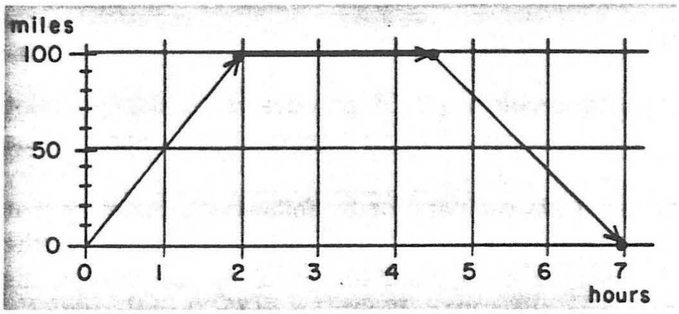
420 ft. with 10 mph wind,

340-350 ft. with 20 mph wind.

Have you a theory to explain why planes require more ground run and ground roll at high altitudes and at high temperatures?

THE AIR IS THINNER - LESS COMPRESSED AT HIGHER ALTITUDES. WARM AIR EXPANDS AND IS THUS THINNED OUT.

Graphs of Trips



The graph above is a summary of distances traveled on a trip from our house to the shore and back.

What can you say?

IT'S 100 MILES TO THE SHORE. WE DROVE IT IN 2 HOURS - AT AN AVERAGE RATE OF 50 MPH. AFTER $2\frac{1}{2}$ HOURS THERE, WE DROVE HOME IN $2\frac{1}{2}$ HOURS AT AN AVERAGE SPEED OF 40 MPH. WE WERE GONE A TOTAL OF 7 HOURS.

The left-hand part of the graph evidently describes the trip to the shore. If the left-hand part had been either of the following, what could you say in each case?

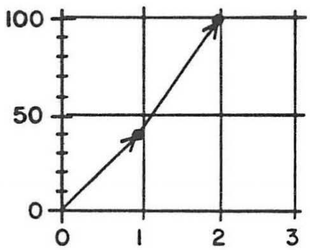


Figure I

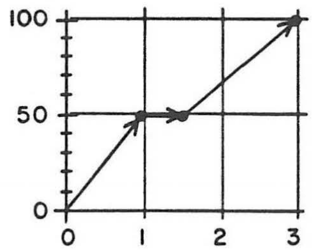


Figure II

What does *Figure I* report? DURING THE FIRST HOUR AN AVERAGE OF 40 MPH WAS MAINTAINED AND 60 MPH DURING THE SECOND HOUR; AN OVERALL AVERAGE OF 50 MPH.

What does *Figure II* report? AVERAGED 50 MPH FOR ONE HOUR AND STOPPED FOR $\frac{1}{2}$ HOUR. DROVE 50 MILES MORE IN $1\frac{1}{2}$ HOURS—AN AVERAGE OF $33\frac{1}{3}$ MPH.

Suppose that the center portion of the graph reported the following. What can you say in each case?

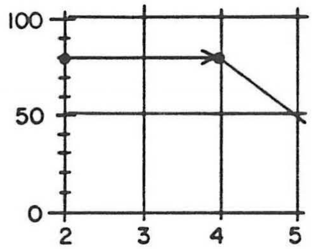


Figure III

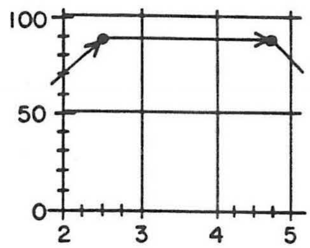


Figure IV

Figure III reports: THE SHORE WAS 80 MILES AWAY AND THE STAY WAS 2 HOURS. THE RETURN TRIP STARTED AT THE RATE OF 30 MPH. AT THAT RATE THEY WILL REACH HOME 6 HOURS AND 40 MINUTES AFTER DEPARTURE.

Figure IV reports: THE SHORE WAS 90 MILES AWAY. THEY DROVE $2\frac{1}{2}$ HOURS TO GET THERE - AN AVERAGE SPEED OF 36 MPH. THEY STAYED THERE FOR 2 HOURS AND 15 MINUTES.

Suppose that the right-hand portion of the graph had reported the following. What can you say?

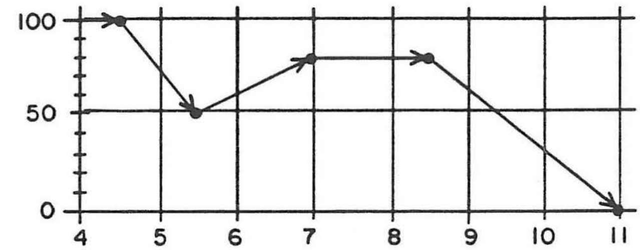


Figure V

Figure V reports: DROVE HALF WAY HOME AT 50 MPH AND THEN DROVE 30 MILES BACK TOWARD THE SHORE AT 20 MPH. STOPPED FOR AN HOUR AND A HALF. DROVE HOME 80 MILES IN $2\frac{1}{2}$ HOURS; AN AVERAGE OF 32 MPH. THEY WERE GONE 11 HOURS.

Tasks: Students study graphic records of trips and write reports on what the graphs represent.

Purpose: To help students learn to describe and compare the slopes of lines on a graphic representation of a real-life situation.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; geometry; measurement.

The Lesson: Up to this point our consideration of graphs has been quite abstract. Also, our axes have used the same scale.

The graph discussed on pupil page 62 concerns a real-life situation. Here, as with most such graphs, the scales are different. What is the significance of the divisions? We notice that the horizontal axis is marked off in hours, from 0 to 7. The vertical axis is marked off in miles, from 0 to 100, and each subdivision represents 10 miles.

As we move from left to right on the graph, we are representing the passage of time from the start of a trip to the end of a trip.

As we move up and down on the graph, the meaning is not quite so obvious. But a little study reveals that movement up in the graph means increasing distance from home and moving down in the graph means decreasing distance from home.

There are three slopes to consider in this graph. Between 0 and 2 hours, the slope is

$$\frac{100 \text{ miles}}{2 \text{ hrs.}}.$$

This slope is obviously the average rate of speed for the trip from home to the shore.

Between 2 hours and 4½ hours, the slope is zero:

$$\frac{0 \text{ miles}}{2\frac{1}{2} \text{ hrs.}} = 0 \text{ mph}.$$

This tells us that we were not changing (in any significant way) our distance from home.

Between 4½ and 7 hours, the slope is

$$\frac{^{-}100 \text{ miles}}{2\frac{1}{2} \text{ hrs.}}, \text{ or } {}^{-}40 \text{ mph}.$$

This is the average rate of speed on the trip from the shore to home. The negative sign of the slope indicates that we are traveling *away* from the shore.

Of course, such a graph generally does not reflect all the details of the actual trip. The slope of 50 mph is only the *average* speed for the first two hours of the trip. Figure *I* indicates 100 miles driven in 2 hours, but the graph is composed of two segments with different slopes:

$$\frac{40 \text{ miles}}{1 \text{ hr.}} \quad \text{and} \quad \frac{60 \text{ miles}}{1 \text{ hr.}}$$

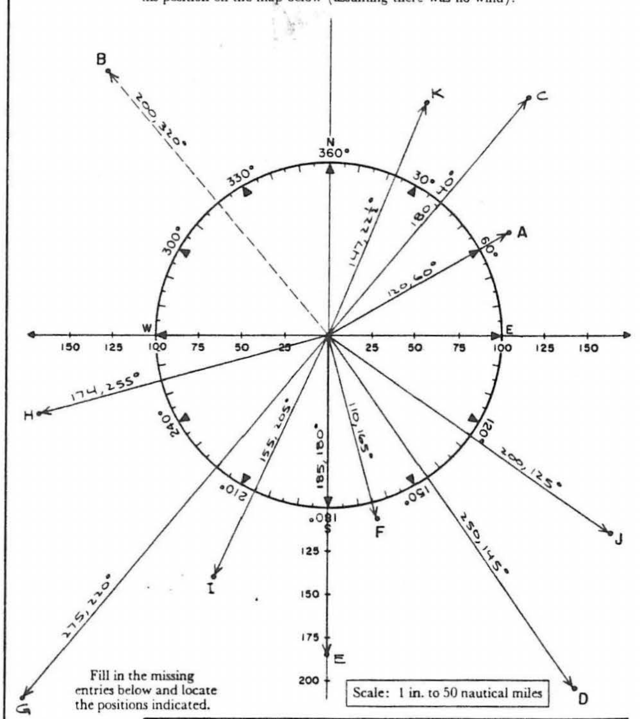
These tell us that the average speed during the first hour was 40 mph and that it was 60 mph during the second hour—an overall average of 100 miles/2 hrs., or 50 mph.

Practice sheet 62 deals with a combination of time, distance, *and* direction. The graph on pupil page 62 is designed so that slope can be interpreted as average rate of speed. On practice sheet 62, the average rate of speed is given or can be computed from information in the chart.

Comparing the two kinds of graphs (pupil page 62 and practice sheet 62) will help students better understand that the kind of graph one uses to represent a situation will vary from case to case.

FPS-62

If a pilot flew a course of 60° at 120 knots, we could indicate his position on the map below (assuming there was no wind).



Fill in the missing entries below and locate the positions indicated.

	A	B	C	D	E	F	G	H	I	J	K
Course	60°	320°	40°	145°	180°	165°	220°	225°	205°	125°	$22\frac{1}{2}$°
Speed in knots	120	100	120	100	$92\frac{1}{2}$	110	100	116	124	120	110
Time elapsed	1 hr.	2 hr.	$1\frac{1}{2}$ hr.	$2\frac{1}{2}$ hr.	2 hr.	1 hr.	$2\frac{3}{4}$ HR.	$1\frac{1}{2}$ HR	$1\frac{1}{4}$ hr.	$1\frac{2}{3}$ hr.	$1\frac{1}{3}$ hr.
Distance from airport in nautical miles	120	200	180	250	185	110	275	174	155	200	147

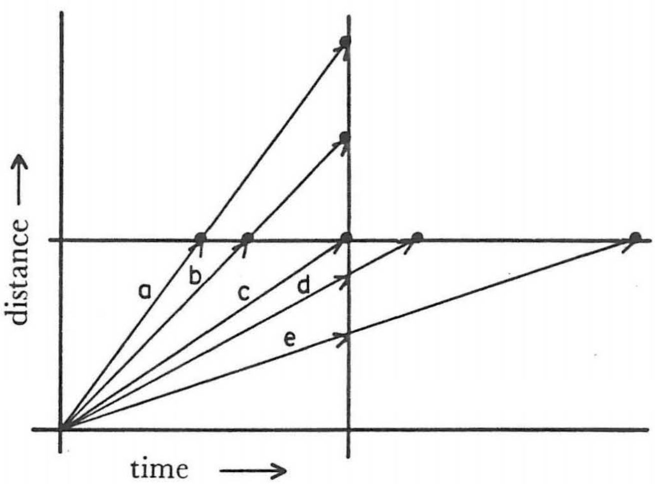
Comparing Speeds

Mr. Allen and Mr. Brown studied records to compare speeds of various cars.

Mr. Allen found out the time it took each to go a certain distance *(see green line)*.

Mr. Brown found out the distance each could go in a certain time *(see blue line)*.

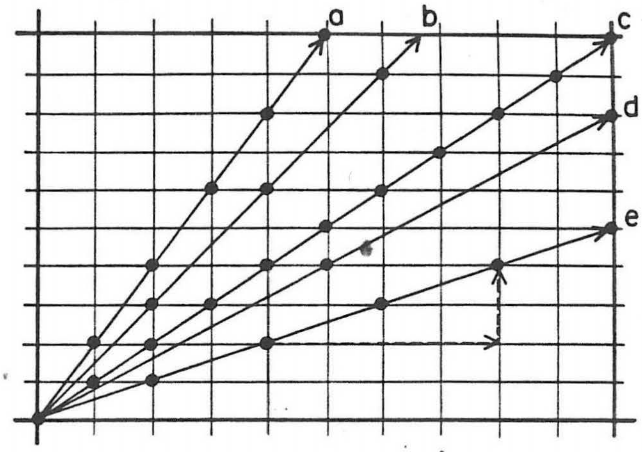
Here is a summary of their results shown in a single chart.



Car a was the fastest and car e was the slowest.

1. Car *a* was about 4 times as fast as car *e*.
2. Car *b* was about 3 times as fast as car *e*.
3. Car *b* was about 2 times as fast as car *d*.
4. Car *a* was about 2 times as fast as car *c*.
5. Car *c* was about $1\frac{1}{4}$ times as fast as car *d*.
6. Car *b* was about $\frac{3}{4}$ times as fast as car *a*.

Let's take a more detailed look at the chart. We shall divide the distance into 10 equal distances and the time into ten equal periods of time.



⟶ over and up ↑

We have marked heavy dots on each black line that passes through crossing points of the lines which indicate units of time and distance.

In each case, how many units must you go to the right and how many units up to move from one point to another? Record your findings:

	a	b	c	d	e
units up	4	3	3	4	2
units over	2	2	3	5	4

Try other points on the black lines to complete this record:

	a	b	c	d	e
units up	2	6	1	8	1
units over	1	4	1	10	2

Try other points to complete this record:

	a	b	c	d	e
units up	8	9	7	4	3
units over	4	6	7	5	6

Summary: Use the closest marked points you can in each case, and complete the following:

	(a)	(b)	(c)	(d)	(e)
$\frac{\text{units up}}{\text{units over}}$	$\frac{2}{1}$	$\frac{3}{2}$	$\frac{1}{1}$	$\frac{4}{5}$	$\frac{1}{2}$

Look back to statements 1 through 6 in the previous column. Would your comments be as true of these five fractions as they were of the speeds? YES

63 Comparing speeds

Tasks: *Students study graphic records of trips and compare slopes.*

Purpose: To help students learn to describe and compare the slopes of lines on a chart representing a real-life situation.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; geometry; measurement.

The Lesson: The scales for the graphs analyzed here are not given. Two men make different measurements to compare the average speeds of five cars. They use the same scale and plot their results on the same chart. Although we cannot tell what the actual speeds of the cars were, we can compare their speeds.

As on pupil page 62, the slopes represent rates of speed; thus it is clear that car *a* was the fastest and car *e* the slowest. The slope for *a* appears to be about twice that for *c*, and the slope for *c* appears to be about twice that for *e*. So, car *a* is about four times as fast as car *e*. These results are obtained by using rough comparisons of time differences or distance differences.

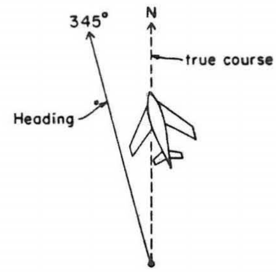
In the right-hand column we enlarge a portion of the chart and draw grid lines. We mark the intersections of the grid lines through which each line passes. It is possible now to make more precise estimates of slopes.

a has a slope of $\frac{4}{2}$ or $\frac{8}{4}$ or $\frac{10}{5}$ . . 2
b has a slope of $\frac{3}{2}$ or $\frac{6}{4}$ or $\frac{9}{6}$ . . . $1\frac{1}{2}$
c has a slope of $\frac{1}{1}$ or $\frac{2}{2}$ or $\frac{3}{3}$ . . . 1
d has a slope of $\frac{4}{5}$ or $\frac{8}{10}$ . . . . $\frac{4}{5}$
e has a slope of $\frac{1}{2}$ or $\frac{2}{4}$ or $\frac{3}{6}$ . . . . $\frac{1}{2}$

Now we can compare the rates of speed very easily and check the original estimates in the left-hand column.

Practice sheet 63 takes up one of the more unfamiliar facts of air navigation: *you don't always travel in the direction you are heading.* This is a notion that is hard for people limited to ground travel to understand. There is almost always a wind blowing—and very seldom is it a direct headwind or a direct tailwind. A direct headwind or tailwind will alter the plane's speed (relative to the earth's surface) but not its direction of travel. A wind in any other direction influences both the speed and the *course* of the plane with respect to the ground.

In problem *III* of practice sheet 63, the pilot wishes to travel to an airport directly north of his starting point. If he wished to travel due north (by using his compass, or by following a highway that runs directly north) and there is a wind from the west, he must head his plane to the west of his destination. "Heading" refers to the direction in which the plane is headed or pointed—not the "true course" it would follow.



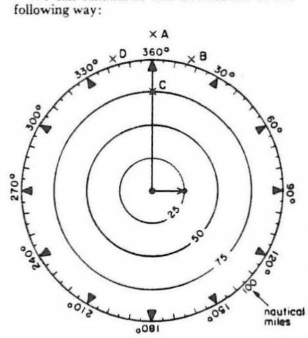
It would seem that if the pilot looked down at a north-south highway, he would soon find his plane west of the highway. Actually, if the wind holds steady, he will remain directly above the highway. The angle of compensation used to take the wind into account is called the *crab angle*.

FPS-63

I

Suppose that a pilot flew for 1 hour at 100 knots on a 360° course at an altitude of 3000 feet and the "winds aloft" weather report was "3-2725" — a wind blowing out of 270° at 25 knots.

We can summarize this information in the following way:

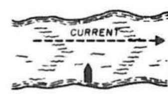


As he flew north (360° or 0°), the wind out of the west would cause him to drift to the east. After flying for 1 hour at 100 knots, his position would be close to one of the crosses labeled *A*, *B*, *C*, and *D*.

Which cross do you think indicates his approximate position after one hour? B (*A*, *B*, *C*, or *D*.)

II

If a river were flowing to the east and a boat were headed straight across, going north, would it land upstream or downstream from the point toward which it was heading?



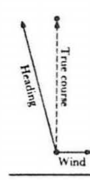
Ans.: DOWNSTREAM
(upstream or downstream)

III

Wise pilots always check the wind reports as they plan their flights. Suppose that a plane's cruising speed is 100 knots and the pilot's destination is 100 nautical miles due north and the wind is out of the west at 25 knots.

He will select a point 25 nautical miles to the west of his destination and set a course for that point. As he flies in that direction, the wind causes him to drift to the east and, as a result, his actual or true course would be due north.

Refer to the sketch in the left column of this page. The pilot would head toward the cross marked *D*, a heading of 345°.

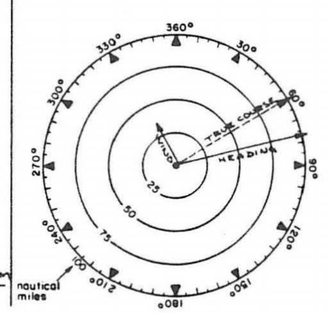


The broken line indicates the path the plane would follow on a map.

IV

True course: 60° Winds: 1535
Cruising speed: 100 knots

The pilot should take a heading of about 75 to 80.

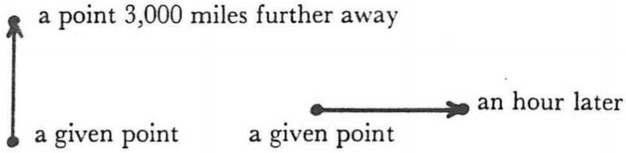


Mixing Time and Space

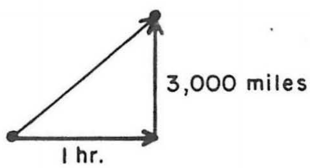
The captain of the space ship reported, "We are now traveling away from Earth at a speed of *3,000 miles per hour.*"

That statement refers to a measure of distance in space and a measure of distance in time.

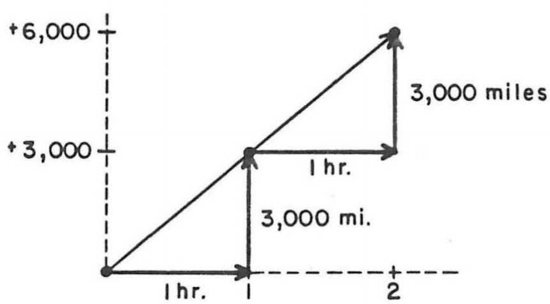
Suppose that we wish to draw a sketch of this event. We can let the length of a vertical arrow represent a change in distance away from Earth, and the length of a horizontal arrow represent a change in time.



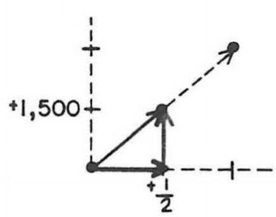
Let's combine these two measurements in this way:

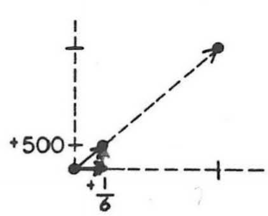


We shall say that the black arrow above suggests the idea that as an hour passed, the space ship traveled 3,000 miles. In two hours, this change would have happened twice.



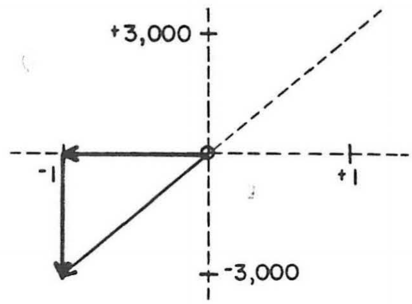
And we could indicate what would have happened in a half-hour, or in 10 minutes.





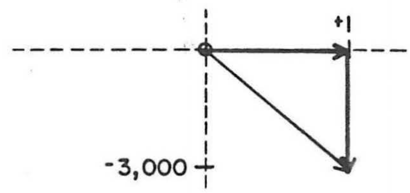
We can also extend our sketch to include some history of the flight.

Where were we an hour ago?

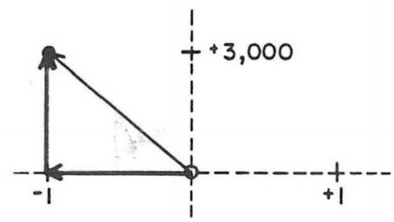


An hour ago (⁻1 hr.) we were 3,000 miles less distant (⁻3,000) from Earth.

If our return journey is at the same speed, as time passes the distance from Earth becomes less.

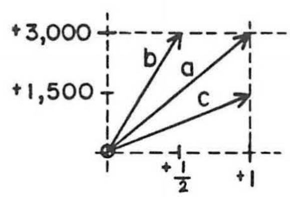


The sketch above shows that in 1 hour we shall be 3,000 miles less distant (⁻3,000) from Earth.



The sketch above suggests the idea that AN HOUR AGO (*an hour ago* or *an hour from now*) we WERE (*were* or *will be*) 3,000 miles FURTHER FROM (*further from* or *closer to*) Earth.

We are space ship *a*. Two other space ships, *b* and *c*, are the same distance we are from Earth and traveling in the same direction.



What information does the diagram suggest? Which space ship will go 3,000 miles in the least time? b Which will go the shortest distance in an hour? C. If our speed in

(continued)

64-65 Mixing time and space

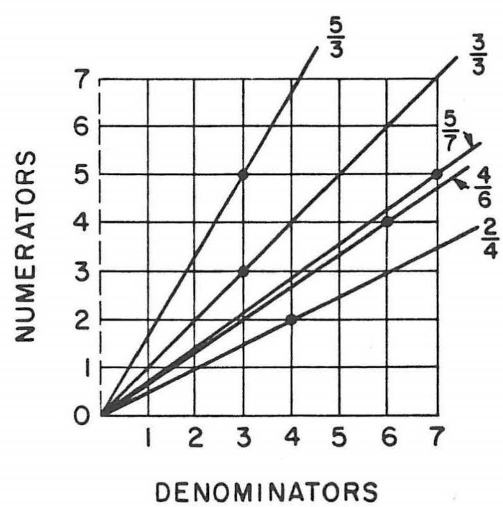
Tasks: *Students study graphic records of a space voyage and report on what the records represent.*

Purpose: To help students learn to describe and compare the slopes of lines on a chart representing a real-life situation.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; geometry; measurement.

The Lesson: On pupil pages 64 and 65, we ask the students to imagine a trip into outer space. The rocket navigator draws a graph to record the information he has collected. The horizontal axis indicates time; 0 is present, $^{+}3$ is 3 hours in the future, $^{-}3$ is 3 hours in the past. The vertical axis represents distance from earth; 0 is our present distance from earth, $^{+}3{,}000$ is 3,000 miles more distant from earth, and $^{-}3{,}000$ is 3,000 miles closer to earth.

You may wish to discuss the idea of slope and its relationship to a comparison of fractions. Let the distances to the right of the vertical axis indicate denominators and distances up from the horizontal axis indicate numerators.



The intersection of a pair of grid lines represents a fraction with a whole-number numerator and a whole-number denominator. Notice that the zero for the horizontal axis has been deleted—zero cannot be used as a denominator.

We have selected a few intersections and drawn the lines through them and the zero-point on the vertical axis. The steeper the slope, the larger the fraction. A quick glance is enough to reveal, for example, that $\frac{5}{7} > \frac{4}{6}$ and that $\frac{5}{3}$ is the largest fraction we selected and $\frac{2}{4}$ is the smallest.

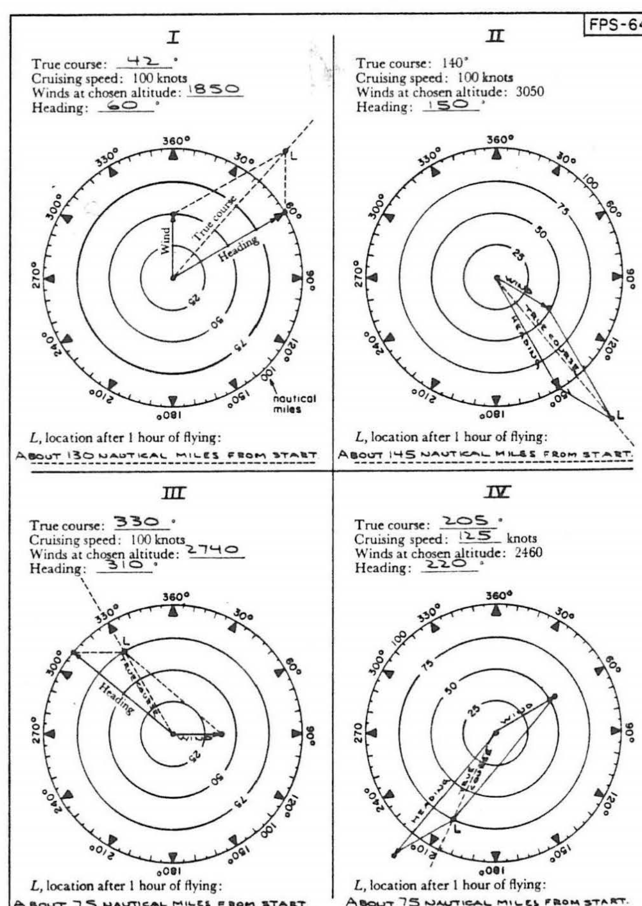
If the diagram were infinitely large horizontally and vertically, there would be an intersection for every fraction (with whole-number denominator and numerator). And if we drew the line for each, we could use the diagram to compare any two fractions.

We can also tell visually whether the fraction we assign to a point is in reduced form. We look to see if the line through (0,0) and our point passes through any intersection points between these two points.

$\frac{2}{4}$ . . . passes through $\frac{1}{2}$
$\frac{4}{6}$ . . . passes through $\frac{2}{3}$
$\frac{3}{3}$ . . . passes through $\frac{2}{2}$, $\frac{1}{1}$

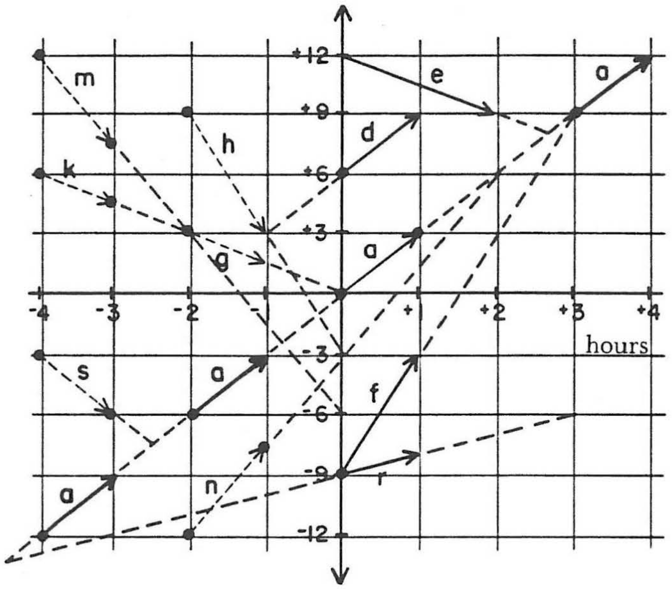
Notice that the intersection points on a "fraction line" indicate equivalent fractions.

(This discussion is continued in the Notes on Pupil Page 65.)



a is 3,000 mph then ship *b* must be traveling about 6,000 mph, and *c* about 1,500 mph.

In the control room of our space ship one of the crew plots some of the information he has about other space ships on the same route. (*The vertical axis shows distances in thousands of miles.*)



The broken-line arrows indicate information plotted 2 hours ago (*h, g, n*) and 4 hours ago (*m, k, s*). Distances from Earth are given in thousands of miles.

We are in space ship *a*. Other space ships going in the same direction as we are d, f, n, AND r. Ships returning to Earth are m, k, s, h, g, ~~AND e~~.

The approximate speeds of the various space ships are:

a.	3,000 mph	*h.*	6,000 mph
d.	3,000 mph	*k.*	1,500 mph
e.	1,500 mph	*m.*	4,500 mph
f.	6,000 mph	*n.*	4,500 mph
g.	1,500 mph	*r.*	1,000 mph

s. 3,000 mph

Have we met any space ships during the past 4 hours? If so, which ones? S, m, k, h, g.

Which space ship did we pass more than 4 hours ago? r

Which ship shall we meet in the next 4 hours? e. Which ships will pass us during the next 4 hours? n AND f.

We met n less than a half-hour ago, and m a little less than an hour ago. It's been 2½ hours since we met ship *s*.

At present, we are 9,000 miles away from *r*, and in 3 hours we shall be 15,000 miles from *r*.

There are evidently two space ships traveling close together. They are k and g. Should we be able to see them at this time? YES.

Ship d is 6,000 miles ahead of us and traveling in the same direction and at the same speed.

What can you say about ships *f* and *r*?

f IS 5,000 MILES AHEAD OF r.

What can you say about

m and *k*? m PASSED k 2 HOURS AGO AND THEY ARE 6,000 MILES APART.

d and *h*? THEY MUST HAVE MET AN HOUR AGO AND ARE 9,000 MILES APART.

e and *k*? THEY ARE TRAVELING AT THE SAME SPEED AND DIRECTION AND ARE 12,000 MILES APART.

the two fastest ships? THEY ARE f AND h. THEY WILL MEET EACH OTHER IN ABOUT 30 MINUTES.

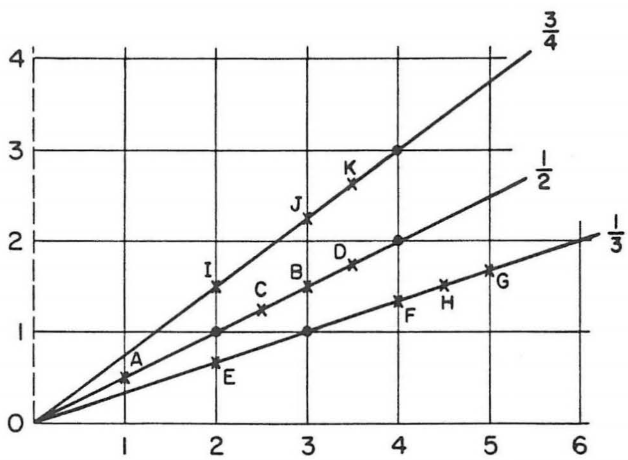
e, g, and *k*? ALL SAME SPEED. WE MET g AND k AND e IS 12,000 MILES AHEAD OF US. k AND g WILL ARRIVE AT EARTH ABOUT 8 HOURS AHEAD OF e.

Tasks: Students read graphic records of space voyages and compare slopes to determine direction, location, speed, and time of space ships.

Purpose: To help students learn to describe and compare the slopes of lines on a graph representing real-life situations.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; geometry; measurement.

The Lesson: So far in our "fraction chart" we have only considered the intersection points of the grid lines. Let's enlarge the scale of the chart, put in some intermediate grid lines, and consider some other points on the "fraction lines."



If all the original intersection points on the ½ line represent equivalent fractions then we are led to suspect that points A, B, C, and D also represent "fractions" equivalent to ½. A is 1 to the right and up ½; B is over 3 and up 1½; C is over 2½ and up 1¼; D is over 3½ and up 1¾. So, since the ½ line has just one slope, we can say that

$$\frac{\frac{1}{2}}{1} = \frac{1\frac{1}{2}}{3} = \frac{1\frac{1}{4}}{2\frac{1}{2}} = \frac{1\frac{3}{4}}{3\frac{1}{2}} = \frac{1}{2}.$$

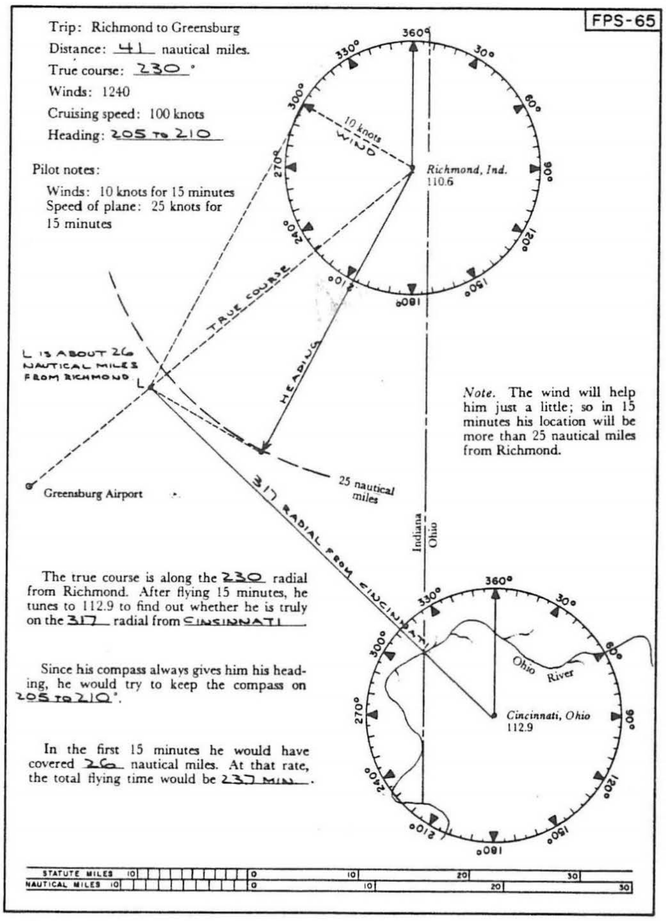
By similar observations, points E, F, G, and H represent fractions equivalent to ⅓:

$$\frac{\frac{2}{3}}{2} = \frac{1\frac{1}{3}}{4} = \frac{1\frac{2}{3}}{5} = \frac{1\frac{1}{2}}{4\frac{1}{2}} = \frac{1}{3}$$

Points I, J, and K lead us to write:

$$\frac{1\frac{1}{2}}{2} = \frac{2\frac{1}{4}}{3} = \frac{2\frac{5}{8}}{3\frac{1}{2}} = \frac{3}{4}$$

We shall continue this discussion in the following chapter, which considers several approaches to handling fractional divisors.



Your authors disagree about what is the most helpful approach to considering

Division by a Fraction.

We call our different approaches by the following names:

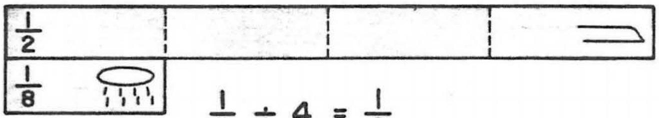
I. Sketches and Common Sense
II. Testing a Theory
III. Undoing Multiplication
IV. Mathematical Reasoning

Rather than selecting one of these approaches as better than the others, we are going to outline all of them for you — and you can decide for yourself which one is most helpful.

However, your authors are in complete agreement about this: that consideration of all four approaches is much better than concentrating on any one by itself.

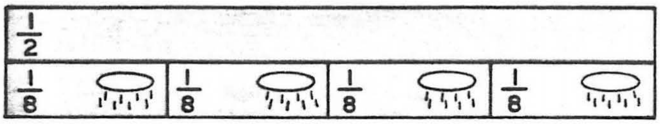
Approach I: Sketches and Common Sense

An ancient Egyptian (see page 33) might ask, "If I took a stick $\frac{1}{2}$ unit long and divided it by 4, how long would each of the pieces be?"



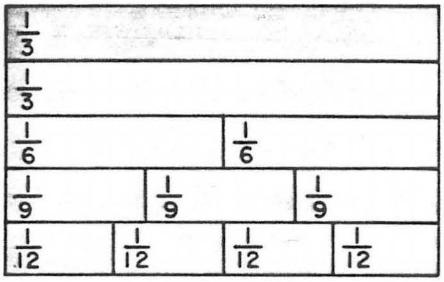
$$\frac{1}{2} \div 4 = \frac{1}{8}$$

Or, he might ask, "How many $\frac{1}{8}$ blocks does it take to make a train as long as a $\frac{1}{2}$ block?"



$$\frac{1}{2} \div \frac{1}{8} = \underline{4}$$

Suppose that an Egyptian carried out a series of experiments to see how many of his smaller blocks it would take to make trains each as long as the $\frac{1}{3}$ unit block. Suppose that the ones he found to fit were:



$$\frac{1}{3} \div \frac{1}{3} = 1 \qquad \frac{1}{3} \div \frac{1}{9} = \underline{3}$$

$$\frac{1}{3} \div \frac{1}{6} = \underline{2} \qquad \frac{1}{3} \div \frac{1}{12} = \underline{4}$$

There is 1 one-third in one-third. There are 2 one-sixths in one-third. There are 3 one-ninths in one-third. Etc.

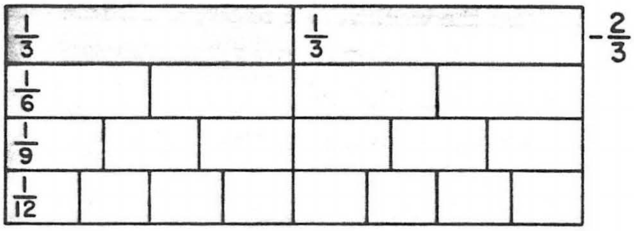
Or, he might have reported the same results in this way:

$$1 \times \frac{1}{3} = \frac{1}{3} \qquad 3 \times \frac{1}{9} = \frac{3}{9} = \frac{1}{3}$$

$$2 \times \frac{1}{6} = \frac{2}{6} = \frac{1}{3} \qquad 4 \times \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$$

To conserve space, let's use smaller sketches.

How many thirds, sixths, ninths, and twelfths are there in two-thirds?



In $\frac{2}{3}$ there are 4 sixths, 6 ninths, etc.

$$\frac{2}{3} \div \frac{1}{6} = \underline{4} \qquad \frac{2}{3} \div \frac{1}{9} = \underline{6} \qquad \frac{2}{3} \div \frac{1}{12} = \underline{8}$$

or

$$\underline{4} \times \frac{1}{6} = \frac{2}{3} \qquad \underline{6} \times \frac{1}{9} = \frac{2}{3} \qquad \underline{8} \times \frac{1}{12} = \frac{2}{3}$$

Look at the sketch above. How many twelfths in $\frac{1}{6}$? How many in $\frac{3}{9}$? In $\frac{3}{6}$?

$$\frac{1}{6} \div \frac{1}{12} = \underline{2} \qquad \frac{3}{9} \div \frac{1}{12} = \underline{4} \qquad \frac{3}{6} \div \frac{1}{12} = \underline{6}$$

or

$$\underline{2} \times \frac{1}{12} = \frac{1}{6} \qquad \underline{4} \times \frac{1}{12} = \frac{3}{9} \qquad \underline{6} \times \frac{1}{12} = \frac{3}{6}$$

66-67 Approach I - sketches and common sense

Tasks: Students complete statements involving division by a fraction and the related multiplication statements.

Purpose: To help students understand division by a fraction by interpreting diagrams.

Unifying Ideas: Structure; multiplication and division; measurement.

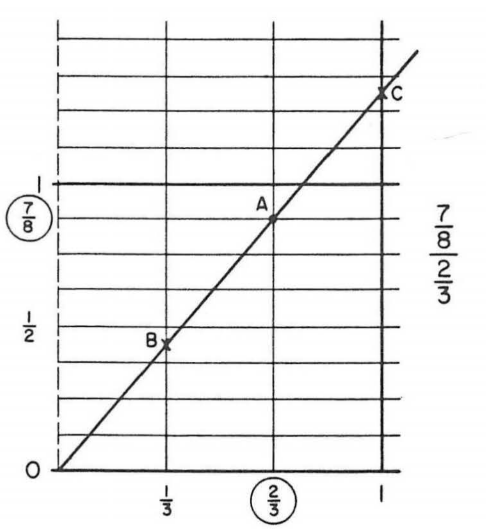
The Lesson: The authors want to emphasize the paragraph in the middle of the left-hand column of pupil page 66. We are fully convinced of the importance of exploring several approaches to division by a fraction. We hope you will also consider the approach for which the basis was laid in the Notes on Pupil Pages 64 and 65—the slope method. This method is perhaps the least familiar to teachers and is not discussed in the text pages. We recommend it only if you feel it will be genuinely useful in your particular class.

$$\frac{7}{8} \div \frac{2}{3} \text{ and } 1\frac{1}{2} \div \frac{2}{3} \text{ and } 2 \div \frac{3}{4}$$

Each of these quotients can be written as a complex fraction, which can be thought of as a slope:

$$\frac{\frac{7}{8}}{\frac{2}{3}} \text{ and } \frac{1\frac{1}{2}}{\frac{2}{3}} \text{ and } \frac{2}{\frac{3}{4}}$$

Let's tackle the first one. We begin with a portion of a fraction grid, marking off the vertical axis in eighths and the horizontal axis in thirds:



We have drawn a line whose slope is the quotient $\frac{7}{8} \div \frac{2}{3}$. Any way of specifying the slope is as good as another. So we try to find on the line a point that yields a simple description. Point A corresponds to $\frac{7}{8}$ over $\frac{2}{3}$. What about point B? This point tells us that the slope can be represented by:

$$\frac{\frac{3\frac{1}{2}}{8}}{\frac{1}{3}} \text{ or } \frac{\frac{7}{16}}{\frac{1}{3}}$$

That's not much better. So, let's try point C—which will give us a whole-number denominator, 1. C is 1 over and

$$1 + \frac{2\frac{1}{2}}{8} \text{ up or } 1\frac{5}{16} \text{ up}$$

So we say that the slope is also equal to

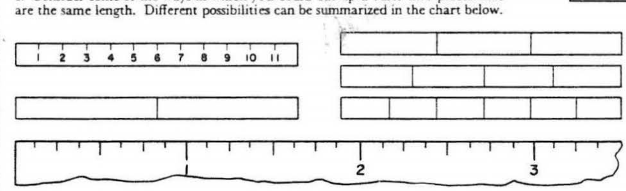
$$\frac{1\frac{5}{16}}{1} \text{ or } 1\frac{5}{16}$$

$$\text{So, } \frac{7}{8} \div \frac{2}{3} = \frac{\frac{7}{8}}{\frac{2}{3}} = 1\frac{5}{16}$$

Practice sheet 67 can be used to develop this approach. (See the Notes on Pupil Page 67.)

FPS-66

I. Consider some of the ways in which you could cut up a ruler into pieces that are the same length. Different possibilities can be summarized in the chart below.



Ways of cutting a 12-inch ruler into pieces of the same length.

	1	2	3	4	5	6	7	8	9	10	11	12
Length of pieces	6 inches	4 inches	12 inches	3 inches	2 inches	1 inch	$\frac{1}{2}$ inch	$\frac{1}{4}$ inch	$\frac{1}{8}$ inch	$\frac{1}{3}$ inch	$\frac{2}{3}$ inch	$\frac{3}{4}$ inch
Number of pieces	2	3	1	4	6	12	24	48	96	36	18	16

II. Complete the following.

1. In twelve dollars there are 24 half-dollars; 48 quarters; 120 dimes; 240 nickels; 1,200 pennies.

2. If you can make a jump on the moon $\frac{1}{4}$ of a mile long, it would take 4 jumps to go 1 mile, 2 jumps to go a half-mile, 1 jumps to go a quarter of a mile, 8 jumps to go two miles, 16 jumps to go 4 miles, and 12 jumps to go 3 miles.

3. If 12 gallons of milk is poured into half-gallon containers, it would fill 24 of them; it would fill 12 one-gallon containers; 48 quart containers; 96 pint containers.

4. If a yardstick is cut up into pieces of equal length, there would be 12 pieces 3 inches long; 72 pieces $\frac{1}{2}$ inch long; 18 pieces 2 inches long; 144 pieces $\frac{1}{4}$ inch long.

III. Write examples to fit the following headlines: *

1. $24 \div \frac{1}{2} = 48$ HOW MANY HALF-INCHES ARE THERE IN TWO FEET?

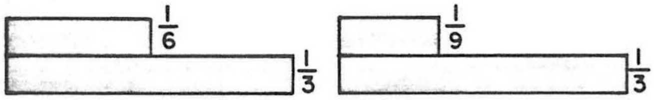
2. $6 = 2 \div \frac{1}{3}$ A TWO INCH LENGTH IS DIVIDED INTO SIX PIECES OF EQUAL LENGTH. THE LENGTH OF EACH PIECE IS ____ INCHES.

3. $4 \div \frac{2}{3} = 6$ A FOUR INCH LENGTH IS DIVIDED INTO EQUAL LENGTHS. HOW MANY PIECES WILL THERE BE IF EACH IS $\frac{2}{3}$ INCHES LONG?

* MANY POSSIBLE ANSWERS.

How many thirds in one-sixth? How many thirds in one-ninth?

$\frac{1}{6} \div \frac{1}{3} = \underline{\frac{1}{2}}$ $\qquad$ $\frac{1}{9} \div \frac{1}{3} = \underline{\frac{1}{3}}$



Clearly, the answer in each case must be less than 1. If we think of our unit as 1 yard then

$\frac{1}{3}$ yd. = 12 in. or 1 ft.; $\frac{1}{6}$ yd = $\underline{6}$ in.;
$\frac{1}{9}$ yd. = $\underline{4}$ in.

How many feet in 6 inches? In 4 inches? Clearly, there's a half-foot in 6 inches and a third of a foot in 4 inches.

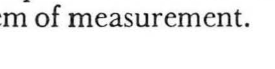
Let's try to restate each question asked about numbers as a question about some familiar and suitable system of measurement.

$2 \div \frac{1}{2} = \underline{4}$; $\frac{3}{4} \div \frac{1}{2} = \underline{1\frac{1}{2}}$; $\frac{1}{4} \div \frac{1}{2} = \underline{\frac{1}{2}}$

Suppose that you think of money. How many half-dollars in two dollars? How many half-dollars in 75¢, or $\frac{3}{4}$ of a dollar? How many half-dollars in a quarter?

$1\frac{1}{2} \div \frac{1}{4} = \underline{6}$; $\frac{3}{4} \div \frac{1}{4} = \underline{3}$; $\frac{1}{3} \div \frac{1}{4} = \underline{1\frac{1}{3}}$

How many quarter-dozens in a dozen and a half? Or, how many groups of 3 eggs in 18 eggs? How many groups of 3 eggs in 9 eggs? How many groups of 3 eggs in 4 eggs? Here is a picture of this last situation.



You can see that there is one group of 3 and a third of a group of three, or one and one-third groups of 3 eggs in 4 eggs.

$2 \div \frac{2}{7} = \underline{7}$; $\frac{6}{7} \div \frac{2}{7} = \underline{3}$; $\frac{1}{7} \div \frac{2}{7} = \underline{\frac{1}{2}}$

How many 2-day periods in 2 weeks? How many 2-day periods in 6 days? How many 2-day periods in 1 day?

$\frac{3}{4} \div \frac{1}{100} = \underline{75}$; $\frac{3}{10} \div \frac{1}{20} = \underline{6}$; $\frac{1}{10} \div \frac{1}{4} = \underline{\frac{2}{5}}$

How many pennies in 75¢? How many nickels in 3 dimes? How many quarters in a dime?

$\frac{1}{2} \div \frac{3}{16} = \underline{\frac{8}{3}}$; $\frac{1}{8} \div \frac{3}{16} = \underline{\frac{2}{3}}$; $\frac{3}{4} \div 1\frac{1}{2} = \underline{\frac{1}{2}}$

How many 3-oz. packages in half a pound? How many 3-oz. packages in 2 oz.? How many pound-and-a-halves in $\frac{3}{4}$ of a pound? How many 24-oz. packages can be made from 12 oz.?

$2\frac{2}{3} \div \frac{2}{3} = \underline{4}$; $1 \div \frac{2}{3} = \underline{1\frac{1}{2}}$; $\frac{7}{8} \div \frac{2}{3} = \underline{1\frac{5}{16}}$

How many 2-ft. lengths in 2 yards and 2 feet? How many 2-ft. strips in a yard?

Since $\frac{7}{8}$ of a yard is not best suited to help think about the third problem, let's talk about $\frac{7}{8}$ of a 24-hour day. How many $\frac{2}{3}$-of-a-day (16 hours) are there in $\frac{7}{8}$ of a day (21 hours)? How many 16's are there in 21?

$\frac{3}{8} \div \frac{1}{4} = \underline{1\frac{1}{2}}$; $\frac{3}{8} \div \frac{1}{2} = \underline{\frac{3}{4}}$; $\frac{3}{8} \div \frac{3}{4} = \underline{\frac{1}{2}}$

Write questions in which the same computing is involved as the mathematical statements above and below. Use familiar measurement situations such as ounces, hours, and inches.

HOW MANY 4 OZ. PACKAGES IN 6 OZ.?
HOW MANY 8 OZ. PACKAGES IN 6 OZ.?
HOW MANY 12 OZ. PACKAGES IN 6 OZ.?
HOW MANY 6 HR. PERIODS, 12 HR. PERIODS, AND 18 HR. PERIODS IN 9 HR?
HOW MANY QUARTER INCHES, HALF INCHES, AND THREE-QUARTER INCHES IN THREE-EIGHTHS OF AN INCH?

$\frac{1}{4} \div \frac{1}{5} = \underline{1\frac{1}{4}}$; $\frac{1}{5} \div \frac{3}{10} = \underline{\frac{2}{3}}$; $\frac{7}{20} \div \frac{7}{100} = \underline{5}$

HOW MANY POUNDS CAN YOU BUY WITH A QUARTER AT 20¢ A POUND?
HOW MUCH OF 30¢ DOES IT TAKE FOR A 20¢ ITEM? HOW MANY 7¢ ITEMS CAN BE PURCHASED WITH 7 NICKELS?

Tasks: Students complete statements involving division by a fraction. They write questions based on measurements that correspond to six statements.

Purpose: To help students understand division by a fraction by interpreting examples through diagrams and real-life measurement situations.

Unifying Ideas: Structure; multiplication and division; geometry; measurement.

The Lesson: Since the text material on pupil page 67 is self-explanatory, we shall discuss some of the examples that use the slope approach of practice sheet 67.

First, scales appropriate for the examples are marked on the axes, and the "slope lines" are drawn through the points labeled A, C, F, and I. (See the copy of practice sheet 67 at the bottom of the next column.) Then the grid lines are drawn.

Second, we find the points where the slope lines cross the first of the vertical grid lines: points A, D, G, and J. These points are on the grid lines corresponding to $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{7}$, and $\frac{1}{2}$, respectively.

Third, we find the points at which the slope lines cross the vertical grid lines that correspond to 1: points B, E, H, and K.

Fourth, we find the distances of these points above the horizontal axis. (We know that these points are 1 unit from the vertical axis.)

B is twice as high as A: $\frac{3}{4} \times 2 = \frac{6}{4} = \frac{3}{2} = 1\frac{1}{2}$
E is four times as high as D: $\frac{1}{6} \times 4 = \frac{4}{6} = \frac{2}{3}$
H is seven times as high as G: $\frac{3}{20} \times 7 = \frac{21}{20} = 1\frac{1}{20}$
K is twice as high as J: $\frac{8}{9} \times 2 = \frac{16}{9} = 1\frac{7}{9}$

Since each of these is the numerator of a fraction that describes the slope for the example, and since the denominator in each case is 1, we can write:

$$\frac{\frac{3}{4}}{\frac{1}{2}} = 1\frac{1}{2} \qquad \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3} \qquad \frac{\frac{3}{5}}{\frac{4}{7}} = 1\frac{1}{20} \qquad \frac{2\frac{2}{3}}{1\frac{1}{2}} = 1\frac{7}{9}$$

In finding the points A, D, G, and J, we, in effect, *divided* the distance between (0,0) and A by 1, between (0,0) and C by 3, between (0,0) and F by 4, and between (0,0) and I by 3. Then we *multiplied* the vertical distance of points A, D, G, and J by 2, 4, 7, and 2, respectively.

Let's write each complex fraction in the usual manner—as one simple fraction divided by another:

$$\overset{*}{\frac{3}{4}} \div \underset{\#}{\frac{1}{2}} = ? \qquad \frac{1}{2} \div \frac{3}{4} = ? \qquad \frac{3}{5} \div \frac{4}{7} = ? \qquad \frac{8}{3} \div \frac{3}{2} = ?$$

Restating what we did:

* We divided 1, 3, 4, and 3, respectively.
* We multiplied by 2, 4, 7, and 2, respectively.

We might describe our manipulations in this way:

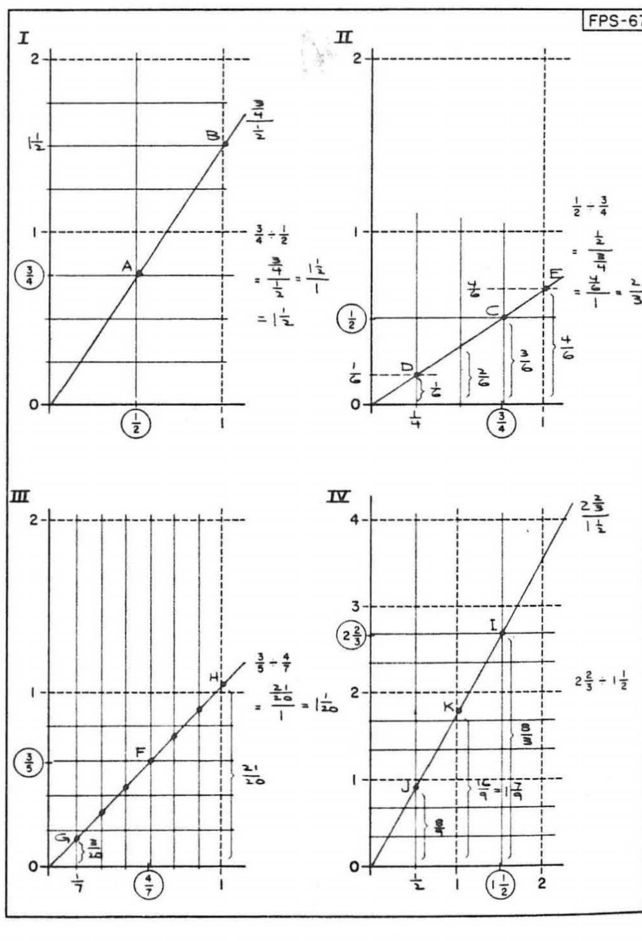
$$\left(\frac{3}{4} \div 1\right) \times 2 = 1\frac{1}{2} \qquad \left(\frac{1}{2} \div 3\right) \times 4 = \frac{2}{3}$$

$$\left(\frac{3}{5} \div 4\right) \times 7 = 1\frac{1}{20} \qquad \left(\frac{8}{3} \div 3\right) \times 2 = 1$$

Another way to call for the same computing is to write:

$$\frac{3}{4} \times \frac{2}{1} = ? \qquad \frac{1}{2} \times \frac{4}{3} = ? \qquad \frac{3}{5} \times \frac{7}{4} = ? \qquad \frac{8}{3} \times \frac{2}{3} = ?$$

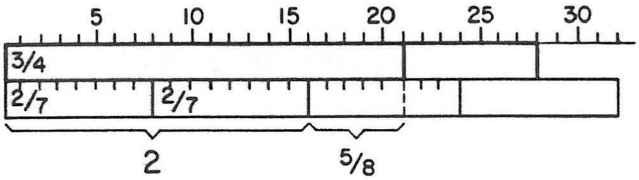
(Look ahead to pupil pages 70 and 71 and practice sheet 71.)



But, suppose that you can't think of any familiar situations where the numbers used would come up naturally?

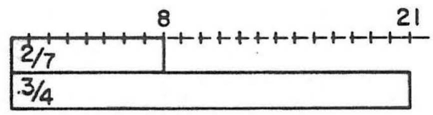
$\frac{3}{4} \div \frac{2}{7} = 2\frac{5}{8}$; $\quad \frac{2}{7} \div \frac{3}{4} = \frac{8}{21}$

Invent a situation. Since 28 is the smallest number that is a multiple of both 4 and 7, start with a stick that's 28 units long. ($\frac{3}{4}$ of 28 is 21, and $\frac{2}{7}$ of 28 is 8.) How many chunks 8 units long are there in a chunk 21 units long?



There are $2\frac{5}{8}$ chunks each 8 units long in a 21-unit chunk. Or, one might say that there are $(21 \div 8)$ chunks.

How many 21-unit chunks are there in an 8-unit chunk; or, how much of a 21-unit chunk is there in an 8-unit chunk?



There are $8 \div 21$, or $\frac{8}{21}$, of a 21-unit chunk in an 8-unit chunk.

Use any method you wish to complete the following.

$\frac{2}{3} \div \frac{1}{9} = 6$	$\frac{3}{4} \div \frac{1}{8} = 6$	$1\frac{2}{5} \div \frac{1}{5} = 7$
$\frac{2}{3} \div \frac{2}{9} = 3$	$\frac{3}{4} \div \frac{1}{4} = 3$	$1\frac{2}{5} \div \frac{2}{5} = 3\frac{1}{2}$
$\frac{2}{3} \div \frac{3}{9} = 2$	$\frac{3}{4} \div \frac{2}{4} = 1\frac{1}{2}$	$1\frac{2}{5} \div \frac{1}{10} = 14$
$\frac{2}{3} \div \frac{4}{9} = 1\frac{1}{2}$	$\frac{3}{4} \div \frac{3}{4} = 1$	$1\frac{2}{5} \div \frac{3}{10} = 4\frac{2}{3}$
$\frac{2}{3} \div \frac{5}{9} = 1\frac{1}{5}$	$\frac{3}{4} \div \frac{5}{4} = \frac{3}{5}$	$\frac{5}{8} \div 2 = \frac{5}{16}$
$\frac{2}{3} \div \frac{6}{9} = 1$	$\frac{3}{4} \div \frac{9}{4} = \frac{1}{3}$	$\frac{5}{8} \div \frac{1}{2} = 1\frac{1}{4}$
$\frac{2}{3} \div \frac{7}{9} = \frac{6}{7}$	$\frac{3}{4} \div \frac{11}{4} = \frac{3}{11}$	$\frac{7}{9} \div 5 = \frac{7}{45}$
$\frac{2}{3} \div \frac{8}{9} = \frac{3}{4}$	$\frac{3}{8} \div \frac{11}{4} = \frac{3}{22}$	$\frac{7}{9} \div \frac{1}{5} = \frac{35}{9}$ OR $3\frac{8}{9}$

Approach II: Testing a Theory

Bill's theory:

"In multiplication or division, you can invert the multiplier or divisor and change the operation sign."

Before we can test that theory, we need to know exactly what he means.

Bill explains. "By 'invert' I mean write the number as a fraction and interchange the numerator and denominator. By 'change the operation sign' I mean change the inverted multiplier to a divisor, or the inverted divisor to a multiplier. According to my theory, the following are true statements, and they illustrate what I mean."

$2 \times \frac{1}{3} = 2 \div \frac{3}{1}$ $\qquad \frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1}$

$7\frac{1}{2} \div 2\frac{1}{2} = 7\frac{1}{2} \times \frac{2}{5}$ $\qquad 6 \times 4 = 6 \div \frac{1}{4}$

Try your hand with Bill's "invert and change the operation sign" theory.

$8 \times \frac{1}{5} = 8 \div 5$ $\qquad \frac{2}{3} \div \frac{2}{3} = \frac{2}{3} \times \frac{3}{2}$

$9 \div 9 = 9 \times \frac{1}{9}$ $\qquad \frac{1}{2} \times 2 = \frac{1}{2} \div \frac{1}{2}$

$\frac{3}{4} \div \frac{2}{7} = \frac{3}{4} \times \frac{7}{2}$ $\qquad \frac{2}{7} \div \frac{3}{4} = \frac{2}{7} \times \frac{4}{3}$

"You see," Bill bragged, "it works every time!"

"Yes," Anne agreed, "but you made up the examples. Let me make some up. I know that $2 \times 3 = 3 \times 2 = 6$. Let me try your theory."

$2 \times 3 = 2 \div \frac{1}{3}$ and $3 \times 2 = 3 \div \frac{1}{2}$

or $2 \times 3 = 2 \div \frac{1}{3} = 3 \times 2 = 3 \div \frac{1}{2} = 6$

"Is that true?"

Bill smiled. "Try some harder examples."

68-69 Approach II - testing a theory

Tasks: Students complete statements involving division by a fraction and translate multiplication problems into division problems, and vice versa.

Purpose: To help students understand division by a fraction by using diagrams to interpret examples. To help students understand division by a fraction by testing a "rule" for division.

Unifying Ideas: Structure; multiplication and division; geometry; measurement.

The Lesson: Before leaving the slope approach for division by a fraction, consider practice sheet 68. This can be understood as the multiplication variation of the activities discussed in the Notes on Pupil Pages 66 and 67. If, however, you have decided against including a discussion of slopes then look at the page as a series of problems in pictorial form.

In exercise *I*, the portion of the vertical axis that represents $\frac{3}{4}$ is divided into 4 parts of equal length. Each must be a fourth of $\frac{3}{4}$, or $\frac{3}{16}$. Knowing this, we can say that the subdivision marks correspond to $\frac{3}{16}$, $\frac{3}{16} \times 2 = \frac{3}{8}$, and $\frac{3}{16} \times 3 = \frac{9}{16}$.

From the slope point of view, the completed diagram for exercise *I* shows that

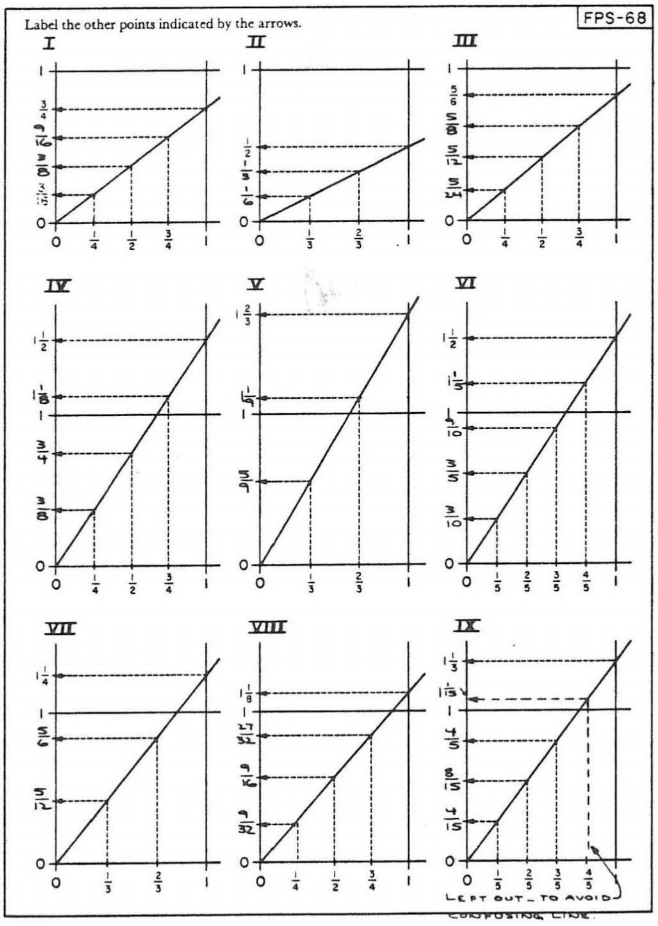
$$\frac{3}{16} \div \frac{1}{4} = \frac{3}{8} \div \frac{1}{2} = \frac{9}{16} \div \frac{3}{4} = \frac{3}{4} \div 1 = \frac{3}{4}$$

Each of the nine exercises on practice sheet 68 can be solved with the same method.

On pupil pages 68 and 69, we attack our basic problem of dividing by a fraction by stating and testing a rule. Very often the end result of an investigation is summarized in a rule or a mnemonic device. Such is the case with the "invert the divisor and multiply" rule. Many "cookbook" authors state this rule and then try to pound it into the students with endless exercises. The student may learn how to use the rule, but this approach gives him little insight into where the rule comes from or why it works. The "invert the divisor and multiply" approach is worthwhile, but it should not be treated as a rule to be accepted and followed without question. Rather, it should be treated as a *theory* to be *tested*. This is *Approach II* to the problem of dividing by a fraction. (See the Notes on Pupil Page 66.)

Bill states his theory briefly, and then explains it. Notice that his theory also covers the problem of multiplying by a fraction—invert the divisor and multiply *or* invert the multiplier and divide. This expansion of the more traditional rule is not especially helpful in computation. However, the expanded rule does focus attention on the fact that multiplication and division are inverse operations.

After Bill has stated his theory, Anne insists on testing it. The division computations that follow can be carried out in any way that makes sense to the students—for example, by *Approach I* or by the slope method.



"All right," Anne agreed.

$$2 \times \frac{1}{2} = \frac{1}{2} \times 2 = 1$$

$$2 \times \frac{1}{2} = 2 \div 2 = \frac{1}{2} \times 2 = \frac{1}{2} \div \frac{1}{2} = 1$$

$$\frac{2}{3} \times \frac{3}{4} = \frac{2}{3} \div \frac{4}{3} = \frac{3}{4} \times \frac{2}{3} = \frac{3}{4} \div \frac{3}{2} = \underline{\frac{1}{2}}$$

$$7 \times 3 = 7 \div \frac{1}{3} = 3 \times 7 = 3 \div \frac{1}{7} = \underline{21}$$

Bill was all smiles. "Let me show you some examples I used to check my theory.

"Since $\frac{7}{8} \times \frac{3}{5} = \frac{21}{40}$ and division undoes multiplication, I am sure that

$$\frac{21}{40} \div \frac{3}{5} = \frac{7}{8} \quad \text{and} \quad \frac{21}{40} \div \frac{7}{8} = \frac{3}{5}.$$

"Since $\frac{9}{4} \times \frac{7}{27} = \frac{63}{108} = \frac{7}{12}$, I know that

$$\frac{7}{12} \div \frac{9}{4} = \frac{7}{27} \quad \text{and} \quad \frac{7}{12} \div \frac{7}{27} = \frac{9}{4} = 2\frac{1}{4}.$$

"Try my theory on these examples."

$$\frac{21}{40} \div \frac{3}{5} = \frac{21}{40} \times \frac{5}{3} = \frac{7}{8} \times \frac{1}{1} = \frac{7}{8}$$

$$\frac{21}{40} \div \frac{7}{8} = \frac{21}{40} \times \frac{8}{7} = \frac{3}{5} \times \frac{1}{1} = \frac{3}{5}$$

$$\frac{7}{12} \div \frac{9}{4} = \frac{7}{12} \times \frac{4}{9} = \frac{7}{3} \times \frac{1}{9} = \frac{7}{27}$$

$$\frac{7}{12} \div \frac{7}{27} = \frac{7}{12} \times \frac{27}{7} = \frac{1}{4} \times \frac{9}{1} = \frac{9}{4} = 2\frac{1}{4}$$

"I really think my theory will always hold up," Bill concluded.

Anne looked puzzled — almost convinced — but not completely sure. "Too bad it doesn't apply where the fractions are decimal fractions," she sighed.

"Oh, but it will," Bill assured her. "Sometimes it is a real shortcut. For example, I find that dividing by .5 is the same as multiplying by 2:

$$27 \div .5 = 27 \div \frac{5}{10} = 27 \times \frac{10}{5} = 27 \times 2 = 54$$

and multiplying by .5 is the same as dividing by 2:

$$27 \times .5 = 27 \times \frac{5}{10} = 27 \div \frac{10}{5} = 27 \div 2 = \underline{13.5}$$

"Multiplying by .25 is the same as dividing by 4, and dividing by .25 is the same as multiplying by 4:

$$36 \times .25 = 36 \times \frac{25}{100} = 36 \times \frac{1}{4} = 36 \div 4 = \underline{9}$$

$$36 \div .25 = 36 \div \frac{25}{100} = 36 \div \frac{1}{4} = 36 \times 4 = \underline{144}$$

"Further, I have less trouble with keeping decimal points in order in examples like this:

$$18.9 \div .07 = 18.9 \div \frac{7}{100} = 18.9 \times \frac{100}{7} = \frac{1890}{7} = \underline{270}$$

"I'm exploring lots of examples using my theory and looking for shortcuts. Also, I'm keeping my eyes open for an example in which my theory wouldn't work. I hope I don't find any.

"Here are some of the shortcuts I've found. I use a double-headed arrow ($\longleftrightarrow$) as an abbreviation for *is the same as.*"

$\times .25 \longleftrightarrow \div 4$	$\div .2 \longleftrightarrow \times 5$
$\div \frac{1}{4} \longleftrightarrow \times 4$	$\times \frac{1}{8} \longleftrightarrow \div 8$
$\div .05 \longleftrightarrow \times 20$	$\times .5 \longleftrightarrow \div 2$
$\div 1\frac{1}{3} \longleftrightarrow \times \frac{3}{4}$	$\div 1.5 \longleftrightarrow \times \frac{2}{3}$
$\div 10\frac{1}{3} \longleftrightarrow \times \frac{3}{31}$	$\div \frac{2}{3} \longleftrightarrow \times \frac{3}{2}$
$\div 1.25 \longleftrightarrow \times \frac{4}{5}$	$\div 3\frac{1}{3} \longleftrightarrow \times \frac{3}{10}$
$\div \frac{3}{17} \longleftrightarrow \times \frac{17}{3}$	$\div 1\frac{5}{9} \longleftrightarrow \times \frac{9}{14}$
$\div 3\frac{1}{7} \longleftrightarrow \times \frac{7}{22}$	$\div 92\frac{2}{3} \longleftrightarrow \times \frac{3}{278}$

Many people refer to part of Bill's theory by saying:

If the divisor is a common fraction, *invert the divisor and multiply.*

Tasks: Students complete statements involving division by a fraction (common and decimal) and investigate the idea that dividing by a fraction and multiplying by it are inverses of each other.

Purpose: To help students understand division by a fraction by testing Bill's theory.

Unifying Ideas: Structure; multiplication and division; functions and relations; measurement.

The Lesson: It soon becomes clear on pupil page 69 that Bill has done some testing of his own before announcing his theory.

He plunges into difficult problems starting with the idea that if $x \times y = z$ then $z \div y = x$ and $z \div x = y$, which defines division for whole numbers ($x \neq 0$, $y \neq 0$). Bill's argument assumes that these relationships hold for fractions as well as for whole numbers. This assumption amounts to a definition of division for fractions.

Any multiplication sentence (not involving 0 and not involving equal factors) leads to two equivalent division sentences, with the product as the dividend in each case. Will Bill's theory produce the desired results? By the definition of division for fractions,

$$\text{since} \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd},$$

$$\frac{ac}{bd} \div \frac{a}{b} = \frac{c}{d}$$

$$\text{and} \quad \frac{ac}{bd} \div \frac{c}{d} = \frac{a}{b}$$

Now does Bill's theory produce the same results? Yes, since by his theory we obtain:

$$\frac{ac}{bd} \div \frac{a}{b} = \frac{\cancel{a}c}{\cancel{b}d}\boxed{\times \frac{\cancel{b}}{\cancel{a}}} = \frac{c}{d}$$

$$\frac{ac}{bd} \div \frac{c}{d} = \frac{a\cancel{c}}{b\cancel{d}}\boxed{\times \frac{\cancel{d}}{\cancel{c}}} = \frac{a}{b}$$

This is the essence of the argument Bill uses to support his rule. He argues from particular examples, but each exemplifies the general statement above.

Another way of testing Bill's theory is to apply it to situations in which we already know the outcome. Does his theory produce the proper results in such situations? For example, if any number divided by itself is 1 then

$$\frac{1}{2} \div \frac{1}{2} = 1 \qquad \frac{3}{8} \div \frac{3}{8} = 1 \qquad 1\frac{3}{5} \div 1\frac{3}{5} = 1$$

$$\frac{1}{2}\boxed{\times \frac{2}{1}} = 1 \qquad \frac{3}{8}\boxed{\times \frac{8}{3}} = 1 \qquad \frac{8}{5}\boxed{\times \frac{5}{8}} = 1.$$

So Bill's theory seems to pass this test.

We also know that

$$24 \div 8 = 3 \qquad 27 \div 3 = 9 \qquad 50 \div 2 = 25$$

$$24\boxed{\times \frac{1}{8}} = 3 \qquad 27\boxed{\times \frac{1}{3}} = 9 \qquad 50\boxed{\times \frac{1}{2}} = 25.$$

"Invert the divisor and multiply" also passes.

As the students put the theory through more tests, they are likely to discover some interesting shortcuts. On pupil page 69, Bill passes on some of the shortcuts he came across.

By the time a student has thought up a great number of tests of Bill's theory, he has a growing conviction that it is sound. He also gets valuable practice in "inverting and multiplying"—a process he will use more often in algebraic manipulation.

Let's talk about FPS-69

	DOLLARS and PIES.	Arithmetic sentences that are suggested by the statements on the left.
1	Half of two dollars is 1 dollar.	$2 \times \frac{1}{2} = 1$ and $1 \div 2 = \frac{1}{2}$
2	In two dollars there are 4 half-dollars.	$2 \div \frac{1}{2} = 4$ and $4 \times \frac{1}{2} = 2$
3	Half of a half-dollar is $\frac{1}{4}$ of a dollar.	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ and $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2}$
4	In a half-dollar there are 2 quarters.	$\frac{1}{2} \div \frac{1}{4} = 2$ and $2 \times \frac{1}{4} = \frac{1}{2}$
5	In two pies there are 12 sixths of a pie.	$2 \div \frac{1}{6} = 12$ and $12 \times \frac{1}{6} = 2$
6	One-fourth of 2 dollars is $\frac{1}{2}$ of a dollar.	$2 \times \frac{1}{4} = \frac{1}{2}$ and $\frac{1}{2} \div \frac{1}{4} = 2$
7	Three-fourths of 2 dollars is $1\frac{1}{2}$ dollars.	$2 \times \frac{3}{4} = 1\frac{1}{2}$ and $1\frac{1}{2} \div \frac{3}{4} = 2$
8	Three-fourths of 2 pies is $1\frac{1}{2}$ pies.	$2 \times \frac{3}{4} = 1\frac{1}{2}$ and $1\frac{1}{2} \div \frac{3}{4} = 2$
9	In $1\frac{1}{2}$ pies there are 3 half-pies.	$1\frac{1}{2} \div \frac{1}{2} = 3$ and $3 \times \frac{1}{2} = 1\frac{1}{2}$
10	In $2\frac{1}{2}$ pies there are 10 quarter-pies.	$2\frac{1}{2} \div \frac{1}{4} = 10$ and $10 \times \frac{1}{4} = 2\frac{1}{2}$
11	One-third of a half-pie is $\frac{1}{6}$ of a pie.	$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ and $\frac{1}{6} \div \frac{1}{3} = \frac{1}{2}$
12	Twice $\frac{3}{4}$ of a dollar is $1\frac{1}{2}$ dollars.	$\frac{3}{4} \times 2 = 1\frac{1}{2}$ and $1\frac{1}{2} \div 2 = \frac{3}{4}$
13	Two-thirds of $4\frac{1}{2}$ dollars is 3 dollars.	$4\frac{1}{2} \times \frac{2}{3} = 3$ and $3 \div \frac{2}{3} = 4\frac{1}{2}$
14	In \$10.00 there are 2 \$5.00.	$10 \div 5 = 2$ and $2 \times 5 = 10$
15	In \$10.00 there are $3\frac{1}{3}$ \$3.00.	$10 \div 3 = 3\frac{1}{3}$ and $3\frac{1}{3} \times 3 = 10$
16	In $1\frac{1}{4}$ pies there are 10 eighths of a pie.	$1\frac{1}{4} \div \frac{1}{8} = 10$ and $10 \times \frac{1}{8} = 1\frac{1}{4}$
17	In $1\frac{1}{4}$ pies there are 5 fourths of a pie.	$1\frac{1}{4} \div \frac{1}{4} = 5$ and $5 \times \frac{1}{4} = 1\frac{1}{4}$
18	In $1\frac{1}{4}$ pies there are $2\frac{1}{2}$ halves of a pie.	$1\frac{1}{4} \div \frac{1}{2} = 2\frac{1}{2}$ and $2\frac{1}{2} \times \frac{1}{2} = 1\frac{1}{4}$
19	In $1\frac{1}{4}$ pies there are $1\frac{2}{3}$ three-fourths of a pie.	$1\frac{1}{4} \div \frac{3}{4} = 1\frac{2}{3}$ and $1\frac{2}{3} \times \frac{3}{4} = 1\frac{1}{4}$
20	In \$1.00 there are $2\frac{1}{2}$ two-fifths of a dollar.	$1 \div \frac{2}{5} = 2\frac{1}{2}$ and $2\frac{1}{2} \times \frac{2}{5} = 1$
21	In \$1.00 there are $1\frac{1}{4}$ four-fifths of a dollar.	$1 \div \frac{4}{5} = 1\frac{1}{4}$ and $1\frac{1}{4} \times \frac{4}{5} = 1$
22	In a half-dollar there are $1\frac{1}{4}$ two-fifths of a dollar.	$\frac{1}{2} \div \frac{2}{5} = 1\frac{1}{4}$ and $1\frac{1}{4} \times \frac{2}{5} = \frac{1}{2}$
23	In \$.75 there are $1\frac{7}{8}$ two-fifths of a dollar.	$\frac{3}{4} \div \frac{2}{5} = 1\frac{7}{8}$ and $1\frac{7}{8} \times \frac{2}{5} = \frac{3}{4}$

Approach III: Undoing Multiplication

Multiplying by a fraction can be considered as "giving instructions to do something."

"How many eggs are there in one-fourth of a dozen?" 3 eggs.

"How many cents are there in one-fourth of a dollar?" 25 cents.

"How many students are there in one-fourth of a class of 28?" 7 students.

Each of these questions asks you to perform an experiment physically or mentally — you are asked to arrange a certain number of things in four groups of the same size and report the number you find in a group.

If we decide to write down the multiplier as the second factor, here are records of the questions:

$12 \times \frac{1}{4} = 3 \quad 100 \times \frac{1}{4} = 25 \quad 28 \times \frac{1}{4} = 7$

The computation you performed could also be described equally well by these records:

$12 \div 4 = 3 \quad 100 \div 4 = 25 \quad 28 \div 4 = 7$

Suppose that the questions had been: What is three-fourths of a dozen eggs; of 100 cents; of 28 students?

You would probably start out in the same way — to find one-fourth and then multiply by 3. We could record your activity in several ways:

$12 \times \frac{1}{4} \times 3 = 9 \quad 100 \times \frac{1}{4} \times 3 = 75 \quad 28 \times \frac{1}{4} \times 3 = 21$

$(12 \div 4) \times 3 = 9 \quad (100 \div 4) \times 3 = 75 \quad (28 \div 4) \times 3 = 21$

and the usual form:

$12 \times \frac{3}{4} = 9 \quad 100 \times \frac{3}{4} = 75 \quad 28 \times \frac{3}{4} = 21$

If we use N in place of the number of things under consideration, and b as the denominator part of the instructions, and a as the numerator part, we can easily describe this way of multiplying by a fraction by writing the formula:

$$N \times \frac{a}{b} = (N \div b) \times a$$

"How many inches in five-ninths of a yard?" In this case, N is 36, a is 5, and b is 9.

$$N \times \frac{a}{b} = (N \div b) \times a\text{; so, } 36 \times \frac{5}{9} = (36 \div 9) \times 5 = 20$$

In multiplying by a fraction, we carry out two instructions: perform a division and perform a multiplication.

If you analyze your thinking in the previous examples, you probably carried out the division part of the instructions first, and then the multiplication: $(N \div b) \times a$. Suppose that you had reversed the order. Would it have affected the outcome?

$$(N \div b) \times a = (N \times a) \div b$$

Are the following statements true?

$(36 \div 9) \times 5 = (36 \times 5) \div 9 = 20$

$(12 \div 4) \times 3 = (12 \times 3) \div 4 = 9$

Test other examples. Are you certain that the order doesn't affect the outcome? YES

In fact, you may have been using this idea whenever dividing first would cause complications.

How many days in two-thirds of a week?

$7 \times \frac{2}{3} = (7 \div 3) \times 2 = (7 \times 2) \div 3 = 4\frac{2}{3}$

In the above formula, could N hold a place for a fraction? Does the following formula work?

$$\frac{c}{d} \times \frac{a}{b} = \left(\frac{c}{d} \div b\right) \times a = \left(\frac{c}{d} \times a\right) \div b$$

Try some examples.

How many dozen eggs in two-thirds of a half-dozen?

$\frac{1}{2} \times \frac{2}{3} = \left(\frac{1}{2} \div 3\right) \times 2 = \left(\frac{1}{2} \times 2\right) \div 3 = \frac{1}{3}$

How many dozen is half of two-thirds of a dozen eggs?

$\frac{2}{3} \times \frac{1}{2} = \left(\frac{2}{3} \div 2\right) \times 1 = \left(\frac{2}{3} \times 1\right) \div 2 = \frac{1}{3}$

70-71 Approach III - undoing multiplication

Tasks: *Students complete statements involving multiplication by a fraction.*

Purpose: To help students explore multiplication by a fraction as a sequence of two operations: a division by the denominator, and a multiplication by the numerator (the order of applying these operations being irrelevant).

Unifying Ideas: Structure; multiplication and division; functions and relations; measurement.

The Lesson: In many discussions and activities we have viewed division as "undoing" multiplication:

$$\text{if } a \times b = c \text{ then } c \div b = a$$

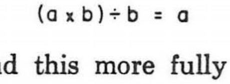
$$(a \times b) \div b = a$$

To understand this more fully where fractions are multipliers or divisors, let us analyze what we mean when we multiply by $\frac{2}{3}$.

What do you think when you answer the question: How many minutes is $\frac{2}{3}$ of an hour? Probably you say that $\frac{1}{3}$ of an hour is 20 minutes ($60 \div 3 = 20$) and $\frac{2}{3}$ of an hour is twice as much, or 40 minutes ($20 \times 2 = 40$).

So you interpreted "$\times \frac{2}{3}$" as a pair of instructions: first "$\div 3$" and then "$\times 2$." Of course, you would get the same results if you carried out those operations in the reverse order. There are 120 minutes in 2 hours ($60 \times 2 = 120$) and 40 minutes in a third of 120 minutes ($120 \div 3 = 40$).

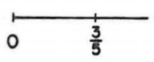
How would you explain the way we locate the point $\frac{3}{5}$ on a number line?



We divide the segment AB into 5 pieces of the same size and consider each subdivision as a fifth. The right-hand end point of the third subdivision we consider as $\frac{3}{5}$. We divided by 5 and multiplied by 3. And we might have reversed the order:

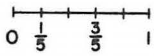
We might have multiplied 1 unit by 3 and divided the result by 5 to locate $\frac{3}{5}$.

Now consider a related problem: The only points labeled on a number line are the points for 0 and $\frac{3}{5}$.



How would you go about finding the unit point?

Simple! You just reverse the process you used to locate $\frac{3}{5}$ in the earlier problem. You divide the given segment by 3 and multiply by 5 (or multiply by 5 and divide by 3):



In the diagram, we have divided by 3 to find $\frac{1}{5}$ and then multiplied by 5 to find 1 unit.

How many minutes in an hour if $\frac{2}{3}$ of an hour is 40 minutes? Divide by 2 and multiply by 3.

We interpret the expressions $1 \times \frac{3}{5}$ and $60 \times \frac{2}{3}$ to mean

$$(1 \div 5) \times 3 = \frac{3}{5} \quad \text{and} \quad (60 \div 3) \times 2 = 40$$

Now, to undo those acts, we apply the inverses of the operations (in either order) to the results:

$$\left(\frac{3}{5} \div 3\right) \times 5 = 1 \quad \text{and} \quad (40 \div 2) \times 3 = 60$$

or

$$\left(\frac{3}{5} \times 5\right) \div 3 = 1 \quad \text{and} \quad (40 \times 3) \div 2 = 60$$

(Continued in the Notes on Pupil Page 71.)

FPS-70

Complete the statements below	. . . and describe a situation in which the same computing is involved.
1. $4 \div \frac{2}{3} = 6$	In a 4-yard length of ribbon there are 6 pieces, each $\frac{2}{3}$ of a yard long.
2. $4 \times \frac{2}{3} = 2\frac{2}{3}$	The lake is 4 miles away. WE HAVE GONE $\frac{2}{3}$ OF THE WAY – $2\frac{2}{3}$ MILES.
3. $2\frac{1}{4} \div \frac{1}{2} = 4\frac{1}{2}$	I have $2.25. AT 50¢ A POUND, I COULD BUY $4\frac{1}{2}$ POUNDS OF MEAT.
4. $1\frac{1}{2} \div \frac{3}{4} = 2$	I have 6 gallons of milk. I COULD FILL TWO 3-QUART CONTAINERS.
5. $\frac{1}{2} \div \frac{3}{4} = \frac{2}{3}$	I have a half-gallon of milk. I COULD FILL A 3-QUART CONTAINER $\frac{2}{3}$ FULL.
6. $\frac{3}{4} \div \frac{1}{2} = 1\frac{1}{2}$	I have three-fourths of a gallon of milk. I COULD FILL $1\frac{1}{2}$ HALF-GALLON CONTAINERS.
7. $1\frac{1}{4} \div \frac{2}{5} = 3\frac{1}{8}$	I wish to buy a book that cost $1.25. I CAN SAVE $\frac{2}{5}$ OF A DOLLAR OR 40¢ A WEEK. IT WILL TAKE $3\frac{1}{8}$ WEEKS
8. $1\frac{1}{4} \div \frac{3}{5} = 2\frac{1}{12}$	The cloth costs 60¢ a yard. WITH $1.25 I CAN BUY 2 YARDS AND 3 INCHES OR $2\frac{1}{12}$ YARDS.

Insert two of these five signs (+, −, ×, ÷, =) *
to make each of the following into a true statement.

A	B	C	D
1. $\frac{14}{2} + \frac{9}{3} = \frac{10}{1}$	1. $\frac{7}{8} + \frac{1}{8} = \frac{1}{1}$	1. $\frac{1}{2} - \frac{1}{8} = \frac{3}{8}$ $\frac{1}{2} = \frac{1}{8} + \frac{3}{8}$	1. $\frac{4}{6} + \frac{1}{3} = \frac{1}{1}$
2. $\frac{4}{2} \times \frac{3}{1} = \frac{6}{1}$	2. $\frac{18}{3} \div \frac{8}{4} = \frac{3}{1}$	2. $\frac{2}{3} \times \frac{2}{6} = \frac{2}{9}$	2. $\frac{4}{2} \times \frac{3}{6} = \frac{1}{1}$
3. $\frac{3}{4} = \frac{1}{1} - \frac{1}{4}$	3. $\frac{1}{4} = \frac{3}{8} - \frac{1}{8}$	3. $\frac{9}{8} - \frac{3}{8} = \frac{3}{4}$	3. $\frac{0}{7} + \frac{3}{9} = \frac{1}{3}$
4. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$	4. $\frac{16}{1} = \frac{8}{1} \div \frac{1}{2}$	4. $\frac{2}{3} = \frac{2}{5} \div \frac{3}{5}$	4. $\frac{7}{8} \div \frac{2}{3} = \frac{21}{16}$

* SEVERAL CORRECT ANSWERS POSSIBLE IN SOME CASES.

$\boxed{\times \frac{a}{b}}$ means $\boxed{\div b \text{ and then } \times \text{ by } a}$

or $\boxed{\times a \text{ and then } \div \text{ by } b}$

That is, multiplying by a fraction is carrying out a pair of operations — a multiplication (indicated by the numerator) and a division (indicated by the denominator) — and the order does not affect the outcome.

What do we mean by $\div \frac{a}{b}$?

Division is "undoing a multiplication."

If $2 \times 5 = 10$ then $10 \div 5 = 2$.

If $r \times 8 = t$ then $t \div 8 = \underline{r}$.

Suppose that we carry out a pair of operations — a multiplication followed by a division. Then, we use another pair of operations to get back to where we started. Here are some examples.

same

$12 \times 2 \div 3 \; 8 \; \ldots\ldots \; 8 \div 2 \times 3 \; 12$

same

$14 \times 3 \div 7 \; 6 \; \ldots\ldots \; 6 \div 3 \times 7 \; 14$

$35 \times 2 \div 5 \; 14 \; \ldots\ldots \; 14 \div 2 \times 5 \; 35$

$42 \times 7 \div 6 \; 49 \; \ldots\ldots \; 49 \div 7 \times 6 \; 42$

Make up your own examples.

same

$30 \times 3 \div 5 \; 18 \; \ldots\ldots \; 18 \div 3 \times 5 \; 30$

same

$15 \times 2 \div 3 \; 10 \; \ldots\ldots \; 10 \div 2 \times 3 \; 15$

$5 \times 8 \div 5 \; 8 \; \ldots\ldots \; 8 \div 8 \times 5 \; 5$

$18 \times 2 \div 9 \; 4 \; \ldots\ldots \; 4 \div 2 \times 9 \; 18$

In general, for $a \neq 0$ and $b \neq 0$,

$r \times a \div b = s \; \ldots\ldots \; s \div a \times b = r.$

To undo a pair of operations consisting of a multiplication and a division, carry out a division to undo the multiplication and a multiplication to undo the division.

To undo $\times \frac{3}{5}$, you can $\times \frac{5}{3}$

Let us agree that $\div \frac{3}{5}$ is simply shorthand for "undo a multiplication by three-fifths." Then, using a double-headed arrow to mean *produces the same result as,* we can write:

$\div \frac{3}{5} \longleftrightarrow \times \frac{5}{3}$ $\qquad$ $\div \frac{4}{1} \longleftrightarrow \times \frac{1}{4}$

$\div \frac{1}{2} \longleftrightarrow \times \frac{2}{1}$ $\qquad$ $\div \frac{2}{3} \longleftrightarrow \times \frac{3}{2}$

$\div \frac{4}{3} \longleftrightarrow \times \frac{3}{4}$ $\qquad$ $\div 1\frac{1}{3} \longleftrightarrow \times \frac{3}{4}$

$\div 2\frac{3}{5} \longleftrightarrow \times \frac{5}{13}$ $\qquad$ $\div \frac{3}{11} \longleftrightarrow \times 3\frac{2}{3}$

$\boxed{\div \frac{a}{b}}$ means $\div a$ and then $\times b$

or $\times b$ and then $\div a$

which can be indicated by $\boxed{\times \frac{b}{a}}$

To undo a multiplication by *a* and a division by *b* —

$$n \times \frac{a}{b} = \frac{n \times a}{b}$$

we can divide by *a* and multiply by *b* —

$$\frac{n \times \cancel{a}}{\cancel{b}} \times \frac{\cancel{b}}{\cancel{a}} = n$$

Use any method you wish to complete the following statements.

$24 \times \frac{2}{3} = 16$	$15 \times \frac{3}{5} = 9$	$12 \times \frac{3}{4} = 9$
$16 \div \frac{2}{3} = 24$	$15 \div \frac{3}{5} = 25$	$9 \div \frac{3}{4} = 12$
$\frac{2}{5} \div \frac{3}{4} = \frac{8}{15}$	$1\frac{1}{2} \div \frac{1}{5} = 7\frac{1}{2}$	$\frac{7}{8} \div \frac{2}{3} = 1\frac{5}{16}$

Tasks: *Students complete examples involving multiplication or division by a fraction.*

Purpose: To help students understand division by a fraction by having them think of the problem as that of "undoing a multiplication."

Unifying Ideas: Structure; multiplication and division; functions and relations.

The Lesson: Let's analyze the way we think about

$$1 \boxed{\times \tfrac{3}{5}} \quad \text{and} \quad 60 \boxed{\times \tfrac{3}{5}}$$

Multiplication by $3/5$ is carried out by dividing by the denominator and then multiplying by the numerator (or by multiplying by the numerator and then dividing by the denominator). In effect, this is a definition of multiplication by $3/5$.

Suppose that we wish to undo multiplication by $3/5$; that is, to undo the result of dividing by 5 and then multiplying by 3. Let's agree to use the division sign "÷" as a signal for that.

$$\boxed{\div \tfrac{3}{5}} \quad \text{means} \quad \boxed{\text{undo } \times \tfrac{3}{5}}$$

But, to undo multiplication by $3/5$, we can simply divide by 3 and then multiply by 5 (or multiply by 5 and then divide by 3). In other words, we can multiply by $5/3$.

$$\boxed{\div \tfrac{3}{5}} \quad \text{means} \quad \boxed{\times \tfrac{5}{3}}$$

We could develop perfectly good algorithms based on this understanding:

$$20 \times \frac{4}{5} = (20 \div 5) \times 4 = 4 \times 4 = 16$$
$$\text{or } (20 \times 4) \div 5 = 80 \div 5 = 16$$

$$16 \div \frac{4}{5} = (16 \div 4) \times 5 = 4 \times 5 = 20$$
$$\text{or } (16 \times 5) \div 4 = 80 \div 4 = 20$$

Another example:

$$\frac{7}{8} \div \frac{2}{3} = \left(\frac{7}{8} \div 2\right) \times 3 = \frac{7}{16} \times 3 = \frac{21}{16}$$
$$\text{or } \left(\frac{7}{8} \times 3\right) \div 2 = \frac{21}{8} \div 2 = \frac{21}{16}$$

But if we multiply $7/8$ by the reciprocal of the divisor—that is, if we invert and multiply—we have an alternate way of giving the same instructions:

$$\frac{7}{8} \times \frac{3}{2} = \left(\frac{7}{8} \div 2\right) \times 3$$
$$\text{or } \left(\frac{7}{8} \times 3\right) \div 2$$

Thus, when we say that $7/8 \div 2/3 = 7/8 \times 3/2$, we are simply indicating two different ways of describing the pair of operations to be carried out on $7/8$.

Practice sheet 71 is another look at the idea of "undoing" a multiplication and a division. When a pair of operations is omitted from the chain reaction, the string of computations need not be repeated—simply start with the final result in the first line and "undo" the operations left out. The final result in exercise 2 can be found as follows:

$$8 \;(\div 5)\;(\times 4)\; 6\tfrac{2}{5}$$

In exercise 3, it can be found as follows:

$$8 \;(\div 5)\;(\times 6)\; 9\tfrac{3}{5}$$

FPS-71

What are the results of

SKIPPING A STEP IN A CHAIN REACTION?

Use any shortcut you can to find the final result. ↓

I

1. 24 (×5)(÷3) 40 (×9)(÷10) 36 (×5)(÷6) 30 (×2)(÷15) 4 (×5)(÷4) 5 (×6)(÷3) 10 (×4)(÷5) 8
2. 24 (×5)(÷3) 40 (×9)(÷10) 36 (×5)(÷6) 30 (×2)(÷15) 4 (×6)(÷3) 8 (×4)(÷5) $6\frac{2}{5}$
3. 24 (×5)(÷3) 40 (×9)(÷10) 36 (×2)(÷15) $4\frac{4}{5}$ (×5)(÷4) 6 (×6)(÷3) 12 (×4)(÷5) $9\frac{3}{5}$
4. 24 (×9)(÷10) $21\frac{3}{5}$ (×5)(÷6) 18 (×2)(÷15) $2\frac{2}{5}$ (×5)(÷4) 3 (×4)(÷5) $2\frac{2}{5}$

II

5. 30 (×2)(÷3) 20 (×5)(÷4) 25 (×7)(÷5) 35 (×2)(÷10) 7 (×8)(÷7) 8 (×3)(÷4) 6 (×7)(÷2) 21
6. 30 (×2)(÷3) 20 (×5)(÷4) 25 (×7)(÷5) 35 (×2)(÷10) 7 (×8)(÷7) 8 (×7)(÷2) 28
7. 30 (×2)(÷3) 20 (×5)(÷4) 25 (×7)(÷5) 35 (×2)(÷10) 7 (×3)(÷4) $5\frac{1}{4}$ (×7)(÷2) $18\frac{3}{8}$
8. 30 (×2)(÷3) 20 (×5)(÷4) 25 (×7)(÷5) 35 (×8)(÷7) 40 (×3)(÷4) 30 (×7)(÷2) 105
9. 30 (×2)(÷3) 20 (×5)(÷4) 25 (×2)(÷10) 5 (×8)(÷7) $5\frac{5}{7}$ (×3)(÷4) $4\frac{2}{7}$ (×7)(÷2) 15
10. 30 (×2)(÷3) 20 (×7)(÷5) 28 (×2)(÷10) $5\frac{3}{5}$ (×8)(÷7) $6\frac{1}{5}$ (×3)(÷4) $4\frac{4}{5}$ (×7)(÷2) $16\frac{4}{5}$
11. 30 (×5)(÷4) $37\frac{1}{2}$ (×7)(÷5) $52\frac{1}{2}$ (×2)(÷10) $10\frac{1}{2}$ (×8)(÷7) 12 (×3)(÷4) 9 (×7)(÷2) $31\frac{1}{2}$

III Make the following true by using any of the five signs: +, −, ×, ÷, =

12. $\frac{6}{2} \times \frac{9}{3} = \frac{54}{6} = \frac{9}{1}$	16. $\frac{8}{2} + \frac{3}{1} = \frac{14}{2} = \frac{7}{1}$	20. $\frac{8}{2} - \frac{3}{1} = \frac{2}{2} = \frac{1}{1}$
13. $\frac{2}{1} \div \frac{1}{2} = \frac{4}{1} = \frac{4}{1}$	17. $\frac{9}{8} - \frac{1}{8} = \frac{64}{64} = \frac{1}{1}$	21. $\frac{18}{3} \div \frac{4}{2} = \frac{36}{12} = \frac{3}{1}$
14. $\frac{3}{4} + \frac{1}{4} = \frac{16}{16} = \frac{1}{1}$	18. $\frac{7}{1} \div \frac{1}{3} = \frac{21}{1} = \frac{21}{1}$	22. $\frac{1}{8} + \frac{1}{8} = \frac{16}{64} = \frac{1}{4}$
15. $\frac{1}{2} - \frac{3}{16} = \frac{10}{32} = \frac{5}{16}$	19. $\frac{2}{3} \times \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$	23. $\frac{2}{7} \times \frac{6}{3} = \frac{12}{21} = \frac{4}{7}$

Approach IV: Mathematical Reasoning

We are going to make some statements and reason from them to a conclusion. The conclusion will give us a procedure for dividing by a fraction.

Here are the statements and some remarks about each one.

(1) For each number a and for each number b (except 0),

$$(a \div b) \times b = a.$$

This statement tells us what is meant by division. If you divide one number by a second and then multiply the answer by the second, the result is the first number.

$$(8 \div 4) \times 4 = 2 \times 4 = 8$$

$$(12 \div 3) \times 3 = \underline{4} \times 3 = \underline{12}$$

$$(3 \div 5) \times 5 = \underline{\tfrac{3}{5}} \times 5 = \underline{3}$$

Another way of looking at statement (1) is this. If you wish to find out the answer to a division problem (divide a first number by a second), just find out what number should be multiplied by the second number to get the first number.

$30 \div 6 = ?$ (30: 1st number; 6: 2nd number) $\qquad ? \times 6 = 30$ (6: 2nd number; 30: 1st number)

$40 \div 5 = \underline{8}$ because $\underline{8} \times 5 = 40$

$\frac{1}{4} \div 3 = \underline{\frac{1}{12}}$ because $\underline{\frac{1}{12}} \times 3 = \frac{3}{12} = \frac{1}{4}$

$5 \div \frac{1}{2} = \underline{10}$ because $\underline{10} \times \frac{1}{2} = 5$

[Notice that we exclude 0 in statement (1) because it is impossible to divide by 0.]

Now for the second statement.

(2) For each number b (except 0), there is exactly one number — called the *reciprocal* of b — such that

$$b \times \text{the reciprocal of } b = 1.$$

For example, consider the number 3. What number can you multiply 3 by to get 1? Statement (2) says that there is such a number, and that it is called the reciprocal of 3. From what you already know about multiplication, you can see that

$$3 \times \frac{1}{3} = 1.$$

So, $\frac{1}{3}$ is the reciprocal of 3.

Complete each of the following statements.

The reciprocal of 5 is $\underline{\frac{1}{5}}$ since $5 \times \underline{\frac{1}{5}} = 1$.

The reciprocal of $\frac{1}{8}$ is $\underline{8}$.

The reciprocal of $\underline{10}$ is .1.

The reciprocal of .1 is $\underline{10}$.

The reciprocal of $\frac{2}{3}$ is $\underline{\frac{3}{2}}$.

The reciprocal of $\frac{4}{5}$ is $\underline{\frac{5}{4}}$.

The reciprocal of $\frac{8}{7}$ is $\underline{\frac{7}{8}}$.

The reciprocal of $\frac{5}{3}$ is $\underline{\frac{3}{5}}$.

What is the reciprocal of 2? $\underline{\frac{1}{2}}$

What is the reciprocal of $\frac{1}{3}$? $\underline{3}$

What is the reciprocal of $\frac{3}{4}$? $\underline{\frac{4}{3}}$

[Do you see why we excluded 0 in statement (2)? Do you think there is a number which is the reciprocal of 0?]

The next two statements are very easy to understand. Check them by testing a few examples.

(3) For all numbers a, b, and c,

$$(a \times b) \times c = a \times (b \times c).$$

Statement (3) tells us that in a multiplication problem you can change the grouping without changing the final answer.

$$(3 \times 5) \times 7 = 15 \times 7 = 105$$

$$3 \times (5 \times 7) = 3 \times 35 = 105$$

Check some examples of statement (3) with fractions and decimals.

(4) For each number a,

$$a \times 1 = a.$$

We doubt that you will need to do any checking to convince yourself of statement (4).

Now let's see what we can get from these four statements.

72-73 Approach IV - mathematical reasoning

Tasks: Students are given four generalizations and check them by testing instances.

Purpose: To give students the premises for an argument whose conclusion is a generalization that leads to a procedure for dividing by a fraction.

Unifying Ideas: Structure; multiplication and division; functions and relations.

The Lesson: Pupil pages 72 and 73 present a fourth approach to division by a fraction. This approach involves starting with certain statements (generalizations), and reasoning (on pupil page 73) from these statements to a conclusion.

Few students will have enough maturity to appreciate a rigorous derivation of statement 5 from statements 1, 2, 3, and 4. Therefore, we are not counting on this approach as the way to teach students how to divide by a fraction. *Approaches I* and *III* should take care of that. However, we do think that students should be *aware* of the fact that there is "mathematical" justification for the procedure used in dividing by a fraction as well as a "physical" justification. So, our intention in these two pages is to give students a *glimpse* of one possible mathematical justification. They will come to master this one or others when they attain more mathematical maturity.

A useful procedure for making sure students understand a generalization such as any of statements 1, 3, and 4 is to have them produce instances of the generalization. Ideally, the generalization should not be written until the students have dealt with many of its instances. Then the statement becomes merely a verbal expression of what they already know. Your students will certainly find this to be so for statements 1, 2, and 4.

Statement 1 tells us what division is, by explaining division in terms of multiplication. It tells us that dividing by a number is the operation which is undone by multiplying by that number. The fact that multiplication is the inverse of division should not be a new idea to students who have worked through *Approach III*. Statement 2 introduces a new term, *reciprocal*, but in the familiar context of multiplication. Statement 2 tells us two things: (a) each nonzero number—whole or fractional—has exactly one reciprocal, and (b) the product of a number and its reciprocal is 1.

As students work exercises *a-k*, some will undoubtedly become aware of the "rule" that tells you that the reciprocal of a fraction is the fraction "turned upside down." There is no need to verbalize such a rule. Doing so might make it difficult for a student to respond to the question "what is the reciprocal of 5?" because there is "nothing to turn upside down." If you ask "how do you know that $3/2$ is the reciprocal of $2/3$?" the only acceptable answer is "because $2/3 \times 3/2 = 1$."

The question "does 0 have a reciprocal?" is answered by noting that the product of 0 and any number is 0, and $0 \neq 1$. If 0 did have a reciprocal, it would have to be a number whose product with 0 was 1. And this is not possible.

Practice sheet 72 involves computations and comparisons in *A-E*. Writing stories for the headlines in *I-VI* is not easy, but it is likely that you will be surprised by the creativity of students when they are faced with this task.

FPS-72

Use =, >, or < to make the following into true statements.

	A	B	C	D	E
1.	$\frac{1}{2} \div \frac{1}{4} = \frac{2}{1}$	$\frac{3}{4} \div \frac{3}{8} > \frac{3}{2}$	$\frac{5}{4} \div \frac{1}{2} = \frac{5}{2}$	$\frac{7}{8} \div \frac{6}{8} > \frac{1}{1}$	$\frac{1}{2} \div \frac{3}{8} > \frac{1}{1}$
2.	$\frac{15}{3} \div \frac{1}{5} > \frac{5}{3}$	$\frac{5}{12} \div \frac{1}{2} < \frac{1}{1}$	$\frac{7}{2} \div \frac{3}{2} > \frac{3}{2}$	$\frac{3}{4} \div \frac{3}{8} = \frac{2}{1}$	$\frac{9}{11} \div \frac{9}{11} > \frac{5}{11}$
3.	$\frac{9}{3} \div \frac{12}{2} = \frac{1}{2}$	$\frac{1}{1} \div \frac{2}{5} < \frac{7}{2}$	$\frac{5}{4} \div \frac{1}{3} < \frac{10}{2}$	$\frac{2}{3} \div \frac{3}{4} < \frac{1}{1}$	$\frac{6}{7} \div \frac{2}{7} = \frac{3}{1}$
4.	$\frac{5}{4} \div \frac{1}{3} < \frac{4}{1}$	$\frac{1}{1} \div \frac{7}{8} > \frac{1}{1}$	$\frac{2}{3} \div \frac{1}{1} < \frac{7}{8}$	$\frac{1}{1} \div \frac{2}{3} > \frac{7}{8}$	$\frac{7}{8} \div \frac{2}{3} > \frac{1}{1}$

Complete the following headlines, or stories, or both.

I $\frac{7}{8} \div \frac{2}{3} > 1$ "I have pails that hold $\frac{2}{3}$ of a gallon each. I want to know whether I can pour $\frac{7}{8}$ of a gallon into 1 pail. I know that I CANNOT because $\frac{7}{8}$ is MORE than $\frac{2}{3}$."

"There are 24 hours in a day, so $\frac{7}{8}$ of a day is 21 hours. Two-thirds of a day is 16 hours. There is certainly MORE (*more* or *less*) than one 16-hour period in 21 hours. In fact, there are 1 and $\frac{5}{16}$ such periods."

II $2 \div \frac{2}{3} = 3$ IN 2 HOURS THERE ARE 3 PERIODS OF 40 MINUTES EACH.

III $2\frac{1}{8} \div \frac{1}{2} \leq 5$ $2\frac{1}{8}$ GALLONS WOULD NOT FILL 5 HALF-GALLON CARTONS.

IV $3 \div 7 < \frac{1}{2}$ IF $3 WERE DIVIDED AS EQUALLY AS POSSIBLE AMONG 7 BOYS, NO ONE WOULD GET AS MUCH AS A HALF-DOLLAR.

V $\frac{2}{1} \div \frac{1}{8} = 16$ IF AN ANT TOOK STEPS $\frac{1}{8}$ OF AN INCH LONG, IT WOULD TAKE HIM 16 STEPS TO GO 2 INCHES.

VI $\frac{2}{1} \div \frac{1}{3} > 4\frac{1}{2}$ OUT OF 2 FEET OF RIBBON, I CAN GET MORE THAN $4\frac{1}{2}$ PIECES IF EACH PIECE IS 4 INCHES, OR $\frac{1}{3}$ OF A FOOT, LONG.

Consider the division problem:

$$15 \div \tfrac{3}{4} = ?$$

According to statement (1), since $\frac{3}{4}$ is not 0, if we multiply the number $(15 \div \frac{3}{4})$ by $\frac{3}{4}$, we should get 15 as the answer. In other words,

$$(15 \div \tfrac{3}{4}) \times \tfrac{3}{4} = 15.$$

Let's now multiply the numbers on both sides of this equation by the reciprocal of $\frac{3}{4}$. We should get the same answer each time since the numbers are equal.

$$[(15 \div \tfrac{3}{4}) \times \tfrac{3}{4}] \times \left(\begin{matrix}\text{the recipro-}\\ \text{cal of } \frac{3}{4}\end{matrix}\right) = 15 \times \left(\begin{matrix}\text{the recipro-}\\ \text{cal of } \frac{3}{4}\end{matrix}\right)$$

Statement (3) tells us that we can change the grouping in a multiplication problem and not change the answer. Applying this idea to the left side of the last equation, we see that

$$(15 \div \tfrac{3}{4}) \times \left[\tfrac{3}{4} \times \left(\begin{matrix}\text{the recipro-}\\ \text{cal of } \frac{3}{4}\end{matrix}\right)\right] = 15 \times \left(\begin{matrix}\text{the recipro-}\\ \text{cal of } \frac{3}{4}\end{matrix}\right).$$

Statement (2) tells you that when you multiply a number by its reciprocal, you get 1. So, this equation becomes:

$$(15 \div \tfrac{3}{4}) \times 1 = 15 \times \left(\begin{matrix}\text{the recipro-}\\ \text{cal of } \frac{3}{4}\end{matrix}\right)$$

Statement (4) tells you that when you multiply a number by 1, you get the number you started with. Therefore,

$$15 \div \tfrac{3}{4} = 15 \times \left(\begin{matrix}\text{the recipro-}\\ \text{cal of } \frac{3}{4}\end{matrix}\right).$$

So, the answer to the original division problem can be found by multiplying 15 by the reciprocal of $\frac{3}{4}$.

From our knowledge of multiplication, we know that $\frac{3}{4} \times \frac{4}{3} = 1$. So, by statement (2), the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$. Thus, we conclude that

$$15 \div \tfrac{3}{4} = 15 \times \tfrac{4}{3}.$$

Dividing 15 by $\frac{3}{4}$ amounts to multiplying 15 by $\frac{4}{3}$. Since $15 \times \frac{4}{3}$ is 20, the answer to the original division problem is 20.

If you look carefully at the preceding discussion you will notice that the particular numbers 15 and $\frac{3}{4}$ did not play an important role in our reasoning until we reached the point where we needed to find the reciprocal of $\frac{3}{4}$. Up to that point we could use the same reasoning starting with *any* numbers (as long as we don't try to divide by 0). This pattern of reasoning is a *proof* of the following statement.

(5) For each number a and for each number b (except 0),

$$a \div b = a \times \text{the reciprocal of } b.$$

The pattern of reasoning shows that statement (5) follows logically from statements (1), (2), (3), and (4).

Statement (5) tells you how to divide by any number (except 0). All you have to do is find the reciprocal of the number and then multiply by it.

For example, suppose that we have the division problem:

$$\tfrac{2}{5} \div \tfrac{7}{8} = ?$$

Statement (5) tells you that (since $\frac{7}{8}$ is not 0) you can find the answer by multiplying $\frac{2}{5}$ by the reciprocal of $\frac{7}{8}$. What is the reciprocal of $\frac{7}{8}$? It must be $\frac{8}{7}$ because $\frac{7}{8} \times \frac{8}{7} = 1$. So,

$$\tfrac{2}{5} \div \tfrac{7}{8} = \tfrac{2}{5} \times \tfrac{8}{7} = \tfrac{16}{35}.$$

Please complete the following statements.

a. $\frac{3}{7} \div \frac{2}{5} = \frac{3}{7} \times \frac{5}{2} = \frac{15}{14} = 1\frac{1}{14}$

b. $1\frac{1}{3} \div \frac{3}{5} = \frac{4}{3} \times \frac{5}{3} = \frac{20}{9} = 2\frac{2}{9}$

c. $5\frac{1}{2} \div \frac{2}{7} = \frac{11}{2} \times \frac{7}{2} = \frac{77}{4} = 19\frac{1}{4}$

d. $\frac{5}{8} \div 1\frac{1}{5} = \frac{5}{8} \div \frac{6}{5} = \frac{5}{8} \times \frac{5}{6} = \frac{25}{48}$

e. $1\frac{1}{2} \div 1\frac{1}{4} = \frac{3}{2} \div \frac{5}{4} = \frac{3}{2} \times \frac{4}{5} = \frac{12}{10} = 1\frac{1}{5}$

f. $9 \div .3 = 9 \div \frac{3}{10} = 9 \times \frac{10}{3} = \frac{90}{3} = 30$

g. $9 \div .03 = 9 \times \frac{100}{3} = \frac{900}{3} = 300$

h. $2 \div 4 = 2 \times \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$

You have now seen several approaches to dividing a fraction by a fraction. No matter which approach you use, you will have to do the same computations. These computations are indicated by:

$$\frac{m}{n} \div \frac{p}{q} = \frac{m}{n} \times \frac{q}{p} = \frac{m \times q}{n \times p}$$

Tasks: Students follow a mathematical argument for a specific example, and then look at the pattern of argument.

Purpose: To illustrate a mathematical argument, and to state a conclusion which leads to a procedure for dividing by a fraction.

Unifying Ideas: Structure; multiplication and division; functions and relations.

The Lesson: We urge you to conclude the work on this chapter with some kind of discussion of the different approaches to division by a fraction.

How would you describe the different approaches?

Which approach gives you the most insight into the problem?

What do the approaches have in common?

How do they differ?

Which appeals most to common sense?

Which depend on arguments you are not sure of?

Do you agree with the authors that looking at a problem from several different points of view is better than limiting your consideration to a single point of view?

On this last question, you may find some disagreement—especially from students who are eager to be told something so they can hurry on to something else. This is a legitimate point of view. We feel, however, that such anxiety is a poor kind of motivation. It may be that the student is eager to get to the end of the mathematics prescribed for him, so that he can drop the subject. If so, then we have failed to achieve one of our goals—that he will elect to study more mathematics than the generation of his parents did. The student may have a feeling that more interesting material lies ahead and want to get to it as soon as possible. We think that his feeling is a healthy one, but we hope he will listen when you tell him that "real mathematics" is already around him.

Our hope is that this chapter may help him see how many approaches can be taken to any important mathematical problem. We hope he will be forced to develop his own judgments about such problems, and thus to develop his own independent tastes in mathematics.

One of our major objectives is to have students do computations with facility, confidence, and genuine understanding.

Practice sheet 73 is designed to make students aware that both multiplication and division distribute over addition and over subtraction.

FPS-73

I. Select any group of three numbers, *a*, *b*, and *c*, and check the following addition-multiplication pattern:

	a	b	c	$(a + b) \times c$	$= (a \times c) + (b \times c)$
1.	3	2	4	$(3 + 2) \times 4 = 5 \times 4 = 20$	$(3 \times 4) + (2 \times 4) = 12 + 8 = 20$
2.	5	7	3	$(5 + 7) \times 3 = 12 \times 3 = 36$	$(5 \times 3) + (7 \times 3) = 15 + 21 = 36$
3.	7	$\frac{1}{2}$	4	$(7 + \frac{1}{2}) \times 4 = 7\frac{1}{2} \times 4 = 30$	$(7 \times 4) + (\frac{1}{2} \times 4) = 28 + 2 = 30$
4.	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$(\frac{3}{4} + \frac{1}{2}) \times \frac{1}{2} = \frac{5}{4} \times \frac{1}{2} = \frac{5}{8}$	$(\frac{3}{4} \times \frac{1}{2}) + (\frac{1}{2} \times \frac{1}{2}) = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}$
5.				$(\ + \ ) \times \ = \ \times \ =$	$(\ \times \ ) + (\ \times \ ) = \ + \ =$

II. Select any group of three numbers, *a*, *b*, and *c*, and check the following subtraction-multiplication pattern:

	a	b	c	$(a - b) \times c$	$= (a \times c) - (b \times c)$
6.	15	12	4	$(15 - 12) \times 4 = 3 \times 4 = 12$	$(15 \times 4) - (12 \times 4) = 60 - 48 = 12$
7.	7	3	$\frac{1}{3}$	$(7 - 3) \times \frac{1}{3} = 4 \times \frac{1}{3} = \frac{4}{3}$	$(7 \times \frac{1}{3}) - (3 \times \frac{1}{3}) = \frac{7}{3} - 1 = \frac{4}{3}$
8.	$^{+}5$	$^{+}9$	$^{-}2$	$(^{+}5 - ^{+}9) \times ^{-}2 = ^{-}4 \times ^{-}2 = ^{+}8$	$(^{+}5 \times ^{-}2) - (^{+}9 \times ^{-}2) = ^{-}10 - ^{-}18 = ^{+}8$
9.				$(\ - \ ) \times \ = \ \times \ =$	$(\ \times \ ) - (\ \times \ ) = \ - \ =$

III. Change multiplication to division in the pattern in I and check the new addition-division pattern.

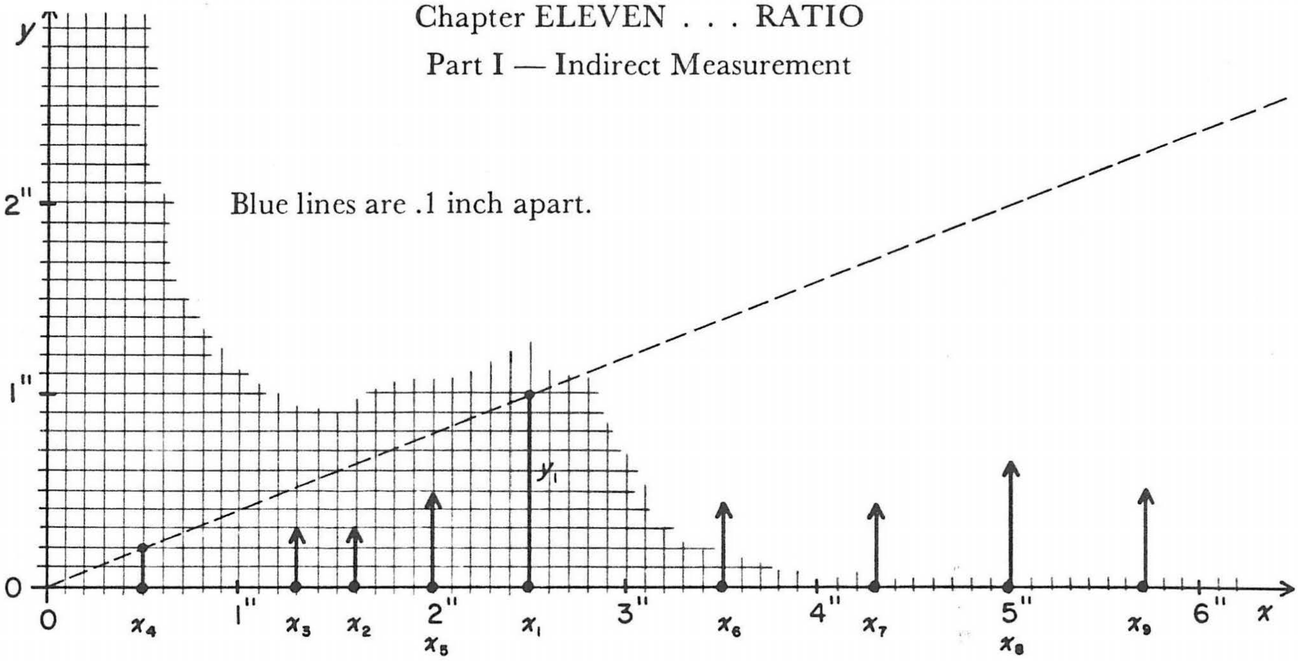
	a	b	c	$(a + b) \div c$	$= (a \div c) + (b \div c)$
10.	12	4	2	$(12 + 4) \div 2 = 16 \div 2 = 8$	$(12 \div 2) + (4 \div 2) = 6 + 2 = 8$
11.	7	3	2	$(7 + 3) \div 2 = 10 \div 2 = 5$	$(7 \div 2) + (3 \div 2) = 3\frac{1}{2} + 1\frac{1}{2} = 5$
12.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$(\frac{1}{2} + \frac{1}{3}) \div \frac{1}{4} = \frac{5}{6} \div \frac{1}{4} = \frac{20}{6}$ or $3\frac{1}{3}$	$(\frac{1}{2} \div \frac{1}{4}) + (\frac{1}{3} \div \frac{1}{4}) = 2 + 1\frac{1}{3} = 3\frac{1}{3}$
13.				$(\ + \ ) \div \ = \ \div \ =$	$(\ \div \ ) + (\ \div \ ) = \ + \ =$

IV. Change multiplication to division in the pattern in II and check the new subtraction-division pattern.

	a	b	c	$(a - b) \div c$	$= (a \div c) - (b \div c)$
14.	12	4	2	$(12 - 4) \div 2 = 8 \div 2 = 4$	$(12 \div 2) - (4 \div 2) = 6 - 2 = 4$
15.	$^{-}6$	$^{-}9$	$^{+}3$	$(^{-}6 - ^{-}9) \div ^{+}3 = ^{+}3 \div ^{+}3 = ^{+}1$	$(^{-}6 \div ^{+}3) - (^{-}9 \div ^{+}3) = ^{-}2 - ^{-}3 = ^{+}1$
16.	$\frac{7}{8}$	$\frac{1}{2}$	$\frac{1}{4}$	$(\frac{7}{8} - \frac{1}{2}) \div \frac{1}{4} = \frac{3}{8} \div \frac{1}{4} = 1\frac{1}{2}$	$(\frac{7}{8} \div \frac{1}{4}) - (\frac{1}{2} \div \frac{1}{4}) = 3\frac{1}{2} - 2 = 1\frac{1}{2}$
17.				$(\ - \ ) \div \ = \ \div \ =$	$(\ \div \ ) - (\ \div \ ) = \ - \ =$

Chapter ELEVEN . . . RATIO

Part I — Indirect Measurement



How long (to the nearest tenth of an inch) would each vertical green line be if it were extended from the "x-axis" until it meets the broken black line?

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
length parallel to y-axis	1"	.64"	.52"	.2"	.8"	1.4"	1.72"	2"	2.28"
length along the x-axis	2.5"	1.6"	1.3"	.5"	2"	3.5"	4.3"	5"	5.7"

Problem: How can you complete the table without drawing a line or making a measurement?

Consider the following cases:

	(1)	(4)	(5)	(8)
y-axis	1"	.2"	.8"	2"
x-axis	2.5"	.5"	2"	5"

You can fill in the lengths in examples (4) and (5) by reading the blue lines on the chart.

Think about these entries as part of a *What's My Rule?* game. What is the relationship between the lengths reported in each entry?

$$\frac{y\text{-axis}}{x\text{-axis}} \qquad \frac{1}{2.5} = \frac{.2}{.5} = \frac{.8}{2} = \frac{2}{5}$$

The length parallel to the y-axis is two-fifths of the length along the x-axis, . . . or it is four-tenths, or forty-hundredths, or any expression in the form of:

$$\frac{2n}{5n}$$

Problem: For all nonzero numbers n, a, b, c, is it the case that

$$\frac{2n}{5n} = \frac{2a}{5a} = \frac{2b}{5b} = \frac{2c}{5c} = \frac{2}{5}\ ?$$

Before answering for sure, convince yourself that common fractions, decimal fractions, and positive and negative numbers would be suitable as replacements for n, a, b, and c.

Problem: How could the idea we are considering here be used to find the height of a flag pole?

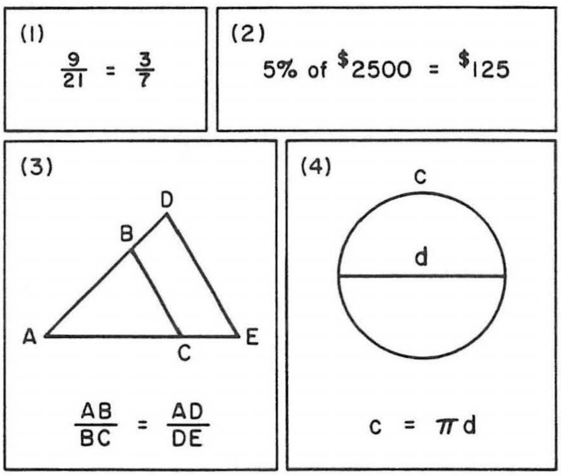
Problem: How could the idea be used to measure the length of a flag pole that was bent over at its base so that it was no longer straight up and down?

Tasks: Students complete two tables and a "chain" equation, and consider three problems.

Purpose: To help students explore the ratio idea in the setting of indirect measurement.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations; geometry; measurement.

The Lesson: Here are four bits of arithmetic and geometry:



Chapter Eleven discusses the common aspect of these apparently dissimilar mathematical ideas, which might be "entitled"

$$\frac{a}{b} = \frac{c}{d}$$

("a is to b as c is to d")

If we rewrite some of the equations, the common aspect may be more apparent:

(1) $\frac{9}{21} = \frac{3}{7}$

(2) $\frac{5}{100} = \frac{125}{2500}$

(3) $\frac{AB}{BC} = \frac{AD}{DE}$

(4) $\frac{c}{d} = \frac{\pi}{1}$.

Students are used to reading these equations as "nine twenty-firsts equals three-sevenths," etc. We hope that they will become familiar with another way of saying the same thing—a language that will help deepen their understanding of many situations. This is the language of *ratio*.

(1) 9 is to 21 as 3 is to 7.

(2) 5 is to 100 as 125 is to 2500.

(3) AB is to BC as AD is to DE.

(4) the circumference is to the diameter as π is to 1.

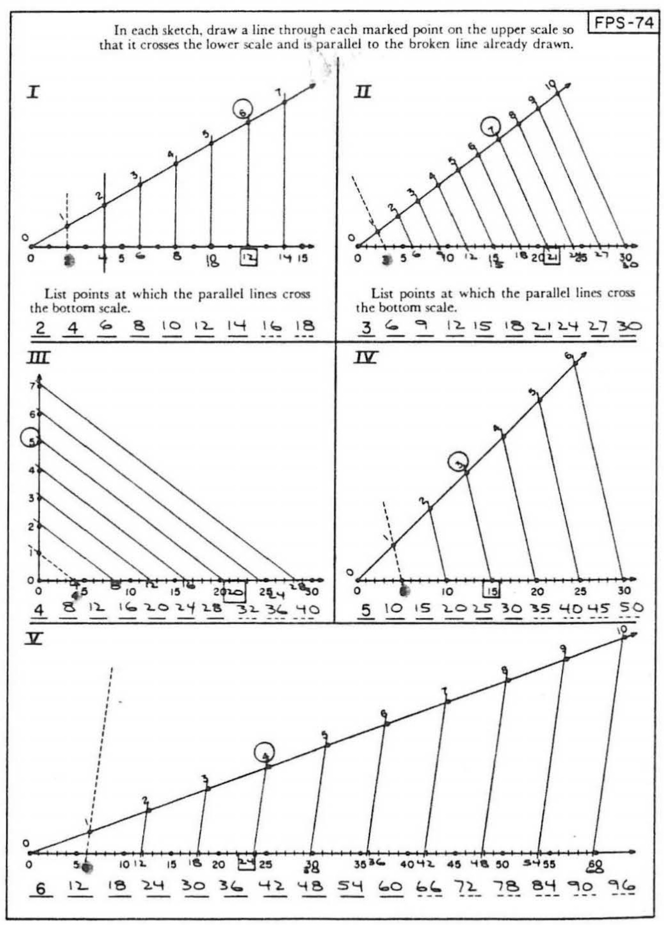
Ratio language is not used in the text until pupil page 78, but feel free to introduce it earlier if you wish.

In the graph on pupil page 74, as the distance along the x-axis increases, the vertical distance to the broken line increases too—and in the same ratio.

In the graphs on practice sheet 74, one might describe the relationship in the first example as follows: 1 is to 2 as 2 is to 4 as 3 is to 6 as 4 is to 8, etc.; and the last one as follows: 1 is to 6 as 2 is to 12 as 3 is to 18, etc. The shorthand for such expressions is quite familiar:

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}, \text{ etc.}$$

$$\frac{1}{6} = \frac{2}{12} = \frac{3}{18}, \text{ etc.}$$



While you are thinking about those problems, let's look at a relationship between any two pairs of entries in our table: (We'll omit the ″ mark).

(1)	(4)
1	2
2.5	.5

(5)	(8)
.8	2
2	5

(1)	(5)
1	.8
2.5	2

Multiply opposite corner entries as indicated by the blue lines.

1 x .5 = .5   .8 x 5 = 4

2.5 x .2 = .5   2 x 2 = 4

1 x 2 = 2

2.5 x .8 = 2

Since (8) has measurements involving only whole numbers, let's use that fact and the idea above to find other entries for the table.

(8)	(2)
2	.64
5	1.6

(8)	(3)
2	.52
5	1.3

(8)	(6)
2	1.4
5	3.5

(8)	(7)
2	1.72
5	4.3

2 x 1.6 = 3.2   2 x 1.3 = 2.6

5 x .64 = 3.2   5 x .52 = 2.6

2 x 3.5 = 7   2 x 4.3 = 8.6

5 x 1.4 = 7   5 x 1.72 = 8.6

Distances along the *x*-axis are indicated in the table below. How long would the green lines have to be to reach up to the broken diagonal line? (All numbers are numbers of inches.)

	(10)	(11)	(12)	(13)	(14)	(15)
y-axis	1.2	2.8	3.6	.12	28	.036
x-axis	3	7	9	.3	70	.09

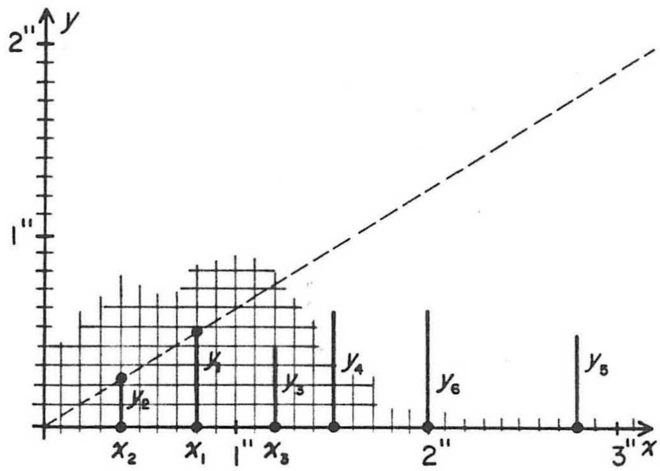
How far out along the *x*-axis would you need to go so that the vertical distance to the broken line is the number of inches indicated?

	(16)	(17)	(18)	(19)	(20)	(21)
y-axis	5	11	14	.05	1.1	1400
x-axis	12.5	27.5	35	.125	2.75	3,500

We hope that you found a shortcut for these examples.

Problem: Does the relationship we have been considering depend on our starting with a 2-to-5 "ratio"? Suppose that we had drawn a graph that had a 5-to-8 ratio — 5 up for every 8 over.

Let's draw such a graph and study it.



Make entries to the nearest hundredth inch.

	(1)	(2)	(3)	(4)	(5)
y-axis	.5	.25	.75	1.0	1.75
x-axis	.8	.4	1.2	1.6	2.8

	(6)	(7)	(8)	(9)	(10)	(11)
y	1.25	5	6.25	11.25	1.5	33.75
x	2	8	10	18	2.4	54

(7)	(5)
5	1.75
8	2.8

(7)	(8)
5	6.25
8	10

(7)	(9)
5	11.25
8	18

(7)	(11)
5	33.75
8	54

5 x 2.8 = 14   5 x 10 = 50

8 x 1.75 = 14   8 x 6.25 = 50

5 x 18 = 90   5 x 54 = 270

8 x 11.25 = 90   8 x 33.75 = 270

For all numbers *a*, *b*, *c*, and *d* ($b \neq 0$ and $d \neq 0$),

if $\frac{a}{b} = \frac{c}{d}$ then $ad = bc$

and

if $ad = bc$ then $\frac{a}{b} = \frac{c}{d}$.

Do you believe that statement? YES

Tasks: *Students complete the tables using the clues given.*

Purpose: To help students recognize and make use of ratio relationships.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations; geometry; measurement.

The Lesson: The central idea on pupil page 75 is that in a statement in the form of $a/b=c/d$, if three of the terms are given, the fourth one can be found.

(1) $\frac{3}{7}=\frac{2}{x}$ (2) $\frac{4}{9}=\frac{x}{3}$

(3) $\frac{x}{4}=\frac{9}{13}$ (4) $\frac{5}{x}=\frac{16}{3}$

At first glance, these examples appear to be quite difficult. However, once one notices that the equations $a/b=c/d$ and $ad=bc$ say the same thing, the examples become easy. In the first example, 3 multiplied by what number equals 7 multiplied by 2? (Answer: $4\frac{2}{3}$.) In the second example, 9 times what number equals 4 multiplied by 3? (Answer: $1\frac{1}{3}$.) The third example shows that $2\frac{10}{13}\times 13=4\times 9$, and the fourth shows that $\frac{15}{16}\times 16=5\times 3$.

We suggest that you put the following statement on the board and discuss it with the class:

$$\frac{a}{b}=\frac{c}{d}$$

says the same thing as:

(1) $a=\frac{bc}{d}$ (2) $b=\frac{ad}{c}$

(3) $c=\frac{ad}{b}$ (4) $d=\frac{bc}{a}$

(5) $ad=bc$

Test each equation:

$$\frac{3}{5}=\frac{12}{20}$$

(1) $3=\frac{5\times 12}{20}$ (2) $5=\frac{3\times 20}{12}$

(3) $12=\frac{3\times 20}{5}$ (4) $20=\frac{5\times 12}{3}$

(5) $3\times 20=5\times 12$

The purpose of making many such tests is not so much to convince the students of the equivalence of the equations, but to increase their familiarity with the many equations that are equivalent to $a/b=c/d$.

Notice that in the graphs we are using the customary designation of the horizontal axis as the x-axis and the vertical axis as the y-axis.

We have also introduced symbols such as x_1, x_2, y_1, y_2, etc.; these are read as "x sub-one," "x sub-two," "y sub-one," "y sub-two," etc. This notation is a useful way to refer to different distances from the origin on a given axis. It is convenient for referring to points in terms of their coordinates. For example, (x_1, y_1) specifies a point, (x_2, y_2) specifies another, etc.

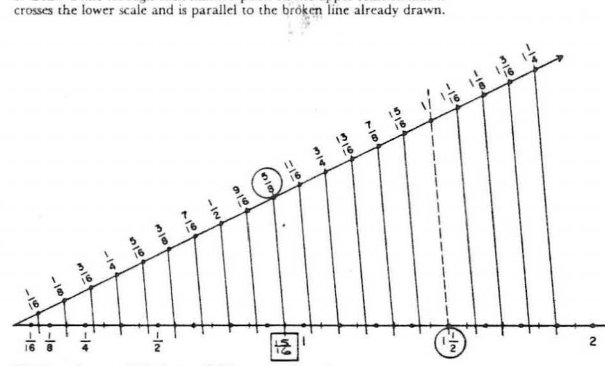
Practice sheet 75 extends the idea of practice sheet 74 by looking at points labeled with fractions. For the first diagram the basic relationship is

$$\frac{\frac{1}{16}}{\frac{3}{32}}=\frac{\frac{1}{8}}{\frac{3}{16}}=\cdots\frac{1}{1\frac{1}{2}}=\cdots$$

The lower section of the practice sheet presents a problem that anticipates the building of a Distance Finder (see pupil page 79).

FPS-75

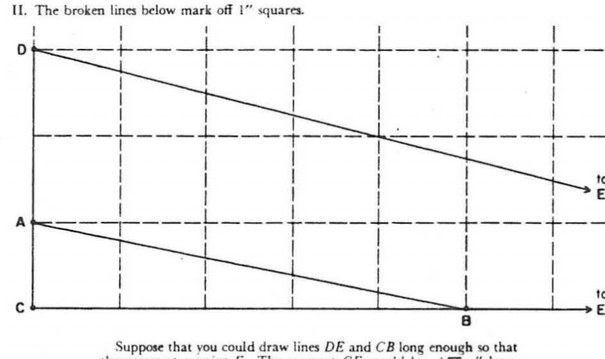
I. Draw a line through each marked point on the upper scale so that it crosses the lower scale and is parallel to the broken line already drawn.



List the points at which the parallel lines cross the scales.

upper scale	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$	$\frac{9}{16}$	$\frac{5}{8}$	$\frac{11}{16}$	$\frac{3}{4}$	$\frac{13}{16}$	$\frac{7}{8}$	$\frac{15}{16}$	1	$1\frac{1}{16}$	$1\frac{1}{8}$	$1\frac{3}{16}$	$1\frac{1}{4}$	$1\frac{5}{16}$
lower scale	$\frac{3}{32}$	$\frac{3}{16}$	$\frac{9}{32}$	$\frac{3}{8}$	$\frac{15}{32}$	$\frac{9}{16}$	$\frac{21}{32}$	$\frac{3}{4}$	$\frac{27}{32}$	$\frac{15}{16}$	$1\frac{1}{32}$	$1\frac{1}{8}$	$1\frac{7}{32}$	$1\frac{5}{16}$	$1\frac{13}{32}$	$1\frac{1}{2}$	$1\frac{19}{32}$	$1\frac{11}{16}$	$1\frac{25}{32}$	$1\frac{7}{8}$	$1\frac{31}{32}$

II. The broken lines below mark off 1" squares.

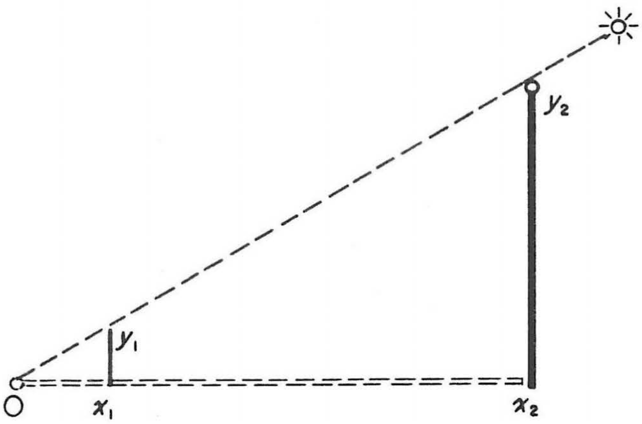


Suppose that you could draw lines DE and CB long enough so that they cross at a point E. The segment CE would be 15" long.

About the height of those flag poles . . .

You have a 5-foot rod and a tape measure.

One method makes use of the shadow of the flag pole. The 5-foot rod is placed on the shadow so that the shadow of its tip and the shadow of the tip of the flag pole are at the same point.



We know that y_1 is 5′. We measure the distance from O (tip of the shadows) to x_1 (bottom of the rod) and find that it is $8\frac{1}{2}$′; and we measure the distance from O to x_2 (bottom of the flag pole) and find it is 51′.

Now we record our information:

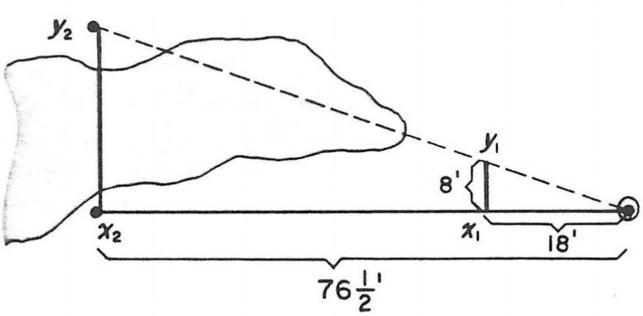
	(1)	(2)
y-axis	5′	30′
x-axis	$8\frac{1}{2}$′	51′

So, the height of the flag pole is 30 ft.

Can you think of a method you could use if the sun weren't out?

Could you use this method to find the height of your room, a building, a tree?

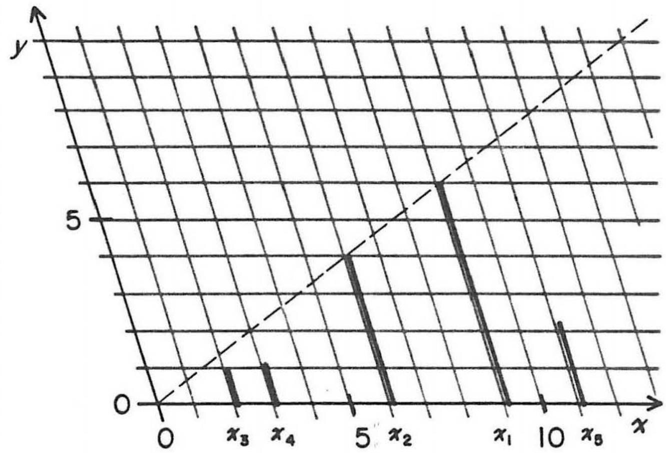
Can you find the distance between two points with a lake between them?



	(1)	(2)
y-axis	8	34
x-axis	18	$76\frac{1}{2}$

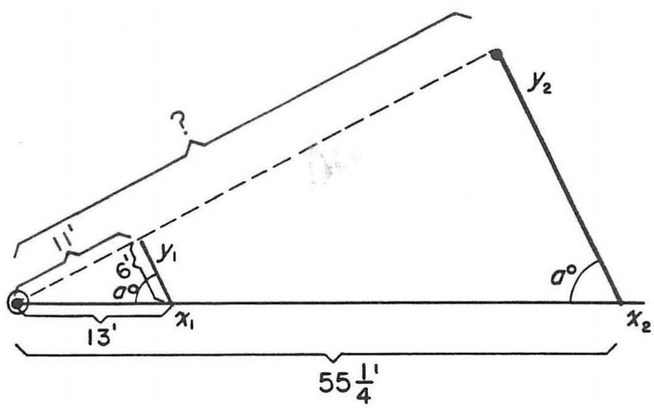
. . . about the bent flag pole . . .

Let's consider a grid in which the x-axis and y-axis are not perpendicular to each other.



	(1)	(2)	(3)	(4)	(5)
y-axis	6	4	$1\frac{1}{3}$	2	$7\frac{1}{3}$
x-axis	9	6	2	3	11

Do your results suggest a way to go about measuring the length of the bent flag pole?



	(1)	(2)
y-axis	6	$25\frac{1}{2}$
x-axis	13	$55\frac{1}{4}$

The length of the flag pole is $25\frac{1}{2}$ ft.

(You would need to take care that the angle of the 6′ rod was the same as that of the flag pole and, sighting from point O, you would need to be sure that the bottoms and tips of the rod and pole were in line.)

Problem: If the point of sight (O) is 11′ from the tip of the 6′ rod, how far is the point of sight from the tip of the flag pole?

Answer: $46\frac{3}{4}$ ft.

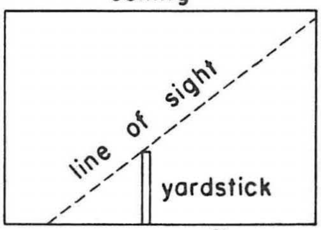
Tasks: Students complete tables and problems on finding the heights of flagpoles by indirect measurement.

Purpose: To help students better understand the ratio idea through the study of relationships in similar triangles.

Unifying Ideas: Structure; multiplication and division; functions and relations; geometry; measurement.

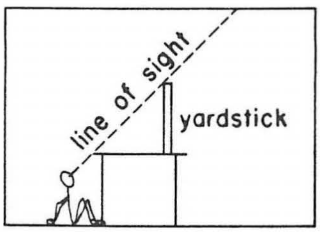
The Lesson: Now to the problems first raised at the bottom of pupil page 74—the heights of flagpoles. If problems about flagpoles seem too contrived, consider translating the problem into a familiar one such as finding the height of the ceiling of a room. The advantage of such "do it yourself" problems is that *practical* problems arise and have to be solved.

Suppose that we decide to use a yardstick placed vertically in some such arrangement as this:



Fine in theory, but how does one get his eye down on the floor to sight across the tip of the yardstick?

Why not put the yardstick on a table and sight past the edge of the table?



Must the line of sight meet the front wall in this arrangement? If not, how can we measure the distance from the yardstick to the point directly below the point where the line of sight meets the ceiling? Further, if the yardstick wavers as we are sighting, the results change appreciably. How can we hold the yardstick stationary and vertical?

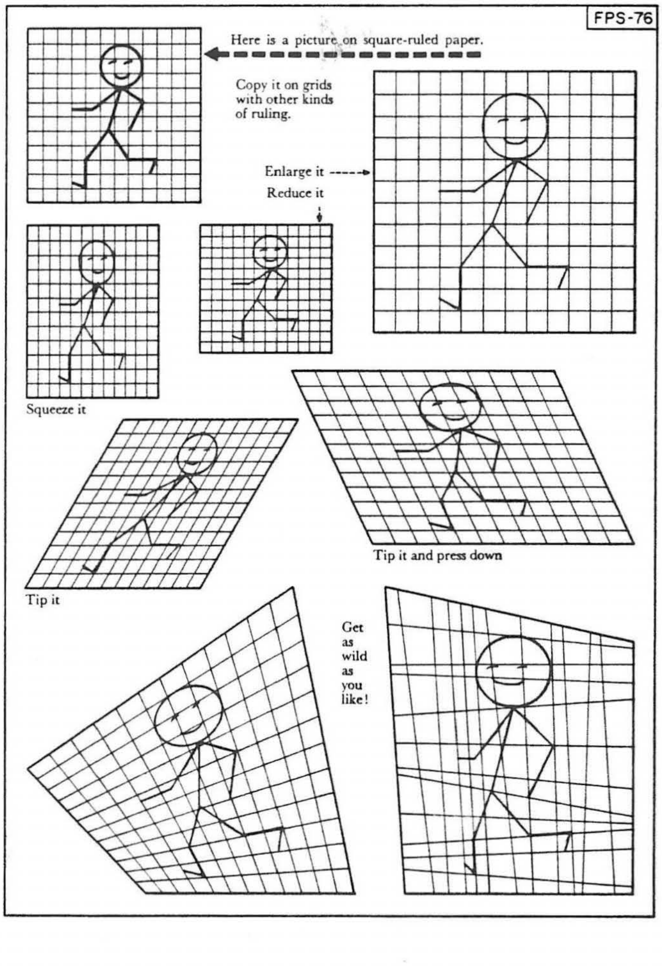
After we solve all these annoying practical problems, make measurements, and compute the height, what distance have we found? The distance between the top of the table and the ceiling, or the distance between the floor and the ceiling?

If several teams of students make independent experiments, their results will vary. What are the sources of these variations? How accurately were ruler measurements made and recorded? What could be done to minimize these variations? Would an "average" of the results have any significance?

Perhaps with the help of a pole or stepladder, a direct measurement can be made to check on the indirect measurements.

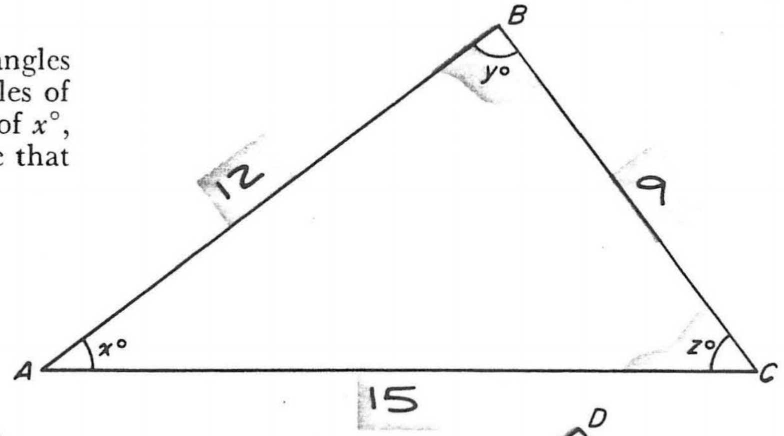
Overcoming the difficulties of carrying out actual experiments sheds light on the relationship between "real situations" and their idealized counterparts.

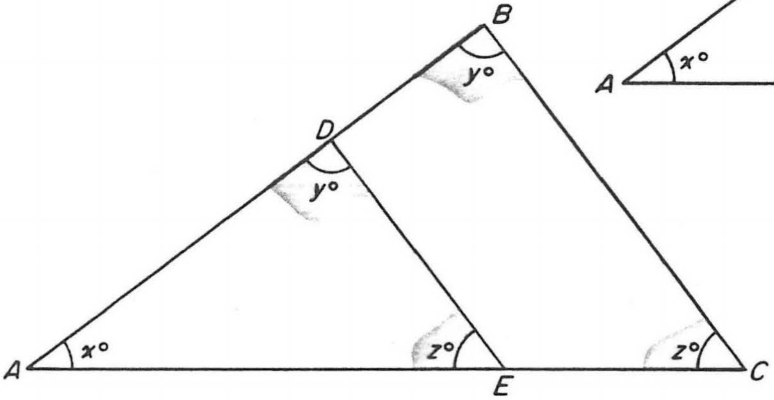
Practice sheet 76 is a bit of fun that may help students feel more confident that the problems of the vertical and the leaning flagpoles are basically the same problem. A right triangle (the case of the vertical flagpole) is simply a special case of triangles in general (the bent pole situation).

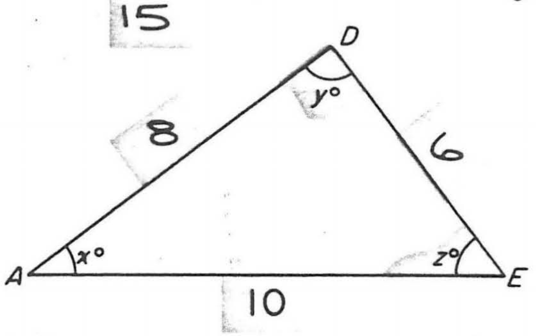


I.

Two triangles are *similar* if the three angles of one are equal in size to the three angles of the other. That is, a triangle with angles of $x°$, $y°$, and $z°$ is similar to any other triangle that has angles of $x°$, $y°$, and $z°$.







In the blue blocks above, record the number of quarter-inches in each side. Use a ruler to find the lengths.

Use the blue blocks below to check the following statements:

II.

$$\frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE} \qquad \frac{12}{8} = \frac{9}{6} = \frac{15}{10}$$

$$\frac{AB}{BC} = \frac{AD}{DE} \qquad \frac{12}{9} = \frac{8}{6}$$

$$\frac{BC}{AC} = \frac{DE}{AE} \qquad \frac{9}{15} = \frac{6}{10}$$

$$\frac{AB}{AC} = \frac{AD}{AE} \qquad \frac{12}{15} = \frac{8}{10}$$

III.

$AB \times DE = AD \times BC$ — $12 \times 6 = 8 \times 9$

$BC \times AE = DE \times AC$ — $9 \times 10 = 6 \times 15$

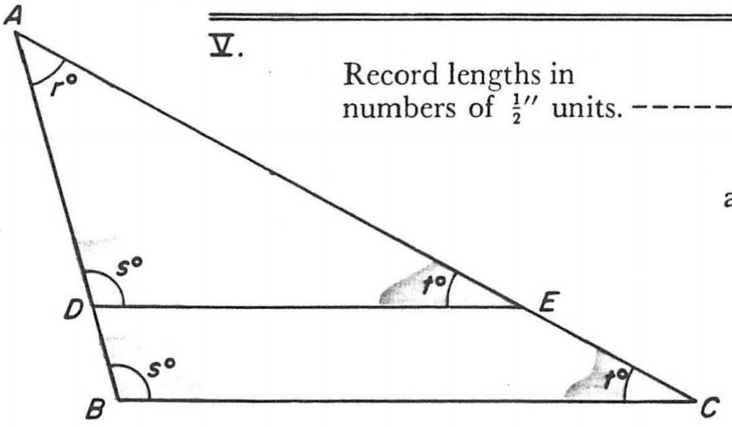
$AB \times AE = AD \times AC$ — $12 \times 10 = 8 \times 15$

IV.

$$\frac{AB - AD}{AC - AE} = \frac{AB + AD}{AC + AE} = \frac{AB}{AC} = \frac{AD}{AE} \qquad \frac{12 - 8}{15 - 10} = \frac{12 + 8}{15 + 10} = \frac{12}{15} = \frac{8}{10}$$

$$\frac{BC - DE}{AC - AE} = \frac{BC + DE}{AC + AE} = \frac{BC}{AC} = \frac{DE}{AE} \qquad \frac{9 - 6}{15 - 10} = \frac{9 + 6}{15 + 10} = \frac{9}{15} = \frac{6}{10}$$

V.



Record lengths in numbers of $\frac{1}{2}''$ units. ------→

AB	AC	BC	AD	AE	DE
4	8	6	3	6	$4\frac{1}{2}$

Are all the statements made above also true about the triangles at the left? Try a few:

$$\frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE} \qquad \frac{4}{3} = \frac{6}{4\frac{1}{2}} = \frac{8}{6}$$

$$\frac{AB}{AC} = \frac{AD}{AE} \qquad \frac{4}{8} = \frac{3}{6}$$

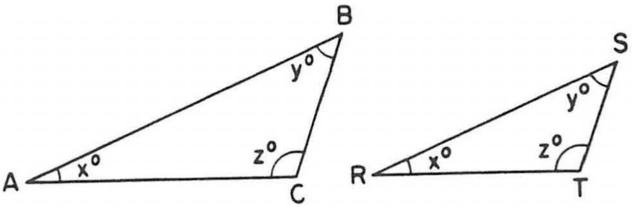
77-78 Similar triangle relationships

Tasks: Students measure the lengths of sides of similar triangles in ¼-inch and ½-inch units. They use these measurements to test the truth of equations that express relationships among the lengths of the sides of these triangles.

Purpose: To help students better understand the ratio idea through the study of relationships among the lengths of the sides of triangles.

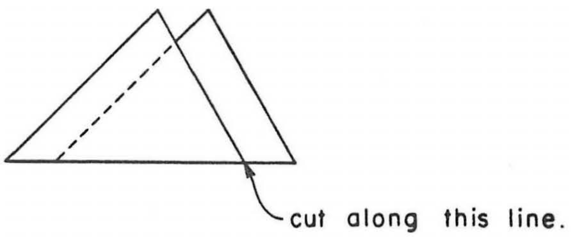
Unifying Ideas: Structure; multiplication and division; functions and relations; geometry; measurement.

The Lesson: If the corresponding angles of two triangles are equal in size, the triangles are *similar*.

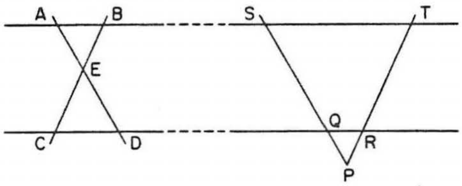


Triangle ABC is similar to triangle RST. From the drawing, it would appear that no two of the measures $x°$, $y°$, $z°$ are equal. If all three angles were the same size, then triangles ABC and RST would be equilateral triangles and, of course, all equilateral triangles are similar.

Suppose that we wish to cut out or draw two triangles that are different in size, but similar. How might we go about it? Most practical suggestions will be based on a useful relationship between similar triangles: you can fit an angle of one triangle in the corresponding angle of the other so that the sides opposite those angles are parallel. We can fold a piece of paper and cut out two triangles at the same time; the resulting triangles will be congruent. If we then cut one of the triangles on a line parallel to any one of its sides, we shall have a smaller but similar triangle. To make a parallel cut, shift one triangle along the corresponding edge of the other, using one triangle as a guide for cutting:



One way to draw a pair of similar triangles is to begin with a pair of parallel lines and then draw a pair of lines that cross each other and both of the parallel lines:

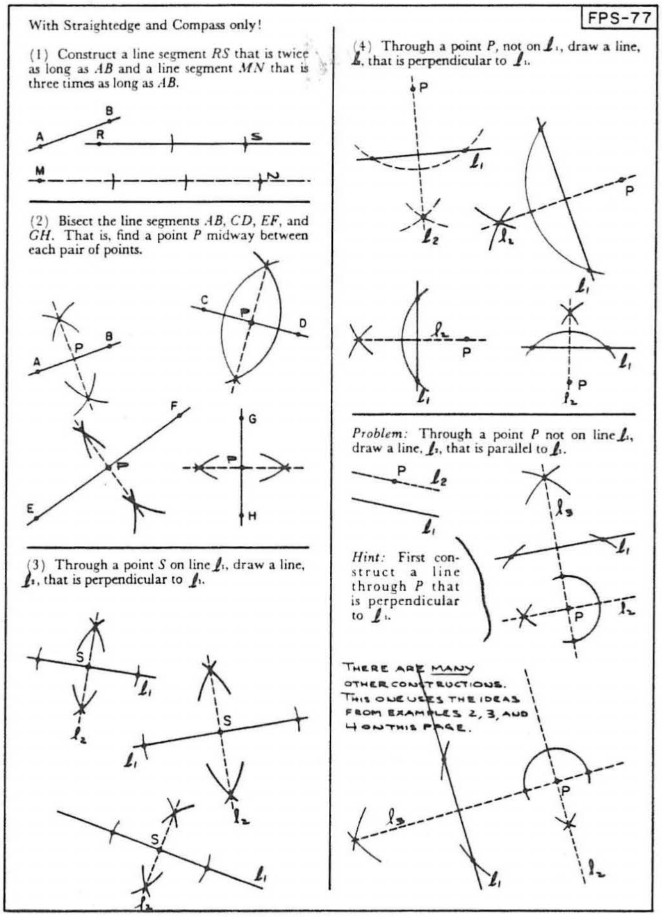


ΔABE ~ (is similar to) ΔDCE

and ΔPQR ~ ΔPST

Discussing ways of cutting out and drawing similar triangles will help students appreciate more fully some of the important relationships between similar triangles.

Practice sheet 77 will be helpful for students who have not used a compass and straightedge to perform simple constructions. The problem on practice sheet 78 depends on a familiarity with these constructions.



FPS-77

With Straightedge and Compass only!

(1) Construct a line segment RS that is twice as long as AB and a line segment MN that is three times as long as AB.

(2) Bisect the line segments AB, CD, EF, and GH. That is, find a point P midway between each pair of points.

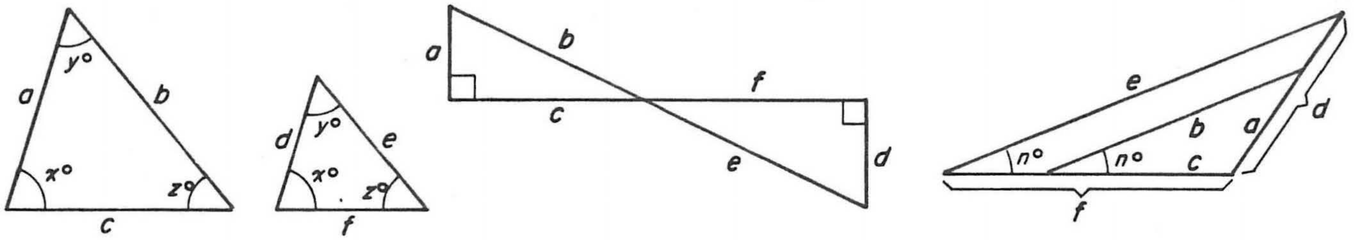
(3) Through a point S on line l_1, draw a line, l_2, that is perpendicular to l_1.

(4) Through a point P, not on l_1, draw a line, l_2 that is perpendicular to l_1.

Problem: Through a point P not on line l_1, draw a line, l_2, that is parallel to l_1.

Hint: First construct a line through P that is perpendicular to l_1.

THERE ARE MANY OTHER CONSTRUCTIONS. THIS ONE USES THE IDEAS FROM EXAMPLES 2, 3, AND 4 ON THIS PAGE.



When the lengths of sides of similar triangles are compared, there are a surprising number of relationships we can point out.

These relationships do not depend on the unit of measure selected nor on the shape of the triangles.

In the similar triangles sketched above, a and d, b and e, and c and f are the numbers of units of measure in pairs of "corresponding" sides.

The basic relationship is expressed by:

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

This is often read as "a is to d as b is to e as c is to f." It is sometimes written:

$$a:d = b:e = c:f$$

We can make several statements about relationships among the numbers $a, b, c, d, e,$ and f.

Do you believe that

$$\text{if } \frac{a}{d} = \frac{b}{e} \text{ then } a \times e = d \times b?$$

Do you believe that

$$\text{if } a \times e = d \times b \text{ or } ae = db$$

$$\text{then } \frac{a}{d} = \frac{b}{e}, \quad \frac{d}{a} = \frac{e}{b},$$

$$\frac{a}{b} = \frac{d}{e}, \quad \frac{b}{a} = \frac{e}{d} \qquad ?$$

$3 \times 21 = 7 \times 9$ . . . is it true that

$$\frac{3}{7} = \frac{9}{21}, \quad \frac{7}{3} = \frac{21}{9},$$

$$\frac{3}{9} = \frac{7}{21}, \quad \frac{9}{3} = \frac{21}{7} \qquad ?$$

Suppose that

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}.$$

Which of the statements in the green box below follow from this?

$$ae = db \qquad bf = ec \qquad af = dc$$

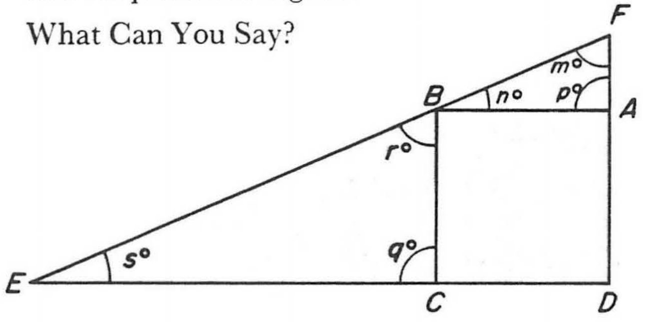
$$\frac{a}{d} = \frac{c}{f} \qquad \frac{b}{c} = \frac{e}{f} \qquad \frac{f}{d} = \frac{c}{a}$$

$$\frac{b-a}{e-d} = \frac{c}{f} \qquad \frac{b+c}{e+f} = \frac{a}{d} \qquad \frac{a+c}{d+f} = \frac{b-c}{e-f}$$

Make up statements expressing relationships among the lengths of sides of the pairs of similar triangles shown above.

A bit of review: $ABCD$ is a square, AF is an extension of DA, CE is an extension of DC, and FE passes through B.

What Can You Say?



True or false:

$q = p = 90$. T F $\quad s + r = 180$. . T F

BA is parallel to ED . . . . T F

$s = n$. . . . T F $\quad r = m$. . . . T F

$\triangle FAB \sim$ (is similar to) $\triangle BCE$. . T F

$\triangle BAF \sim \triangle EDF$ . . . . T F

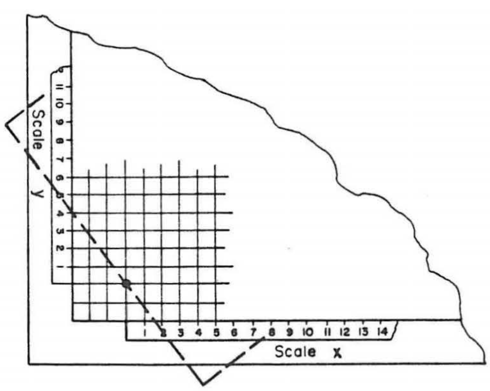
Tasks: *Students judge mathematical statements and discover relationships among similar triangles.*

Purpose: To help students understand the ratio idea.

Unifying Ideas: Structure; multiplication and division; functions and relations; geometry; measurement.

The Lesson: Here is a device that may dramatize some of the relationships we are talking about. On a large piece of cardboard, draw two perpendicular lines, each about 2 inches in from an edge. Draw lines parallel to each of the perpendicular lines and spaced 1 inch apart—so that you have ruled off an area about 8 inches by 8 inches. Make a pair of paper scales marked off in inches, halves, quarters, or as fine as you wish.

Put a thumbtack in the cardboard at any point of intersection of lines in your grid. Then place the paper scales as shown below.



Rest a piece of cardboard against the thumbtack. As it turns, it will sweep across the two scales.

In the case pictured, we can describe the position of the thumbtack (and the scales) as 3 over and 2 up. Make a table of the readings on scales y and x for each position of the cardboard.

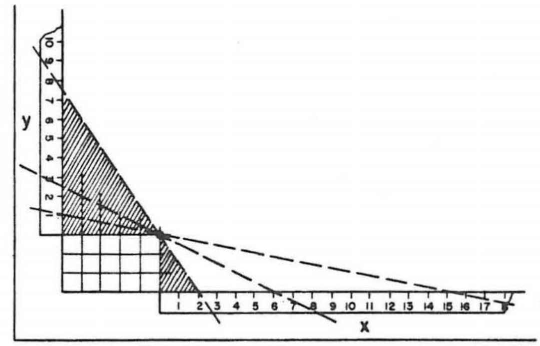
y	1	3	4	2
x	6	2	$1\frac{1}{2}$	3

etc. (and $x \times y = 6$)

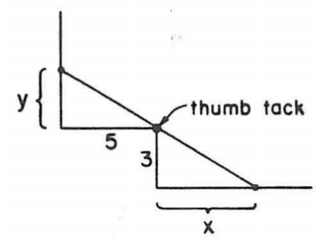
Move the thumbtack to a different position, such as 5 over and 3 up, and move the paper scales accordingly.

y	$7\frac{1}{2}$	$2\frac{1}{2}$	1	3	$3\frac{3}{4}$
x	2	6	15	5	4

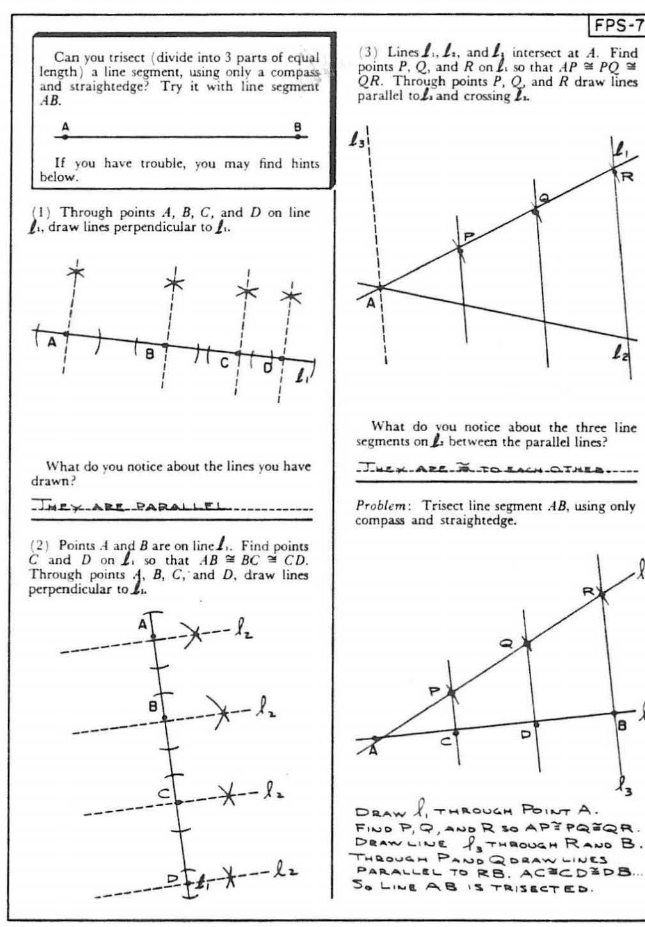
(and $x \times y = 15$)



This result is easily explained by noting the following:



$$\frac{y}{5} = \frac{3}{x} \quad \text{or} \quad x \times y = 15$$


FPS-78

Can you trisect (divide into 3 parts of equal length) a line segment, using only a compass and straightedge? Try it with line segment AB.

If you have trouble, you may find hints below.

(1) Through points A, B, C, and D on line ℓ_1, draw lines perpendicular to ℓ_1.

What do you notice about the lines you have drawn?

THEY ARE PARALLEL

(2) Points A and B are on line ℓ_1. Find points C and D on ℓ_1 so that $AB \cong BC \cong CD$. Through points A, B, C, and D, draw lines perpendicular to ℓ_1.

(3) Lines ℓ_1, ℓ_2, and ℓ_3 intersect at A. Find points P, Q, and R on ℓ_1 so that $AP \cong PQ \cong QR$. Through points P, Q, and R draw lines parallel to ℓ_3 and crossing ℓ_2.

What do you notice about the three line segments on ℓ_2 between the parallel lines?

THEY ARE ≅ TO EACH OTHER

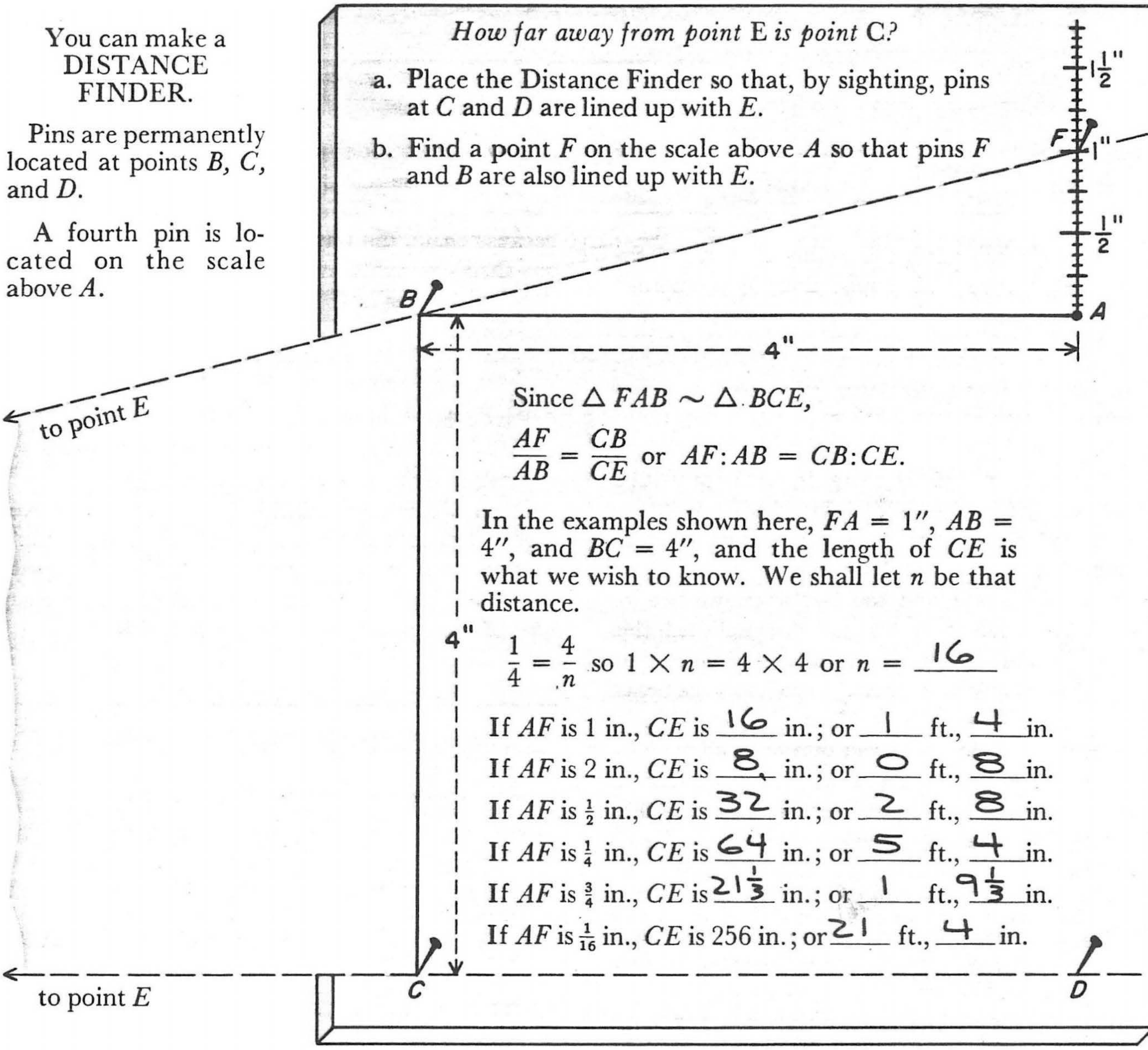
Problem: Trisect line segment AB, using only compass and straightedge.

DRAW ℓ_1 THROUGH POINT A. FIND P, Q, AND R SO AP≅PQ≅QR. DRAW LINE ℓ_3 THROUGH R AND B. THROUGH P AND Q DRAW LINES PARALLEL TO RB. AC≅CD≅DB... SO LINE AB IS TRISECTED.

You can make a DISTANCE FINDER.

Pins are permanently located at points B, C, and D.

A fourth pin is located on the scale above A.



How far away from point E *is point* C?

a. Place the Distance Finder so that, by sighting, pins at C and D are lined up with E.

b. Find a point F on the scale above A so that pins F and B are also lined up with E.

Since $\triangle FAB \sim \triangle BCE$,

$$\frac{AF}{AB} = \frac{CB}{CE} \text{ or } AF:AB = CB:CE.$$

In the examples shown here, $FA = 1''$, $AB = 4''$, and $BC = 4''$, and the length of CE is what we wish to know. We shall let n be that distance.

$$\frac{1}{4} = \frac{4}{n} \text{ so } 1 \times n = 4 \times 4 \text{ or } n = \underline{16}$$

If AF is 1 in., CE is 16 in.; or 1 ft., 4 in.

If AF is 2 in., CE is 8 in.; or 0 ft., 8 in.

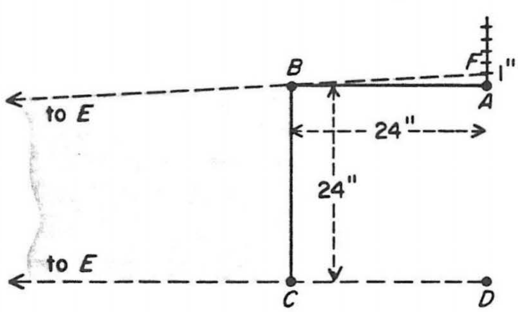
If AF is $\frac{1}{2}$ in., CE is 32 in.; or 2 ft., 8 in.

If AF is $\frac{1}{4}$ in., CE is 64 in.; or 5 ft., 4 in.

If AF is $\frac{3}{4}$ in., CE is $21\frac{1}{3}$ in.; or 1 ft., $9\frac{1}{3}$ in.

If AF is $\frac{1}{16}$ in., CE is 256 in.; or 21 ft., 4 in.

For greater accuracy and longer distances, consider a model in which AB and BC are 2 feet. Now, when AF is 1″, we have this situation:



AF is 1″, AB is 24″, CB is 24″, CE is n''

$$\frac{1}{24} = \frac{24}{n}, n = 24 \times 24 = \underline{576} \text{ inches}$$

or 48 ft., 0 in.

Here is a table for use with the Distance Finder for longer distances described on the left. *(Please complete the table.)*

AF	CE ft.	CE in.
1"	48	0
$\frac{15}{16}''$	51	2+
$\frac{7}{8}''$	54	10+
$\frac{13}{16}''$	59	1
$\frac{3}{4}''$	64	0
$\frac{11}{16}''$	69	9+
$\frac{5}{8}''$	76	10-
$\frac{9}{16}''$	85	4

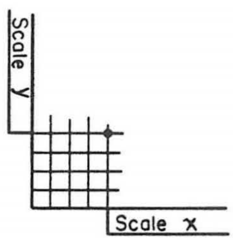
AF	CE ft.	CE in.
$\frac{1}{2}''$	96	0
$\frac{7}{16}''$	109	9-
$\frac{3}{8}''$	128	0
$\frac{5}{16}''$	153	7+
$\frac{1}{4}''$	192	0
$\frac{3}{16}''$	256	0
$\frac{1}{8}''$	384	0
$\frac{1}{16}''$	768	0

Tasks: *Students complete a group of if-then statements and tables based on comparisons of similar triangles.*

Purpose: To help students better understand the ratio idea through the study of relationships between the lengths of the sides of similar triangles.

Unifying Ideas: Structure; multiplication and division; functions and relations; geometry; measurement.

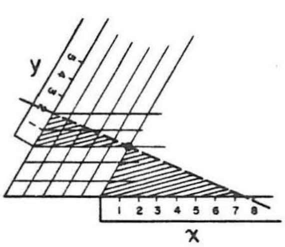
The Lesson: The Distance Finder depends on precisely the same relationships as the apparatus described in the Notes on Pupil Page 78.



and $x \times y = 16$

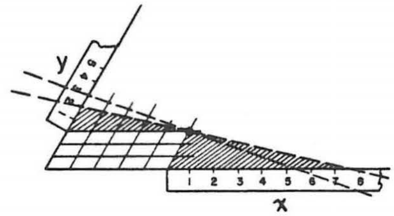
so $y = \frac{16}{x}$

But to gain deeper insight, let's consider the role played by the angle formed by the first two lines we drew on the apparatus. We drew these lines *perpendicular* to each other. What if they are not drawn perpendicular? If all else remains unchanged, we would have this situation:



We find that the relationships hold in this situation. Our results depend on the *similarity* of the triangles and on nothing else.

Suppose that one set of parallel lines were spaced differently from the other set, and the corresponding scales were also different. We would then have a situation such as this:



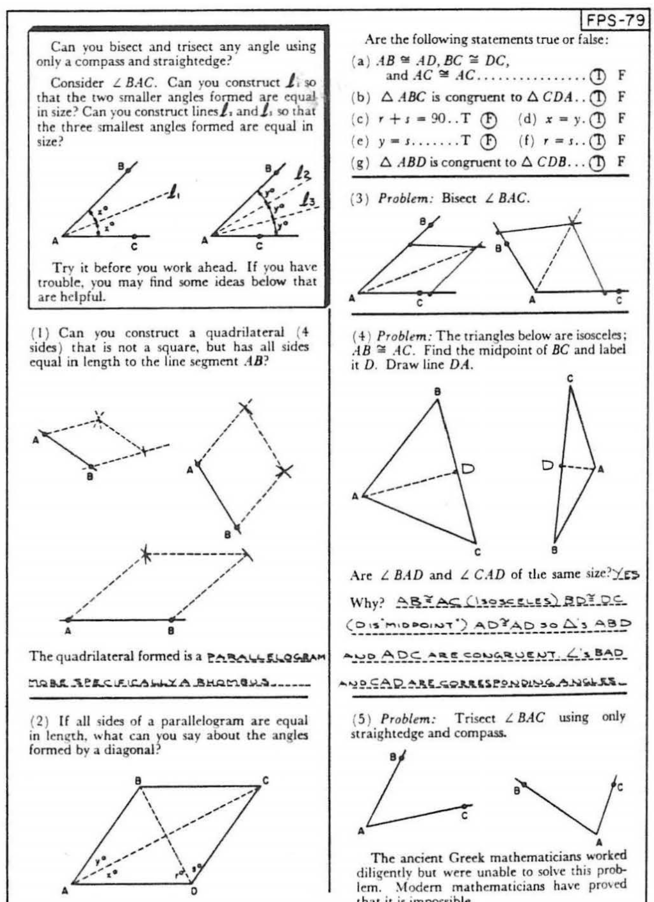
If we let u be the unit measured by the y-scale and U be the unit measured by the x-scale, we might describe what we notice in the following shorthand:

$$\frac{y(u)}{5(U)} = \frac{3(u)}{x(U)}$$

$$xy(u \times U) = 15(u \times U)$$

$$\text{so; } xy = 15$$

This kind of variation on our original apparatus is often necessary for recording relationships conveniently. Thus, in the same graph, time can be measured in hours on one scale and distance measured in miles on another.



FPS-79

Can you bisect and trisect any angle using only a compass and straightedge?

Consider $\angle BAC$. Can you construct l_1 so that the two smaller angles formed are equal in size? Can you construct lines l_2 and l_3 so that the three smallest angles formed are equal in size?

Try it before you work ahead. If you have trouble, you may find some ideas below that are helpful.

(1) Can you construct a quadrilateral (4 sides) that is not a square, but has all sides equal in length to the line segment AB?

The quadrilateral formed is a PARALLELOGRAM MORE SPECIFICALLY A RHOMBUS

(2) If all sides of a parallelogram are equal in length, what can you say about the angles formed by a diagonal?

Are the following statements true or false:

(a) $AB \cong AD$, $BC \cong DC$, and $AC \cong AC$ .... T F
(b) $\triangle ABC$ is congruent to $\triangle CDA$.. T F
(c) $r + s = 90$..T F (d) $x = y$. T F
(e) $y = s$.......T F (f) $r = s$.. T F
(g) $\triangle ABD$ is congruent to $\triangle CDB$... T F

(3) *Problem:* Bisect $\angle BAC$.

(4) *Problem:* The triangles below are isosceles; $AB \cong AC$. Find the midpoint of BC and label it D. Draw line DA.

Are $\angle BAD$ and $\angle CAD$ of the same size? YES

Why? AB ≅ AC (ISOSCELES) BD ≅ DC (D IS "MIDPOINT") AD ≅ AD SO △'S ABD AND ADC ARE CONGRUENT. ∠'S BAD AND CAD ARE CORRESPONDING ANGLES

(5) *Problem:* Trisect $\angle BAC$ using only straightedge and compass.

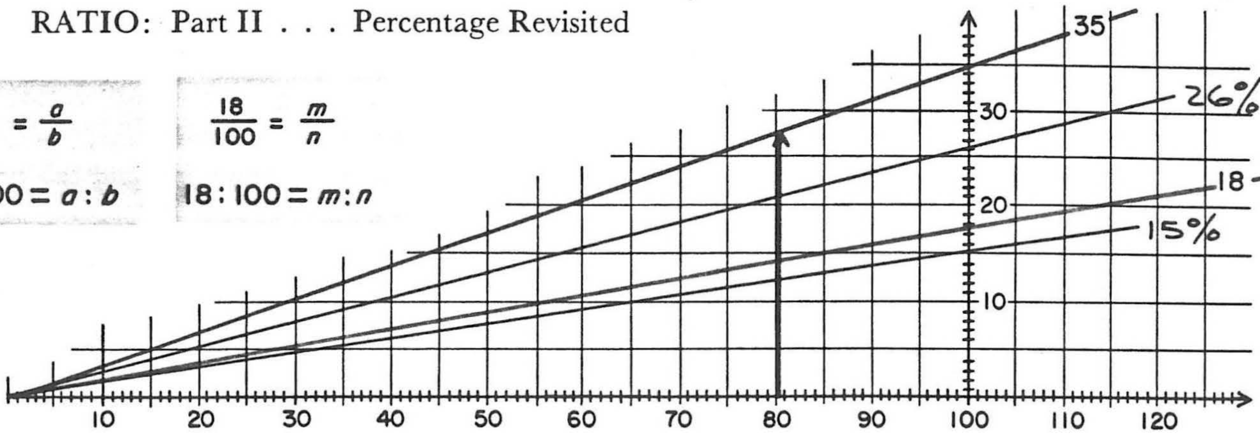
The ancient Greek mathematicians worked diligently but were unable to solve this problem. Modern mathematicians have proved that it is impossible.

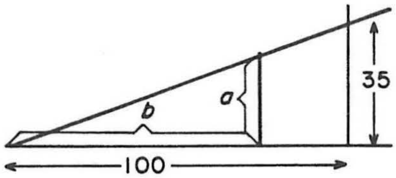
$\frac{35}{100} = \frac{a}{b}$

$35:100 = a:b$

$\frac{18}{100} = \frac{m}{n}$

$18:100 = m:n$





***35%** . . . at the same rate as 35 to 100*

There were 80 problems. Jim missed them at the rate of 35 out of 100. How many (a) out of 80 is the same rate as 35 out of 100?

$a:80 = 35:100$ or $\frac{a}{80} = \frac{35}{100}$

Read the graph. Jim must have missed about 27-29 problems.

If $\frac{a}{80} = \frac{35}{100}$ then $100a = 80 \times 35$
or $a =$ 28.

In another set of examples, Jane missed 21. This is at the rate of 35 out of 100 . . . 35%. How many problems (a) were there in the set?

$21:a = 35:100$ or $\frac{21}{a} = \frac{35}{100}$

If $\frac{21}{a} = \frac{35}{100}$ then $35a = 21 \times$ 100
or $a =$ 60.

Use the graph to complete the following:

35% of 55 is about 18-20.

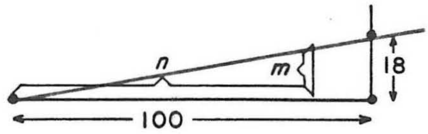
35% of 42 is about 14-16.

35% of 105 is about 36-38.

35% of 41-43 is about 15.

35% of 84-86 is about 30.

35% of 24-26 is about 9.



***18%** . . . at the same rate as 18 to 100*

$18:100 = m:n$

Read the graph to complete the following:

18% of . . .	35	70	110	55	125	100
is about . . .	6-6½	12½	20	10	22½	18

18% of . . .	67	672	6.68	6694	134
is about . . .	12	120	1.2	1200	24

18% of . . .	83	8.3	830	1660	16.6
is about . . .	15	1.5	150	300	3

18% of . . .	112	224	450	22.5	1.125
is about . . .	20	40	81	4	.2

With the use of a ruler (or other straight-edge) and the graph above, complete the following:

26% of . . .	75	120	195	65	100	265
is about . . .	20	31	51	17	26	69

15% of . . .	67	125	100	33	80	113
is about . . .	10	19	15	5	12	17

Did you notice shortcuts to help you check your results?

80-83 Part II - percentage revisited

Tasks: Students use the graph to complete statements and tables that deal with percentage problems.

Purpose: To help students better understand the ratio idea through its application to percentage problems.

Unifying Ideas: Structure; multiplication and division; functions and relations; geometry; measurement.

The Lesson: One of the more unfortunate results of the well-intentioned efforts in the early 1960's to "clean up" elementary school arithmetic was to deepen the already existing chasm between the world of arithmetic and the real world we live in. More traditional programs had accomplished this divorce by the middle or intermediate grades, but self-styled "modernists" managed to make the break even sooner—often before primary children had any idea how arithmetic is woven into our culture.

There are an indispensable handful of pure mathematicians who work very diligently to increase that body of mathematical knowledge which, as far as we know now, has nothing in common with the real world about us. These men are the poets of mathematics. Without them, the subject would not flourish. But their work should not *determine* the content of elementary school arithmetic.

Our rapid technological advances, however, have in no small measure been made possible by the more numerous group of "applied mathematicians" whose concern is to help solve problems that arise as technology accelerates its pace.

The rest of us use mathematics to varying degrees in our everyday lives, in our professions, and in our studies. There are many fields in which the individual's facility in using mathematical ideas determines how far he can progress.

The authors of MATH WORKSHOP have designed this program to provide greater understanding for *all* students of the crucial role of mathematics in the twentieth century world.

With no sense of guilt, we have added apples to apples because people really do add apples to apples. And we speak about games won divided by games played, because that's the way teams are ranked on the sports page. We shall divide miles by minutes because that's the way scientists and engineers lay out their computations. If something was observed to travel 7 miles in 15 seconds, we might write:

$$\frac{7\text{ mi.}}{15\text{ sec.}} = \frac{x\text{ mi.}}{60\text{ sec.}} = \frac{60x\text{ mi.}}{1\text{ hr.}}$$

Particularly when we employ the idea of ratio, such labels along with numbers help materially to remind us what it is we are talking about.

Pupil pages 80-83 and the corresponding practice sheets deal in many contexts with the idea of percent as ratio. In the comments to teachers for these pages, we shall try to indicate some of the broader uses of ratio in thinking about the real world.

FPS-80

"Percentage" is a method of making comparisons. The main problem is:

What is being compared and to what?

The basic idea of percentage is

a number is to *100*
as
something is to *something*
or

$$\frac{\textit{a number}}{\textit{100}} = \frac{\textit{something}}{\textit{something}}.$$

As soon as you know what the "somethings" are, the rest is simple arithmetic. Consider some typical uses of percentage.

Dr. Allen told Mr. Brown: "You ought to increase your weight by 10%."

We can translate this as:

$$\frac{10}{100} = \frac{\text{the amount of gain}}{\text{Mr. Brown's weight}}$$

If his weight were any of the following amounts — 180 lbs., 200 lbs., 150 lbs., 175 lbs. — we could write:

$$\frac{10}{100} = \frac{18}{180} \qquad \frac{10}{100} = \frac{20}{200}$$

$$\frac{10}{100} = \frac{15}{150} \qquad \frac{10}{100} = \frac{17\frac{1}{2}\text{ or }17.5}{175}$$

and know that Dr. Allen was recommending a gain of 18, 20, 15, or $17\frac{1}{2}$ lbs.

And, if Mr. Brown were to follow the doctor's recommendation, he would increase his weight to 198, 220, 165, or $192\frac{1}{2}$ lbs.

Suppose that Dr. Allen had given different advice to Mr. Case.

Dr. Allen told Mr. Case: "You weigh only 80% of what you should weigh."

Translating:

$$\frac{80}{100} = \frac{\text{your weight}}{\text{what you should weigh}}$$

If Mr. Case's weight were any of the following — 160 lbs., 180 lbs., 150 lbs., or 165 lbs. — we could write:

$$\frac{80}{100} = \frac{160}{200} \qquad \frac{80}{100} = \frac{180}{225}$$

$$\frac{80}{100} = \frac{150}{187\frac{1}{2}} \qquad \frac{80}{100} = \frac{165}{206\frac{1}{4}}$$

and know that Dr. Allen is recommending an increase to a weight of 200, 225, $187\frac{1}{2}$, or $206\frac{1}{4}$ lbs.

To follow the doctor's advice, Mr. Case must gain 40, 45, $37\frac{1}{2}$, or $41\frac{1}{4}$ lbs.

Dr. Allen advised Mr. Drew to cut his weight by 15 pounds.

"What percent cut is that?" Mr. Drew asked.

We translate Mr. Drew's question into this comparison:

$$\frac{?}{100}\text{ or }\frac{x}{100} = \frac{\text{advised reduction}}{\text{Mr. Drew's weight}}$$

If his weight were any of the following — 210, 195, 240, or 200 lbs. — we could write:

$$\frac{x}{100} = \frac{15}{210} \qquad \frac{x}{100} = \frac{15}{195}$$

$$\frac{x}{100} = \frac{15}{240} \qquad \frac{x}{100} = \frac{15}{200}$$

The doctor would be suggesting that he cut his weight by $7\frac{1}{7}$, [illegible], $6\frac{1}{4}$, or $7\frac{1}{2}$ %.

If Mr. Drew's weight were any of the above weights, and he followed the doctor's advice, he would reduce his weight to 195, 180, 225, or 185 lbs.

I. "We pay 4% interest per annum (each year) on all savings accounts."

$\frac{4}{100} = \frac{\text{interest}}{\text{savings}}$, $.04 \times \text{savings} = \text{interest}$

Interest at 4%

Amount	1 year	2 years	8 months
$15.00	$.60	$1.20	$.40
$100.00	$4.00	$8.00	$2.67
$115.00	$4.60	$9.20	$3.07
$30.00	$1.20	$2.40	$.80
$45.00	$1.80	$3.60	$1.20
$400.00	$16.00	$32.00	$10.68
$515.00	$20.60	$41.20	$13.75
$560.00	$22.40	$44.80	$14.95
$7.50	$.30	$.60	$.20
$22.50	$.90	$1.80	$.60
$11.25	$.45	$.90	$.30
$18.75	$.75	$1.50	$.50
$37.50	$1.50	$3.00	$1.00

II. 15% off all regular prices

$\frac{15}{100} = \frac{\text{reduction}}{\text{reg. price}}$, $.15 \times \text{reg. price} = \text{reduction}$

Regular Price	Reduction	Sale price
$100.00	$15.00	$85.00
$80.00	$12.00	$68.00
$20.00	$3.00	$17.00
$8.00	$1.20	$6.80
$92.00	$13.80	$78.20
$192.00	$28.80	$163.20
$96.00	$14.40	$81.60
$920.00	$138.00	$782.00
$15.00	$2.25	$12.75
$45.00	$6.75	$38.25
$90.00	$13.50	$76.50
$135.00	$20.25	$114.75
$18.00	$2.70	$15.30
$2.60	$.39	$2.21
$15.20	$2.28	$12.92

III. The won-lost records of baseball teams compare games won with games played. The percent is figured to the closest tenth of a percent, and written as thousandths.

$\frac{n}{100}$ or $\frac{10n}{1000} = \frac{\text{games won}}{\text{games played}}$

In 1965, each National League baseball team played 162 games in the regular season (except Philadelphia and St. Louis; each played only 161 games).

Complete the season's records given below in part.

	Games		
	Won	Lost	Rating
Los Angeles	97	65	.599
San Francisco	95	67	.586
Pittsburgh	90	72	.556
Cincinnati	89	73	.549
Milwaukee	86	76	.531
Philadelphia	85	76	.528
St. Louis	80	81	.497
Chicago	72	90	.444
Houston	65	97	.401
New York	50	112	.309

IV. The National Geographic Society reports that about 71%, or 139,434,000 square miles, of the earth's total surface area is water. This is about 61% of the Northern Hemisphere and about 81% of the Southern Hemisphere.

$$\frac{71}{100} = \frac{139{,}434{,}000 \text{ square miles}}{\text{total surface area of the earth}}$$

The total surface area of the earth is about 196,386,000 square miles. In the Northern Hemisphere, there are about 38,295,000 square miles of land; in the Southern Hemisphere, there are about 18,657,000 square miles of land.

The total land surface area of the earth is about 56,952,000 square miles.

Tasks: Students work four problems and complete tables and statements about them.

Purpose: To help students better understand the ratio idea through its application to percentage problems.

Unifying Ideas: Multiplication and division; functions and relations.

The Lesson: "Mr. Brown drives 110 miles in $2\frac{1}{4}$ hours. At this rate, how long will it take him to travel 200 miles?"

As soon as we write the question down in the form of an equation of ratios, we can see the problem more clearly:

$$\frac{110 \text{ miles}}{2\frac{1}{4} \text{ hours}} = \frac{200 \text{ miles}}{x \text{ hours}}$$

or:

$$\frac{110}{2\frac{1}{4}} \text{ miles per hour} = \frac{200}{x} \text{ miles per hour}$$

By using a little arithmetic, we go from:

$$\frac{110}{2\frac{1}{4}} = \frac{200}{x}$$

to: $110x = 450$

or: $x = 4\frac{1}{11}$

So it will take him about 4 hours and $5\frac{1}{2}$ minutes to travel 200 miles at the given rate. We start with a notation that reminds us of the real situation. Since both ratios refer to a rate expressed in miles per hour, we can drop the labels. Once the equation is solved, we use the "miles per hour" label to answer in the language of the original question.

We hope that students will feel completely free to use this or any other method to keep the main ideas of the examples in mind and not lose them as the arithmetic computations get under way.

Another way to express the given problem by an equation of ratios is this:

$$\frac{110 \text{ mi.}}{200 \text{ mi.}} = \frac{2\frac{1}{4} \text{ hr.}}{x \text{ hr.}}$$

Here we are saying that, for a given rate of travel, the ratio of the distances traveled is the same as the ratio of the times of travel. Since the distances are expressed in the same unit and the times are expressed in the same unit, we can drop the labels and write:

$$\frac{110}{200} = \frac{2\frac{1}{4}}{x}$$

When we have solved this equation, we must be sure to reinstate the appropriate label.

"A chemical costs \$2.18 per pound. The amount of a purchase was \$3.97. How much of the chemical was purchased?"

$$\frac{2.18}{1} \text{ dollars per lb.} = \frac{3.97}{x} \text{ dollars per lb.}$$

$$\frac{2.18}{1} = \frac{3.97}{x}$$

If we solve this equation, we find that about 1.82 pounds were purchased.

FPS-81

A. "25% discount"

$$\frac{25}{100} = \frac{\text{amount of reduction}}{\text{regular price}}$$

Consider 4 items the regular prices of which are \$128, \$56, \$18, and \$3.80.

$\frac{25}{100} = \frac{32}{128}$ $\frac{25}{100} = \frac{14}{56}$

$\frac{25}{100} = \frac{4.50}{18}$ $\frac{25}{100} = \frac{.95}{3.80}$

The amounts of reduction would be \$32, \$14, \$4.50, and \$.95. The sale prices are \$96, \$42, \$13.50, and \$2.85.

B. "Pay only 85% of the Regular Price"

$$\frac{85}{100} = \frac{\text{price you pay}}{\text{REGULAR PRICE}}$$

Regular	\$6.00	80¢	\$18.	\$1.02
Sale Price	\$5.10	68¢	\$15.30	.8670
Savings	90¢	12¢	\$2.70	.1530

$\frac{85}{100} = \frac{510}{600}$ $\frac{85}{100} = \frac{68}{80}$

$\frac{85}{100} = \frac{15.30}{18.00}$ $\frac{85}{100} = \frac{.8670}{102}$

C. "Big Savings on Every Item"

Which item has the largest percent of savings?

$$\frac{x}{100} = \frac{\text{savings}}{\text{REGULAR PRICE}}$$

Regular	\$5.00	\$9.50	\$1.50	\$85.
Sale Price	\$4.	\$7.41	\$1.23	\$64.60
Savings	\$1.00	\$2.09	27¢	\$20.40
% Savings	20%	22%	18%	24%

$\frac{20}{100} = \frac{1.00}{5.00}$ $\frac{22}{100} = \frac{2.09}{9.50}$

$\frac{18}{100} = \frac{.27}{1.50}$ $\frac{24}{100} = \frac{20.40}{85.00}$

D. The Rockets have a .875 record.

In sports, a team's record compares its games won to the total number of games played.

$$\frac{875}{1000} = \frac{87.5}{100} = \frac{\text{games WON}}{\text{games PLAYED}}$$

Complete the following record of some of the possible won-lost combinations that would explain the Rocket's record.

Won	21	7	49	28	98	133
Lost	3	1	7	4	14	19
Played	24	8	56	32	112	152

$\frac{875}{1000} = \frac{21}{24}$ $\frac{875}{1000} = \frac{7}{8}$ $\frac{875}{1000} = \frac{49}{56}$

$\frac{875}{1000} = \frac{28}{32}$ $\frac{875}{1000} = \frac{98}{112}$ $\frac{875}{1000} = \frac{133}{152}$

E. "We pay 4% interest per year"

This is the way banks and other savings institutions announce the amount of interest they pay.

$$\frac{4}{100} = \frac{\text{amount of interest}}{\text{principal}}$$

For each \$100 in the bank for 1 year, the bank pays \$4 interest — or increases the principal from \$100 to \$104.

With the interest rate at 4%, there is enough information so that you can complete the records below.

Interest	\$30	\$13	\$1.75	\$390
Principal	\$750	\$325	\$43.75	\$9,750

$\frac{4}{100} = \frac{30}{750}$ $\frac{4}{100} = \frac{13}{325}$

$\frac{4}{100} = \frac{1.75}{43.75}$ $\frac{4}{100} = \frac{390}{9,750}$

Two more examples:

$\frac{4}{100} = \frac{35.02}{875.50}$ $\frac{4}{100} = \frac{691.20}{17,280}$

Over 7 and up 1, for $\frac{1}{7}$, does not lead to as much accuracy as over 70 and up 10. And over 132 and up 180, for $\frac{180}{132}$, takes us off the chart; but over 66 and up 90 can be handled.

ratio	I look up	approx. equiv. to
$\frac{1}{7}$	$\frac{10}{70}$	$\frac{14}{100}$
$\frac{180}{132}$	$\frac{90}{66}$	$\frac{136}{100}$
$\frac{4}{9}$	$\frac{40}{90}$	$\frac{44}{100}$
$\frac{123}{112}$	$\frac{61\frac{1}{2}}{56}$	$\frac{110}{100}$
$\frac{3}{11}$	$\frac{24}{88}$	$\frac{27}{100}$
$\frac{150}{110}$	$\frac{75}{55}$	$\frac{136}{100}$
$\frac{7}{8}$	$\frac{70}{80}$	$\frac{88}{100}$
$\frac{15}{8}$	$\frac{150}{80}$	$\frac{188}{100}$
$\frac{1}{6}$	$\frac{15}{90}$	$\frac{17}{100}$
$\frac{5}{6}$	$\frac{75}{90}$	$\frac{83}{100}$
$\frac{7}{6}$	$\frac{70}{60}$	$\frac{117}{100}$
$\frac{5}{8}$	$\frac{50}{80}$	$\frac{63}{100}$
$\frac{9}{7}$	$\frac{90}{70}$	$\frac{129}{100}$
$\frac{7}{13}$	$\frac{49}{91}$	$\frac{54}{100}$
$\frac{220}{178}$	$\frac{110}{89}$	$\frac{124}{100}$
$\frac{3}{57}$	$\frac{3}{57}$	$\frac{5}{100}$

ratio	I look up	approx. equiv. to
$\frac{3}{8}$	$\frac{30}{80}$	$\frac{38}{100}$
$1\frac{3}{8}$	$\frac{110}{80}$	$\frac{138}{100}$
$\frac{7}{19}$	$\frac{35}{95}$	$\frac{37}{100}$
$\frac{26}{19}$	$\frac{130}{95}$	$\frac{137}{100}$
$\frac{64}{360}$	$\frac{16}{90}$	$\frac{18}{100}$
$\frac{1\frac{1}{2}}{21}$	$\frac{6}{84}$	$\frac{7}{100}$
$\frac{2\frac{2}{3}}{30}$	$\frac{8}{90}$	$\frac{9}{100}$
$\frac{1}{200}$	$\frac{\frac{1}{2}}{100}$	$\frac{\frac{1}{2}}{100}$
$\frac{.5}{200}$	$\frac{.25}{100}$	$\frac{.25}{100}$

$$\frac{.5}{200} = \frac{n}{100}$$

$$200 \times n = .5 \times 100$$

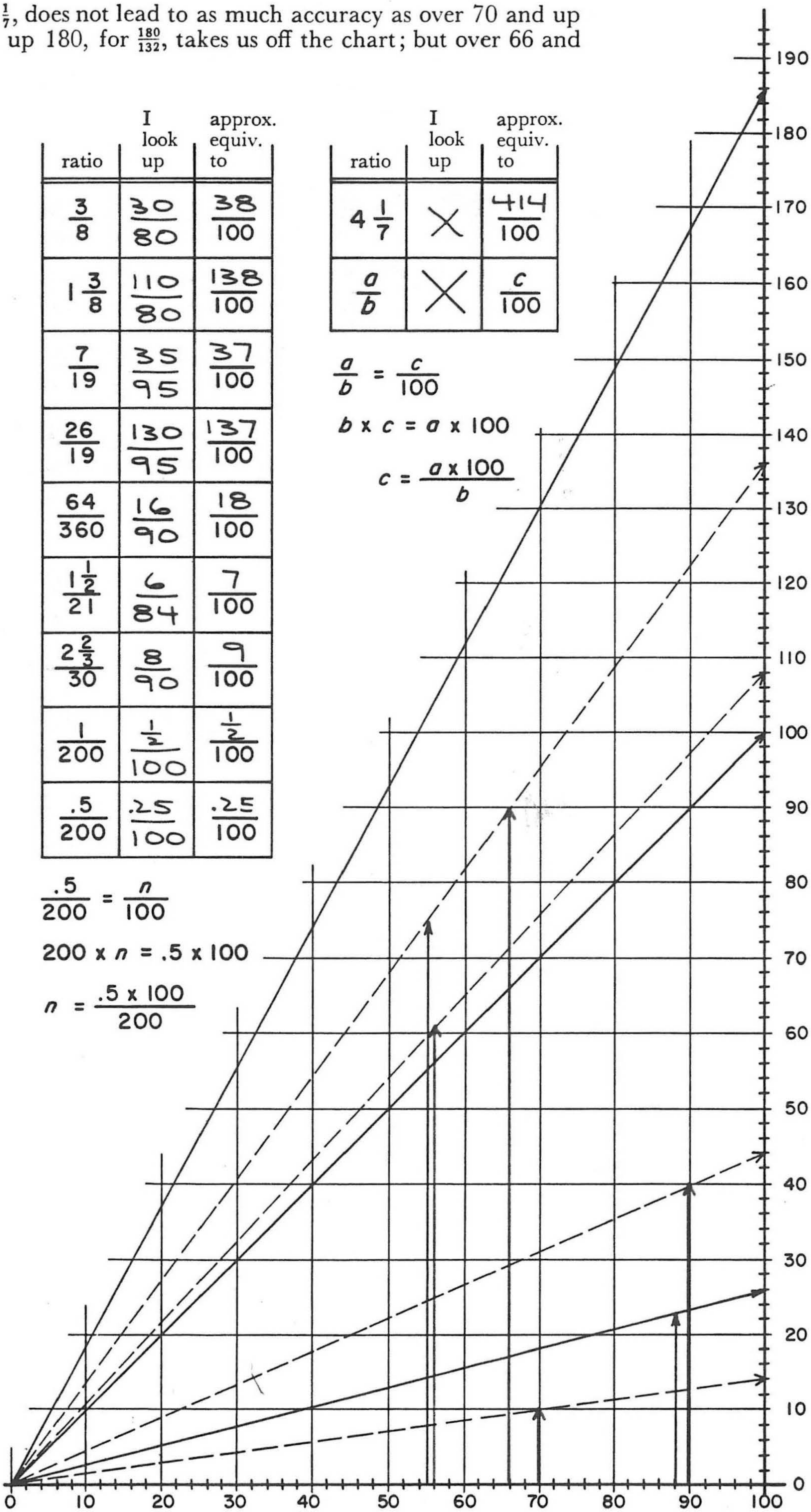
$$n = \frac{.5 \times 100}{200}$$

ratio	I look up	approx. equiv. to
$4\frac{1}{7}$	X	$\frac{414}{100}$
$\frac{a}{b}$	X	$\frac{c}{100}$

$$\frac{a}{b} = \frac{c}{100}$$

$$b \times c = a \times 100$$

$$c = \frac{a \times 100}{b}$$



Tasks: *Students use a graph to complete a table of equivalent ratios.*

Purpose: To help students better understand the ratio idea through its application to percentage problems.

Unifying Ideas: Multiplication and division; functions and relations; geometry; measurement.

The Lesson: "A car that travels 8 miles in 14 minutes is going ___ miles per hour."

$$\frac{8}{14}\text{ miles per minute} = x\text{ miles per hour}$$

Our job is to express the rate in miles per hour.

Approach I:
$$\frac{8\text{ miles}}{14\text{ minutes}} = \frac{x\text{ miles}}{1\text{ hour}} = \frac{\frac{x}{60}\text{ miles}}{1\text{ minute}}$$
$$\frac{8}{14} = \frac{\frac{x}{60}}{1}$$
$$\frac{x}{60} \times 14 = 8 \times 1$$
$$x = \frac{60 \times 8 \times 1}{14} = 34\tfrac{2}{7}$$

Approach II:
$$\frac{8\text{ miles}}{14\text{ minutes}} = \frac{8\text{ miles}}{\frac{14}{60}\text{ hour}} = \frac{x\text{ miles}}{1\text{ hour}}$$
$$\frac{8}{\frac{14}{60}} = \frac{x}{1}$$
$$\frac{14 \times x}{60} = 8 \times 1$$
$$x = \frac{8 \times 1 \times 60}{14} = 34\tfrac{2}{7}$$

Answer: The car is traveling $34\frac{2}{7}$ miles per hour.

"If a model train travels 20 feet in 6 seconds, it is traveling____miles per hour." We begin with:

$$\frac{20\text{ ft.}}{6\text{ sec.}} = \frac{x\text{ mi.}}{1\text{ hr.}}$$

We notice that we have a double job of conversion, feet into miles and seconds into hours. We might do this directly:

$$\frac{\frac{20}{5{,}280}\text{ mi.}}{\frac{6}{3{,}600}\text{ hr.}} = \frac{x\text{ mi.}}{1\text{ hr.}}$$
$$\tfrac{6}{3{,}600}\,x = \tfrac{20}{5{,}280}$$
$$31{,}680\,x = 72{,}000$$
$$x = \text{a little less than } 2.3$$

Or, we might use the following equations for the conversions required:

$$\frac{20\text{ ft.}}{6\text{ sec.}} = \frac{200\text{ ft.}}{1\text{ min.}} = \frac{12{,}000\text{ ft.}}{1\text{ hr.}}$$

We find that 12,000 feet is a little less than 2.3 times the 5,280 feet in a mile. Answer: The speed of the train is about 2.3 miles per hour.

Urge students to invent their own lines of reasoning as they work toward the solution. This approach places the major emphasis on understanding the problem and the steps in its solution. Here are some other problems you can use:

"If a man runs the 100-yard dash in 10 seconds, what is his speed in miles per hour?" Answer: about $20\frac{1}{2}$ miles per hour.

"If a pump moves 28 quarter-gallons of water per minute, it moves ___ gallons per hour." (Answer: 420 gallons per hour.)

FPS-82

An interest of _____% per annum is paid on a principal of \$_____. The amount of interest for a year is \$_____. The amount due is \$_____.

If any two blanks in the above statement are filled in, the other two can be filled in. Suppose that $x\%$ is the annual interest rate. Then

$$\frac{x}{100} = \frac{\text{interest}}{\text{principal}},$$

amount due = principal + interest,

$$\frac{100 + x}{100} = \frac{\text{amount due}}{\text{principal}}.$$

Enough information is given in the record below so that you can find the missing entries.

Rate	Principal	Interest	Amt. Due
7%	\$ 500	\$ 35.	\$ 535
25%	3,500	875.	4375.
3%	125.	3.75	128.75
9%	812.	73.08	885.08
$4\frac{1}{2}$%	812	36.54	848.54
75%	600.	450.	1,050
31%	15.00	4.65	19.65
1%	179.	1.79	180.79
14%	128.	17.92	145.92
$\frac{1}{2}$%	9710	48.55	9758.55

Here are a few hints:

$$\frac{100+7}{100} = \frac{535}{\text{PRINCIPAL}} \qquad \frac{25}{100} = \frac{\text{INTEREST}}{3500}$$

PRINCIPAL + \$3.75 = \$128.75

$$\frac{x}{100} = \frac{73.08}{812} \qquad \frac{100+75}{100} = \frac{1050}{\text{PRINCIPAL}}$$

Bill worked 85% of the problems correctly.

$$\frac{85}{100} = \frac{\text{NUMBER OF PROBLEMS DONE CORRECTLY}}{\text{number of problems done}}$$

We've gone 60% of the way.

$$\frac{60}{100} = \frac{\text{DISTANCE ALREADY TRAVELED}}{\text{TOTAL DISTANCE}}$$

What percent of the games were lost?

$$\frac{x}{100} = \frac{\text{GAMES LOST}}{\text{GAMES PLAYED}}$$

Of his allowance, he saved the same percent each month.

$$\frac{x}{100} = \frac{\text{AMOUNT SAVED}}{\text{AMOUNT OF ALLOWANCE}}$$

What rate of interest did the bank charge on the \$5,000 loan?

$$\frac{x}{100} = \frac{\text{INTEREST}}{5000}$$

What percent of the pages of the book has Mary finished?

$$\frac{x}{100} = \frac{\text{PAGES READ}}{\text{TOTAL PAGES OF THE BOOK}}$$

My age is what percent of my sister's?

$$\frac{x}{100} = \frac{\text{MY AGE}}{\text{MY SISTER'S AGE}}$$

In arranging to buy a house, Mr. and Mrs. Kilmer agreed to make a 15% down payment.

$$\frac{15}{100} = \frac{\text{DOWN PAYMENT}}{\text{PRICE OF THE HOUSE}}$$

A. Mr. Baxter received an annual interest payment of $25.50 on an investment of $850. What was the annual rate of interest?

$$\frac{n}{100} = \frac{\text{interest}}{\text{investment}} \quad \text{or } n:100 = \text{int.}:\text{inv.}$$

The annual rate of interest is 3 %.

B. Forty-five students are absent. That is 5% of the enrollment. What is the enrollment?

$$\frac{5}{100} = \frac{\text{absentees}}{\text{enrollment}} \quad 5:100 = \text{abs.}:\text{enr.}$$

The total enrollment is 900 students.

C. Mrs. Allen earns a commission of 12%. In a certain period, she reported $1,525.00 sales. What is her commission for that period?

$$\frac{12}{100} = \frac{\text{commission}}{\text{sales}}$$

Mrs. Allen is entitled to $183.00 in commission.

D. Mr. Martin borrowed some money for a year. He paid $22 interest at the rate of 4%. How much did he borrow?

$$\frac{4}{100} = \frac{\$22}{n}$$

He must have borrowed $550.00.

E. Barry gave correct answers to 34 out of 40 problems. What percent did he get correct, and what percent did he miss?

$$\frac{n}{100} = \frac{34}{40}$$

Barry had 85 % of the answers correct.
He missed 15 % of the questions.

F. 120 is 30% of what?

$$\frac{30}{100} = \frac{120}{n}$$

120 is 30% of 400.

G. 150% of 700 is 1,050.

$$\frac{150}{100} = \frac{1{,}050}{700}$$

H. 12 is 75 % of 16.

$$\frac{n}{100} = \frac{12}{16}$$

I. 16 is $133\frac{1}{3}$ % of 12.

$$\frac{n}{100} = \frac{16}{12}$$

J. $\frac{80}{100} = \frac{28}{35}$

K. $\frac{14}{100} = \frac{91}{650}$

L. $\frac{7}{100} = \frac{29.40}{420}$

M. $\frac{65}{100} = \frac{228.80}{352}$

N. $\frac{19}{100} = \frac{9.5}{50}$

O. $\frac{31}{100} = \frac{155}{500}$

P. $\frac{125}{100} = \frac{35}{28}$

Q. $\frac{200}{100} = \frac{62.5}{31.25}$

R. $\frac{175}{100} = \frac{437.50}{250}$

S. $\frac{113}{100} = \frac{88.14}{78}$

T. $\frac{180}{100} = \frac{52.20}{29}$

U. $\frac{300}{100} = \frac{51}{17}$

V. $\frac{5}{100} = \frac{17.50}{350}$

W. $\frac{2.5}{100} = \frac{8.75}{350}$

X. $\frac{19}{100} = \frac{95}{500}$

Y. $\frac{9\frac{1}{2}}{100} = \frac{47.5}{500}$

Z. $\frac{25}{100} = \frac{31.25}{125}$

a. $\frac{12\frac{1}{2}}{100} = \frac{31.25}{250}$

b. $\frac{130}{100} = \frac{982.8}{756}$

c. $\frac{130}{100} = \frac{491.4}{378}$

d. 50 % is 25 % of 200 %

Tasks: Students complete percentage problems.

Purpose: To help students better understand the ratio idea through its application to percentage problems.

Unifying Ideas: Multiplication and division; functions and relations.

The Lesson: Standardized achievement tests usually include several one-step "word problems" such as: "A passenger car pays a toll of $2.80 for 176 miles. What is the cost per mile (to the nearest tenth of a cent)?"

One way to handle the problem is:

$$\frac{280¢}{176\text{mi.}} = \frac{x¢}{1\text{mi.}}$$

$$x = \text{about } 1.6$$

More difficult word problems often require two steps for their solution. For example: "If Mr. Jones paid $3.42 for $2\frac{1}{4}$ pounds of seed, how much would he pay for 5 pounds?"

$$\frac{\$3.42}{2\frac{1}{4}\text{ lbs.}} = \frac{x\text{ dollars}}{5\text{ lbs.}}$$

$$2\tfrac{1}{4}x = 17.10$$

$$x = 7.60$$

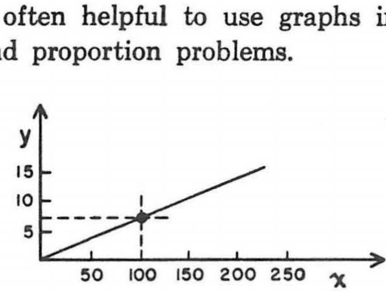
He would pay $7.60 for 5 pounds.

We suggest that students be asked to make up their own word problems and then state them in the form of equal ratios. Setting up the equations is good practice in itself. Solving the equations is worthwhile only if the students actually need practice in computation.

Whenever we refer to "7 percent," "150%," or "x percent," we indicate a ratio:

$$\frac{7}{100},\ \frac{150}{100},\ \text{or: } \frac{x}{100}$$

It is often helpful to use graphs in discussing ratio and proportion problems.



This graph shows the line for the ratio named by "7%"—7 per hundred. When we then talk about "interest," we shall assign the dollar-amount of principal to the horizontal scale and the dollar-amount of interest to the vertical scale. Thus we have the equation:

$$\frac{7}{100} = \frac{\text{interest}}{\text{principal}}$$

FPS-83

1. Total amount sold this week is 10% larger than last week.

$$\frac{110}{100} = \frac{\text{this week's sales}}{\text{last week's sales}}$$

2. What percent of cars are white?

$$\frac{x}{100} = \frac{\text{WHITE CARS}}{\text{all cars}}$$

3. If principal and rate of interest (x%) are known, how can you find the total amount due?

$$\frac{100+x}{100} = \frac{\text{AMOUNT DUE}}{\text{PRINCIPAL}}$$

4. There were x% fewer fatal accidents this year than last year.

$$\frac{100-x}{100} = \frac{\text{no. this year}}{\text{no. last year}}$$

5. Mr. Bailey made a down payment of x% on a new car. How much does he have left to pay?

$$\frac{100-x}{100} = \frac{\text{LEFT TO PAY}}{\text{price of car}}$$

6. If the rate of interest is x% for a year, what is the amount of interest due in 6 months?

$$\frac{\frac{1}{2}x}{100} = \frac{\text{6 MO. INTEREST}}{\text{principal}}$$

7. If the average interest rate on x million dollars worth of bonds is 3.6%, how much is the annual interest payment?

$$\frac{3.6}{100} = \frac{\text{INTEREST}}{x\text{ million}}$$

8. What percent of the games were lost?

$$\frac{x}{100} = \frac{\text{GAMES LOST}}{\text{GAMES PLAYED}}$$

9. A percent (x%) is added to the price to cover shipping. What is the total amount?

$$\frac{100+x}{100} = \frac{\text{TOTAL AMOUNT}}{\text{PRICE}}$$

10. A salesman is entitled to x% commission on a sale. What is his commission?

$$\frac{x}{100} = \frac{\text{COMMISSION}}{\text{SALE}}$$

11. I spent x% of my savings. How much have I left?

$$\frac{100-x}{100} = \frac{\text{savings left (after spending)}}{\text{savings (before spending)}}$$

12. Mr. Jones borrowed some money from the bank at an annual interest rate of $4\frac{1}{2}$%. How much was the interest for the first year?

$$\frac{4\frac{1}{2}}{100} = \frac{\text{INTEREST}}{\text{PRINCIPAL}}$$

13. Mr. Woods wished to buy a house. The bank would loan only 40% of the purchase price. What is the largest loan they would make?

$$\frac{40}{100} = \frac{\text{LOAN}}{\text{PURCHASE PRICE}}$$

14. What percent of the road is paved?

$$\frac{x}{100} = \frac{\text{LENGTH OF PAVED ROAD}}{\text{TOTAL LENGTH OF ROAD}}$$

$\begin{array}{r} 3\frac{1}{5} \\ ?\,\overline{)\;?\;} \end{array}$ 22% are absent

When we select a particular way to express a ratio, we often lose a little bit of history.

Consider the way we report results of an example involving computation.

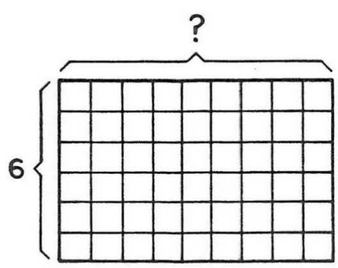
What was the division problem that has $3\frac{1}{5}$ as its quotient?

How many students are absent if 22% are absent?

Remainders

It is usually considered good form to report a remainder as a fraction in its "lowest terms."

How many columns of tile will there be if we try to arrange 52 tiles, 6 to a column.



Consider three ways of reporting results:

(a) 8 columns and 4 tiles left over.

(b) 8 and $\frac{2}{3}$ columns.

(c) 8 full columns and 4 out of 6 in another column.

Which way of reporting gives the most information?

Now consider these three reports of results of another grouping of tiles.

(a) 4 columns and 2 tiles left over.

(b) 4 and $\frac{1}{4}$ columns.

Neither report by itself provides enough information to know just what the activity was.

However, consider this report:

(c) 4 full columns and 2 out of 8 in another column.

Now you know that there were 34 tiles to be arranged, 8 to a column.

The two usual ways of reporting remainders are:

$\begin{array}{r} 4\text{ R}3 \\ 12\overline{)51} \end{array}$ and $\begin{array}{r} 4\frac{1}{4} \\ 12\overline{)51} \end{array}$

If this were a "tile" example, the ratio of 3 out of 12 for the remainder column is changed to its "simplest form" — and consequently a bit of its history is lost.

To preserve this history, we might agree to make a full report:

$$51 \div 12 = 4\tfrac{3}{12}$$

Or, we might decide to give both of the other reports for a division example. If you think about it, you can reconstruct the following examples:

a. $\begin{array}{r} 12\text{ R}4 \\ 8\overline{)100} \end{array}$ and $\begin{array}{r} 12\frac{1}{2} \\ 8\overline{)100} \end{array}$

b. $\begin{array}{r} 8\text{ R}6 \\ 9\overline{)78} \end{array}$ and $\begin{array}{r} 8\frac{2}{3} \\ 9\overline{)78} \end{array}$

c. $\begin{array}{r} 24\text{ R}4 \\ 7\overline{)172} \end{array}$ and $\begin{array}{r} 24\frac{4}{7} \\ 7\overline{)172} \end{array}$

d. $\begin{array}{r} 13\text{ R}9 \\ 12\overline{)165} \end{array}$ and $\begin{array}{r} 13\frac{3}{4} \\ 12\overline{)165} \end{array}$

In the following, the remainder describes the full history. Can you reconstruct the examples?

e. $\begin{array}{r} 14\frac{3}{9} \\ 9\overline{)129} \end{array}$

f. $\begin{array}{r} 17\frac{7}{14} \\ 14\overline{)245} \end{array}$

g. $\begin{array}{r} 24\frac{15}{25} \\ 25\overline{)615} \end{array}$

h. $\begin{array}{r} 15\frac{12}{15} \\ 15\overline{)237} \end{array}$

84-85 Part III - reporting remainders

Tasks: Students study and reconstruct remainder problems.

Purpose: To help students better understand the idea of ratio through the consideration of the problem of how remainders should be reported in division.

Unifying Ideas: Numeration; multiplication and division; measurement.

The Lesson: "Mrs. McGuire used 6 of the 9 eggs she had."

"Mrs. McGuire used ⅔ of the eggs she had."

Which statement contains more information? Clearly, the first statement does; from the information contained in it, you could derive the second statement. However, from the second you cannot derive the first.

"The Beavers won 16 out of 20 games."

"The Beavers won 80% of their games."

We can say the same thing about these two statements: the first contains more information.

Thus, when we report an event using the idea "at the rate of . . . ," we are often sacrificing particular information and focusing on a rate or a ratio. Our purpose is clear—we are interested in getting our information into a form that will facilitate a comparison with other information. "Mrs. McGuire used ⅔ of the eggs she had" is, by itself, probably a very useless bit of information.

We must not overlook the fact that reducing remainders to lowest terms in a division problem robs us of some history of the computation. Somewhere along the way, we should focus attention on what a remainder is, what information it contains, and on the fact that we do lose something when we reduce it.

If we were to agree that an asterisk (*) after a remainder was an abbreviation for "I left the remainder as it occurred" then the quotient would contain enough information for us to reconstruct both the dividend and the divisor (provided there is a remainder other than zero). For example, we could tell that the quotients $7\frac{2}{3}$* and $7\frac{10}{15}$* came from quite different examples:

$$3\overline{)23}\;\;7\tfrac{2}{3} \qquad \text{and} \qquad 15\overline{)115}\;\;7\tfrac{10}{15}$$

This loss of information that we have been discussing is usually inconsequential, and we are not suggesting the adoption of the asterisk notation. Nevertheless, discussion of the notion will give students a deeper insight into how remainders arise in computation and what they mean.

Practice sheets 84 and 85 consider remainders from a very different point of view.

What numbers have a remainder of 1 when divided by 2? The odd numbers.

What numbers have a remainder of 1 when divided by 3? 1 (since $1=0\times 3+1$), 4, 7, etc.

What numbers will appear in both lists?

This kind of problem is found on practice sheet 84.

FPS-84

List numbers which —

	have remainders of	when divided by	a	b	c	d	e	f	g	h	i	j
1.	2	5	17	32	2	7	12	22	27	37	42	47
2.	4	5	4	9	14	19	24	29	34	39	44	49
3.	3	4	3	7	11	15	19	23	27	31	35	39
4.	1	3	1	4	7	10	13	16	19	22	25	28
5.	1	2	1	3	5	7	9	11	13	15	17	19
6.	1	6	1	7	13	19	25	31	37	43	49	55
7.	4	9	22	49	67	4	13	31	40	58	76	85
8.	6	7	20	76	13	27	34	41	48	55	62	69
9.	8	11	30	19	52	41	63	74	85	96	107	118
10.	5	8	69	101	5	13	21	29	37	45	53	61
11.	7	10	37	17	67	27	47	57	77	87	97	107

List numbers which satisfy both of two conditions:

(Take your time, but keep an eye open for patterns that help.)

	have remainders of	when divided by	AND have remainders of	when divided by	A	B	C	D	E	F	G
12.	1	2	1	3	7	13	19	25	31	37	43
13.	2	3	2	6	2	8	14	20	26	32	38
14.	3	5	3	4	3	23	43	63	83	103	123
15.	2	10	1	2	7	17	27	37	47	57	67
16.	7	10	1	6	7	37	67	97	127	157	187
17.	3	4	3	9	3	39	75	111	147	183	219
18.	5	7	5	11	5	82	159	236	313	390	467

22% are absent

In a similar way, reports such as the one above do not include their history.

How many students are there if all are present? We might be reporting that

11 out of 50 are absent;
22 out of 100 are absent;
99 out of 450 are absent;
154 out of 700 are absent;
385 out of 1750 are absent.

Thus, a report that "55 out of 250 are absent" tells us more than the report that "22% are absent."

In another school, the report was:

114 out of 475 are absent

That is a full report . . . but which school has the higher rate of absence?

(a) 55 out of 250, or
(b) 114 out of 475

These are the full reports, but they are hard to compare. We know that school (a) has a rate of 22 out of 100. What is the rate at school (b)?

(a) 55:250 = 22:100
(b) 114:475 = 24:100

If we preserve the full history in our reports, it may be hard to compare them with others.

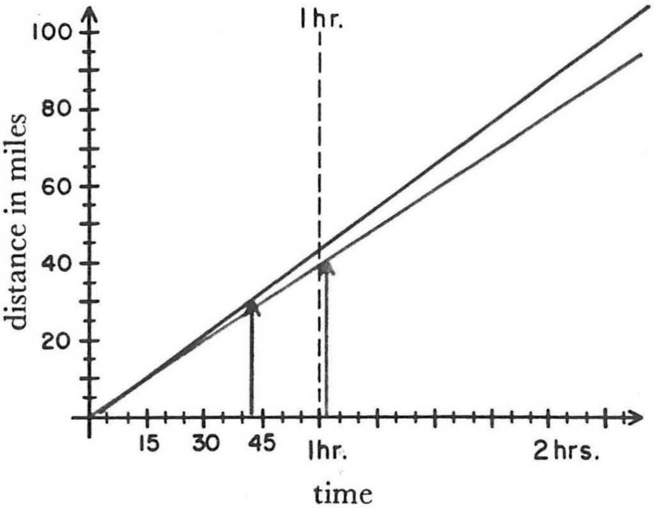
If we report in terms of "per cents" then comparisons will usually be easier.

Team A has won 9 out of 20 games and team B has won 11 out of 25 games. Which has the better record?

Answer: A, because A-45%, B-44%

② AT THE $1\frac{1}{2}$ HR. MARK A, B, AND C HAD TRAVELED 90, $67\frac{1}{2}$, AND $52\frac{1}{2}$ MILES RESPECTIVELY.

Consider a graph showing distances traveled in specified times.



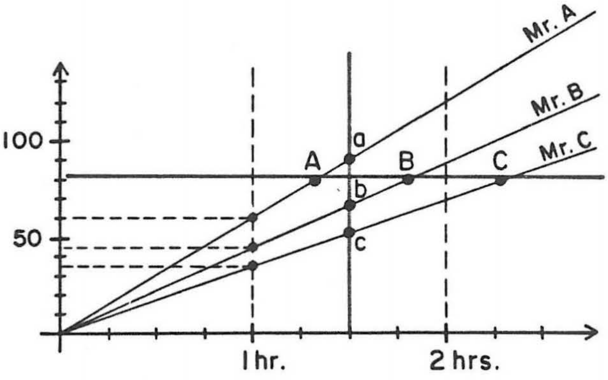
Mr. Newton traveled 30 miles in 40 minutes, and Mr. Walsh traveled 40 miles in 1 hour and 2 minutes.

The lines in the graph show the rates at which the two are traveling. We can tell that Mr. Newton is traveling at about 45 miles per hour, and Mr. Walsh at about 37 miles per hour.

By looking at the vertical arrows, we can tell the exact nature of the trip each reported — the time he had to travel the number of miles given.

But here is another report.

Mr. Albert was traveling at 60 miles per hour, Mr. Bailey at 45 miles per hour, and Mr. Collins at 35 miles per hour.



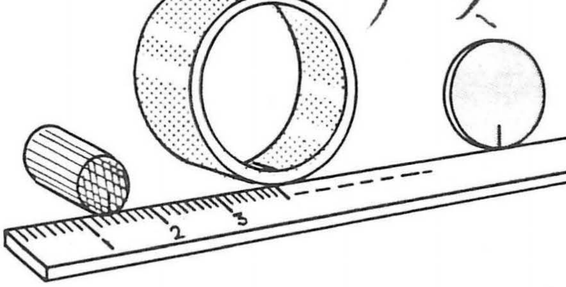
Now we know nothing but their rates of travel . . . but we have lots of information available.

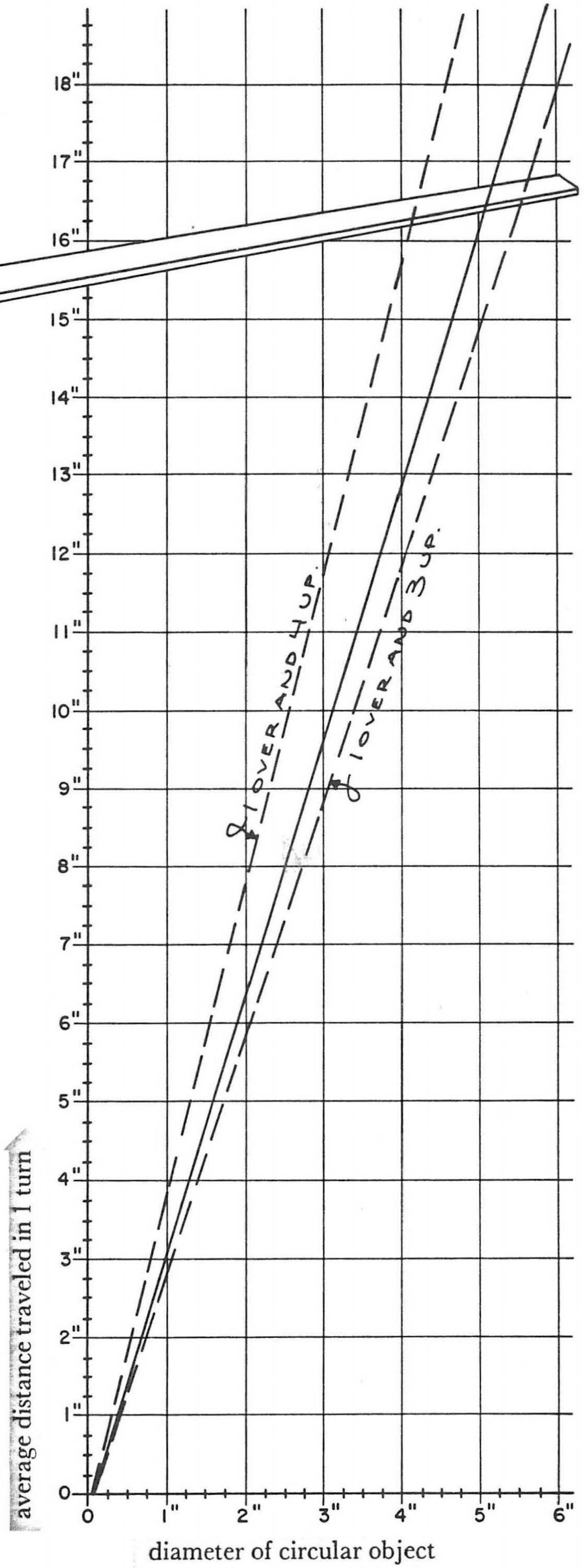
1. What do points *A, B,* and *C* on the green line suggest?

2. What do points *a, b,* and *c* on the blue line suggest?

① MR. A TRAVELED 80 MI. IN $1\frac{1}{3}$ HR.
MR. B " " " $1\frac{7}{9}$ HR.
MR. C " " " $2\frac{2}{7}$ HR.

2 1/3
3)7
6
1





You will need a yardstick or a tape measure.

Out of cardboard, cut wheels with diameters of 1″, 2″, 3″, 3½″, 4″, 5″, and 6″. Near their outside edges, put a mark so you can tell when they have moved through one turn. Roll them along the yardstick. Record in the table below and the graph on the right the distances each rolls in four experiments. Add the results and divide by four to find "average distance."

All measurements are in inches.

diam.	distance in 1 turn 1st	2nd	3rd	4th	avg.
1					
2					
3					
$3\frac{1}{2}$					
4					
5					
6					

Continue experimenting with other objects such as tin cans, rolls of tape, wheels — anything that is circular. Record as many of the results as possible in the graph.

Describe what you notice in the graph.

IT RISES AT MORE THAN 3 TO 1 – BUT NOT MUCH MORE.

86-88 Part IV - a very special ratio - pi

Tasks: Students collect data in experiments involving rolling cardboard wheels or other circular objects along a yardstick. For each object, they record in a table and a graph the distance traveled in one turn.

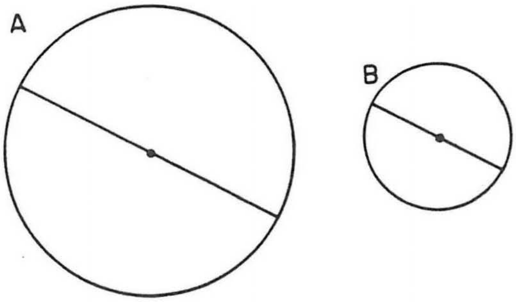
Purpose: To help students discover that the ratio of the circumference of a circle to its diameter is a constant, π (pi), or approximately $22/7$.

Unifying Ideas: Multiplication and division; functions and relations; geometry; measurement.

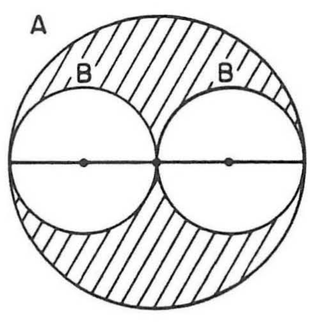
The Lesson: It is intuitively clear that the longer the diameter of a circle is, the longer its circumference will be. It is not intuitively clear that the ratio of circumference to diameter is a constant. This constant ratio is denoted by π (pi), and its value is approximately $22/8$. In fact, for every circle,

$$\frac{\text{circumference}}{\text{diameter}} \text{ is approximately equal to } \frac{22}{7}.$$

If the diameter of circle A is twice that of circle B, is the circumference of A twice that of B? Is the area of circle A twice that of circle B?



It is easy to see that the area of A is more than twice that of B, as the following diagram shows:



At this stage, there are two things we wish the student to discover: first, that the ratio of circumference to diameter is constant; second, that we can get a rough approximation for this ratio.

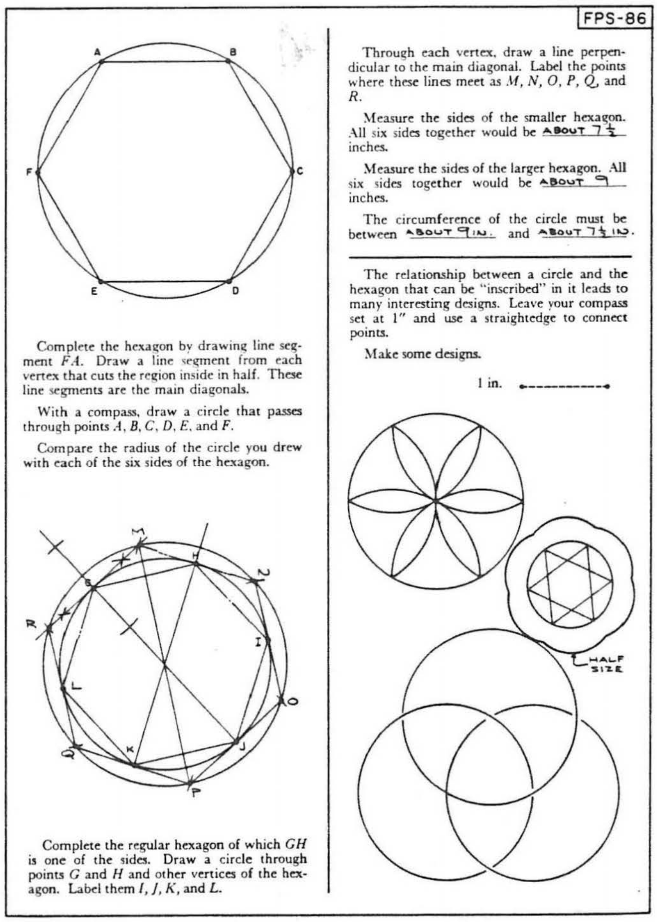
One kind of experiment is suggested on pupil page 86. We have provided a grid so that results can be recorded in graphic form. We suggest that some experiments be discussed and performed in front of the group and that other results be reported. This will enable each student to have many entries on his graph. If measurements are made with care, it will soon be clear that the points on the graph lie along a straight line. This indicates that the ratio of circumference to diameter is constant.

We have suggested using a diameter of $3\frac{1}{2}$ inches in some experiments, since the circumference is very close to 11 inches. A common approximation to π is $3\frac{1}{7}$, and the student can note that

$$\frac{\text{circumference}}{\text{diameter}} = \frac{11}{3\frac{1}{2}} = \frac{22}{7}, \text{ or } 3\frac{1}{7}.$$

You may wish to incorporate the results of all the individual experiments into one large graph.

Practice sheet 86 reintroduces the construction of a regular hexagon inside a circle and outside a circle. We find a variation of the idea on pupil page 87.


FPS-86

Complete the hexagon by drawing line segment FA. Draw a line segment from each vertex that cuts the region inside in half. These line segments are the main diagonals.

With a compass, draw a circle that passes through points A, B, C, D, E, and F.

Compare the radius of the circle you drew with each of the six sides of the hexagon.

Complete the regular hexagon of which GH is one of the sides. Draw a circle through points G and H and other vertices of the hexagon. Label them I, J, K, and L.

Through each vertex, draw a line perpendicular to the main diagonal. Label the points where these lines meet as M, N, O, P, Q, and R.

Measure the sides of the smaller hexagon. All six sides together would be ABOUT $7\frac{1}{2}$ inches.

Measure the sides of the larger hexagon. All six sides together would be ABOUT 9 inches.

The circumference of the circle must be between ABOUT 9 IN. and ABOUT $7\frac{1}{2}$ IN.

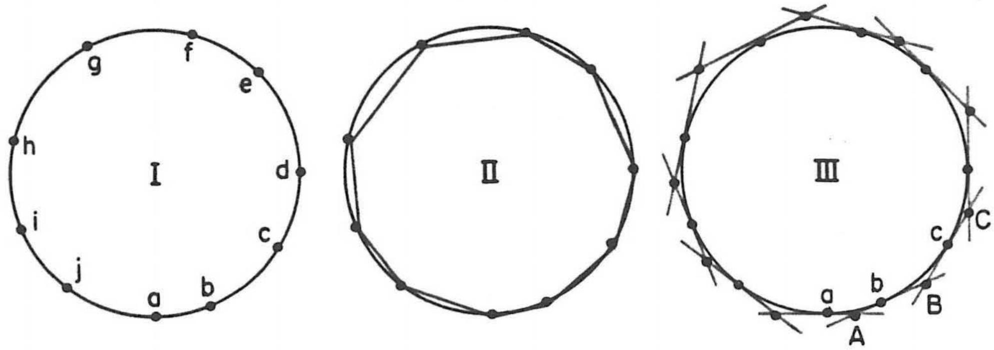
The relationship between a circle and the hexagon that can be "inscribed" in it leads to many interesting designs. Leave your compass set at 1" and use a straightedge to connect points.

Make some designs.

1 in.

Another Experiment.

Draw a circle any size. Mark some points on the circle, any distance apart — they need not be equally spaced (see I).



Connect these points with line segments (see II).

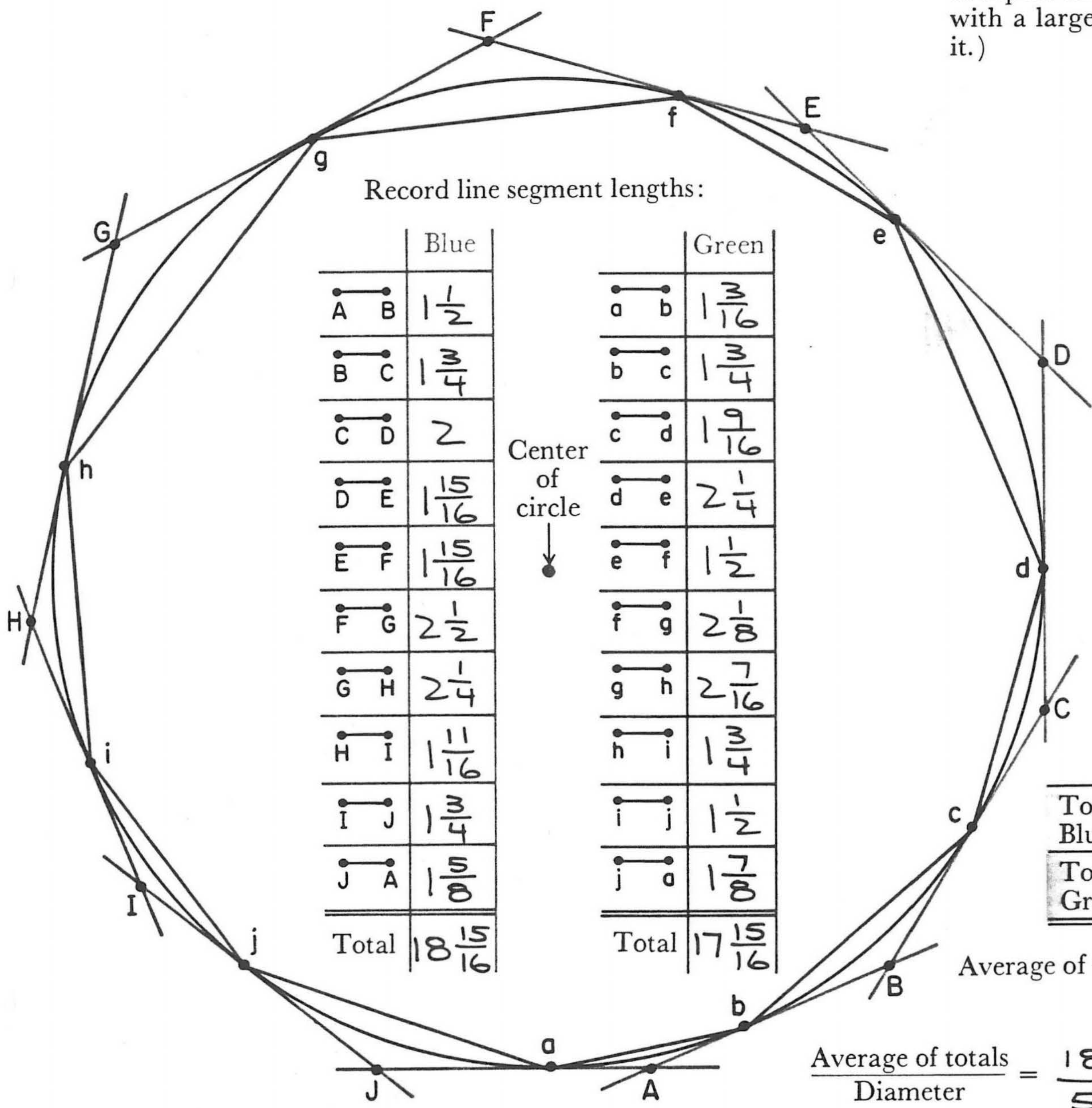
Draw line segments outside the circle that touch it only at the points you have already indicated. Where they cross each other, locate more points (see III).

(Below is a single, enlarged drawing that we shall use.)

Measure each green line segment and add the results. Using the second group of points, measure each blue line segment and add the results.

Why must the distance around the circle — called its *circumference* — be somewhere between these two totals?

(Perhaps you would like to repeat this experiment with a larger circle. Try it.)



Record line segment lengths:

	Blue
AB	$1\frac{1}{2}$
BC	$1\frac{3}{4}$
CD	2
DE	$1\frac{15}{16}$
EF	$1\frac{15}{16}$
FG	$2\frac{1}{2}$
GH	$2\frac{1}{4}$
HI	$1\frac{11}{16}$
IJ	$1\frac{3}{4}$
JA	$1\frac{5}{8}$
Total	$18\frac{15}{16}$

	Green
ab	$1\frac{3}{16}$
bc	$1\frac{3}{4}$
cd	$1\frac{9}{16}$
de	$2\frac{1}{4}$
ef	$1\frac{1}{2}$
fg	$2\frac{1}{8}$
gh	$2\frac{7}{16}$
hi	$1\frac{3}{4}$
ij	$1\frac{1}{2}$
ja	$1\frac{7}{8}$
Total	$17\frac{15}{16}$

Total of Blue	$18\frac{15}{16}$
Total of Green	$17\frac{15}{16}$
Average of totals	$18\frac{7}{16}$

$$\frac{\text{Average of totals}}{\text{Diameter}} = \frac{18\frac{7}{16}}{5\frac{13}{16}} = 3.172$$

Tasks: Students measure and record the sums of the measurements of the sides of two irregular polygons—one inside and one outside of a circle—and average these totals.

Purpose: To help students discover that the ratio of the circumference of a circle to its diameter is a constant.

Unifying Ideas: Multiplication and division; functions and relations; geometry; measurement.

The Lesson: There is a unique aspect to the experiment described on pupil page 87 that is worth discussing. Notice that the first step is to select points *at random* on the circle.

In most such experiments, equally spaced points are selected. The result is a regular polygon inscribed in a circle which is itself inscribed in a larger regular polygon with the same number of sides. This leads to a shorter experiment, since the perimeter of each polygon can be found by measuring the length of a side and multiplying by the number of sides. Such a procedure has the disadvantage of multiplying any errors in measurement and construction by a factor equal to the number of sides.

On the other hand, in our experiment we use points selected at random, and this leads to two *irregular* polygons. Of course, to find the perimeters of these polygons we must make many measurements, and each measurement will be an approximation. But the errors in the measurements will tend to offset one another rather to accumulate as they do when regular polygons are used.

The general idea of the experiment is not at all unusual—rather, it is classic: to trap a number (of units of measurement) between a larger number and a smaller number. In this case, the trapping is done by using polygons with more and more sides. As the number of sides increases, the gap between the perimeters decreases, and we close in on the number we are looking for.

With only ten points and a circle 6 inches in diameter, we can obtain a very close approximation for π. The number π cannot be written exactly as a common decimal fraction. (Thus π is an irrational number.) We can, however, approximate it as closely as we please. (See pupil page 88.)

What kind of experiment would give a closer approximation than that resulting from the measurements and computations on pupil page 87? Among other things, we might

- use a larger circle,
- select more points,
- take greater care in using instruments.

We might also use experiments such as those on pupil page 86 and on practice sheet 87.

After all different approaches have been tried, we might average the results of all the experiments.

How close were the various results to the usual approximations?

EPS-87

Experiment I

Find different size wheels or cylindrical objects, such as tin cans. Put a mark on each so that you can tell when it has made a complete revolution.

Roll each object over a long distance, counting the revolutions. Carefully measure the distance the object traveled and the diameter of the object. The total distance divided by the number of revolutions will be a good approximation to the circumference.

	Name of object	Number of revolutions	Distance rolled	Circumference: $\frac{\text{distance}}{\text{no. of rev.}}$	Diameter of object	Circumference divided by diameter
1						
2						
3						
4						
5						
6						
7						
8						

Add the entries in the right-hand column and divide the total by the number of entries to find the AVERAGE. _____________

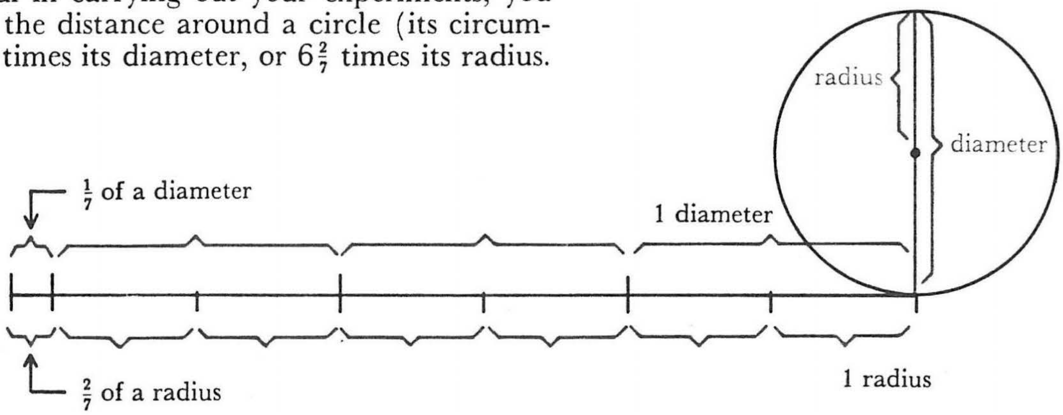
Experiment II

Use the same or other cylindrical objects and a supply of fine thread. Wrap thread neatly around the cylinder a large number of times. Count the turns and measure the length of the thread used. The length of thread divided by the number of turns will be a close approximation to the circumference.

	Name of object	Number of turns	Length of thread used	Circumference: $\frac{\text{thread length}}{\text{no. of turns}}$	Diameter of object	Circumference divided by diameter
1						
2						
3						
4						
5						
6						
7						
8						

Add the entries in the right-hand column and divide the total by the number of entries to find the AVERAGE. _____________

If you were careful in carrying out your experiments, you probably found that the distance around a circle (its circumference) is about $3\frac{1}{7}$ times its diameter, or $6\frac{2}{7}$ times its radius.



3.14159265358979323846264338327950288419716939937510 $\approx \pi$

The ratio $\frac{\text{circumference}}{\text{diameter}}$ cannot be exactly expressed, no matter how many decimal places we use.

About 300 B.C., some Egyptian mathematicians found that 3.1416 was a close approximation to this ratio. In about 1579, a Roman by the name of Vieta, found an even closer approximation: 3.1415926535.

Modern computers have written down the first 10,000 decimal place digits . . . the first 50 are given above.

So, we simply say that

$$\frac{\text{circumference}}{\text{diameter}} = \pi.$$

π is a Greek letter. We pronounce it as "pie."

Then, we use as close an approximation as we need. Often we use $3\frac{1}{7}$; for more accuracy, we use 3.1416 — but astronomers and engineers sometimes need closer approximations.

Suppose that we consider the diameter of the earth as 8,000 miles and use three different approximations:

(a) $3\frac{1}{7} \times 8{,}000 =$ 25,142.86

(b) $3.1416 \times 8{,}000 =$ 25,132.80

(c) $3.14159265 \times 8{,}000 =$ 25,132.74

The difference between (a) and (b) is about 10.06 miles.

The difference between (b) and (c) is about 320.80 feet.

Using d for diameter, r for radius, c for circumference, and π as the ratio of the circumference of a circle to its diameter, we write:

$c : d = \pi : 1$ or $\frac{c}{d} = \frac{\pi}{1} = \pi$

$2r = d$

$c : 2r = \pi : 1$ or $\frac{c}{2r} = \frac{\pi}{1} = \pi$

Remembering our investigation on page 78, we can write:

$c = d \times$ π $\qquad c = 2r \times$ π

The last two statements are usually given as:

$c = \pi d = 2\pi r$ and it follows

that $d = \frac{c}{\pi}$ and $r = \frac{c}{2\pi}$.

Using $3\frac{1}{7}$ as an approximation to π, check the results you graphed on page 86 and check your final result on page 87. What would you report in general?

THE RATIO OF THE CIRCUMFERENCE OF A CIRCLE TO ITS DIA. IS THE CONSTANT π.

Tasks: Students consider and test several approximations of π and then consider several relationships involving this ratio.

Purpose: To help students discover that the ratio of the circumference of a circle to its diameter is a constant.

Unifying Ideas: Multiplication and division; functions and relations; geometry; measurement.

The Lesson: To emphasize the fact that we are using approximations to π, we suggest that you ask students to consider $3\frac{1}{8}$ as an approximation to π. Is it closer than $3\frac{1}{7}$? No, it's not as close. Does it often lead to faster computations? Yes, since the improper fraction form of $3\frac{1}{8}$ is $\frac{25}{8}$, or $\frac{100}{32}$, and 8 has more computational advantages than 7.

As an example, consider the problem at the bottom of the first column:

$$3\frac{1}{8}, \text{ or } \frac{25}{8}, \times 8{,}000 \text{ miles} = 25{,}000 \text{ miles}$$

You can do this arithmetic in your head. And notice that the diameter of the earth is itself an approximation and rounded off to 8,000 miles. Further, we usually refer to the circumference of the earth as about 25,000 miles at the equator. (Our best information at present is that the polar diameter is about 7,899.88 miles while the equatorial diameter is about 7,026.68 miles.) In this particular case, an approximation of $3\frac{1}{8}$, taken together with an approximation of 8,000 miles as the diameter, just happens to give a better approximation of the earth's circumference than $3\frac{1}{7}$.

This ought to be enough evidence to persuade some of the students to champion the cause of $3\frac{1}{8}$ as being more useful than $3\frac{1}{7}$. Quite clearly, the only argument for $3\frac{1}{7}$ is that it is closer than $3\frac{1}{8}$. What value shall we place on accuracy?

How much closer to 3.1416 is $3\frac{1}{7}$ than $3\frac{1}{8}$?

3.1428 is .0013 too large.
3.1250 is .0166 too small.

What is the percent of error?

$$\frac{x}{100} = \frac{.0013}{3.1416} \qquad \frac{x}{100} = \frac{.0166}{3.1416}$$

. . . about .04% too large and .5% too small, respectively. What is the difference between $3\frac{1}{8}$ and $3\frac{1}{7}$?

$$3\frac{1}{7} - 3\frac{1}{8} = \frac{22}{7} - \frac{25}{8} = \frac{1}{56}$$

There is, of course, no answer to the question of which is a more useful approximation. However, the discussion is healthy and may provide some useful practice in computation.

A companion problem to those on practice sheet 88 is, "How smooth is the earth's surface?" The highest peak is Mt. Everest, which is 29,028 feet, or $5\frac{1}{2}$ miles, high. If a scale model of the earth were made 2 inches in diameter, how high would we make the point representing Mt. Everest?

$$\frac{5\frac{1}{2} \text{ mi.}}{4{,}000 \text{ mi.}} = \frac{x}{1 \text{ in.}}$$

$$x = .001\tfrac{1}{3} \text{ in., approximately}$$

. . . a little over $\frac{1}{1{,}000}$ of an inch high. By comparison, the earth is even smoother than a billiard ball!

FPS-88

Problem I. Imagine that the surface of the world is perfectly smooth and that a wire is laid around the equator — 25,000 miles long.

Now suppose that you wish to raise this wire so that it is 10 feet off the globe all the way around the equator. You would, of course, need more wire. How much more? (We are, of course, assuming the wire would not stretch and that it won't sag between the poles holding it 10 feet off the surface.)

Make a guess: will you need 25 feet more, 60 feet more, 250 feet more, 1000 feet more, a mile more, 3 miles more, 100 miles more, 1000 miles more, 5000 miles more? Which of the above lengths is closest to what you will need? 60 FEET (make a guess)

Before continuing, you might check your guess. If you have trouble, the following discussion might help.

Use 3.14, or $3\frac{1}{7}$, as an approximation to π, and use the formula for finding the circumference to complete the following table.

radius	circumference	
7 inches	43.96 OR 44	in.
8 inches	50.24 OR $50\frac{2}{7}$	in.
1 foot	6.28 OR $6\frac{7}{25}$	ft.
10 feet	62.8 OR $62\frac{6}{7}$	ft.
70 feet	439.6 OR 440	ft.
80 feet	502.4 OR $502\frac{6}{7}$	ft.
700 feet	4396 OR 4400	ft.
710 feet	4458.8 OR $4462\frac{6}{7}$	ft.
1 mile	6.28 OR $6\frac{2}{7}$	mi.
7 mies	43.96 OR 44	mi.

Use $\frac{22}{7}$ as an approximation to π in completing the next table.

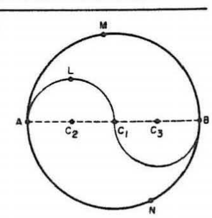
radius	circumference	
1400 feet	8800	ft.
1410 feet	$8862\frac{6}{7}$	ft.
2156 feet	13,552	ft.
2166 feet	$13{,}614\frac{6}{7}$	ft.
2156 miles	13,552	mi.
2156 mi. + 10 ft.	13,552 mi. + $62\frac{6}{7}$	ft.
4000 miles	$25{,}142\frac{6}{7}$	mi.
4000 mi. + 10 ft.	$25{,}142\frac{6}{7}$ mi. + $62\frac{6}{7}$	ft.
Radius of Earth	25,000	miles
Radius of Earth + 10 ft.	25,000 mi. + $62\frac{6}{7}$	ft.

If the wire were 10 ft. above the surface of the earth, it would make a circle with a radius 10 ft. more than the radius of the earth.

So, the last entry in the table above provides the solution to our problem:

You would need $62\frac{6}{7}$ more wire.

Problem II.



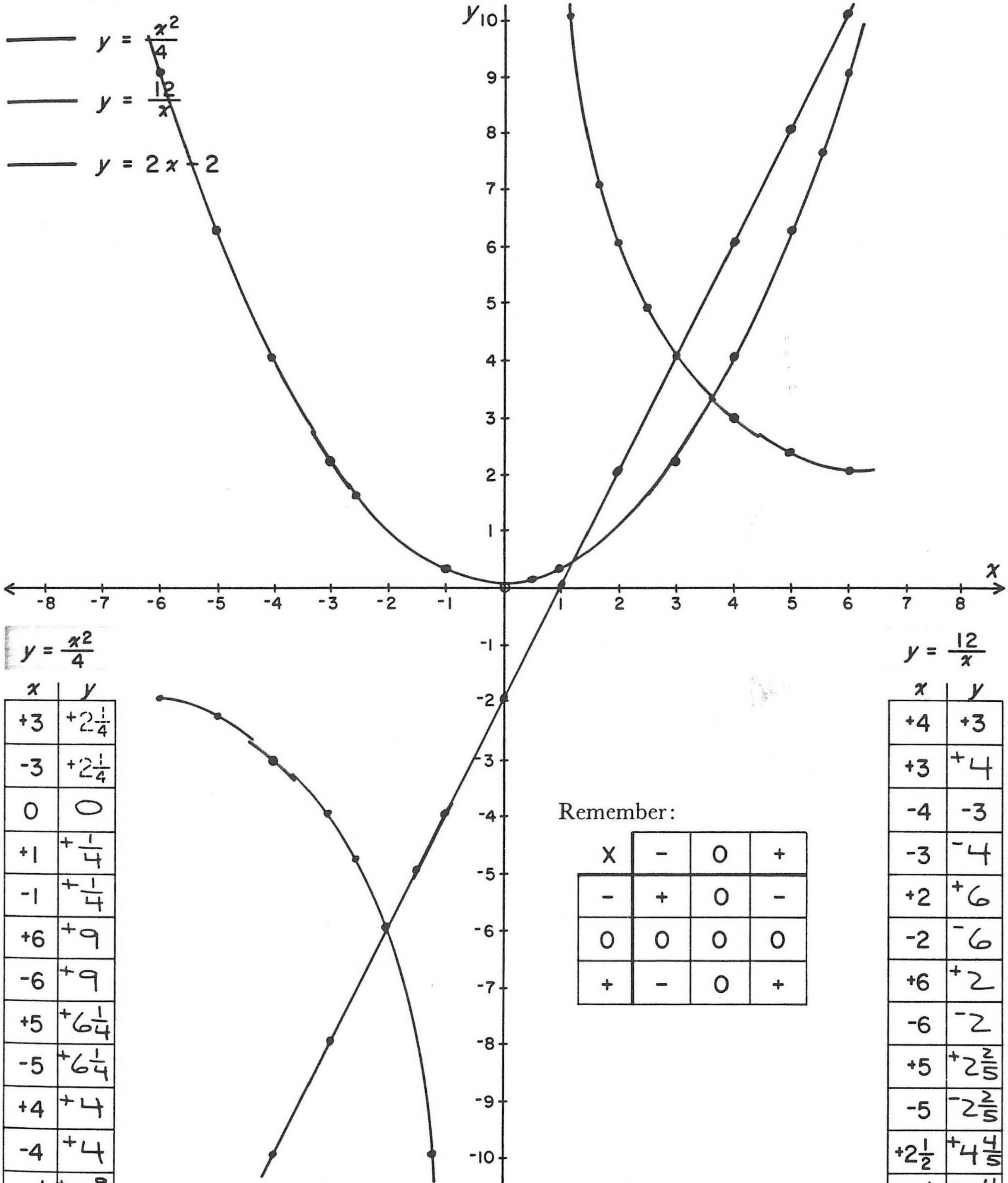
An ant wishes to crawl from A to B along one of the solid lines — AMB, ANB, or ALB. C_1 is the center of the larger circle, C_2 and C_3 are centers of the two semicircles.

Obviously, the paths from A to B through M and through N are the same length. What about the path through L. Is it the same length as the others, longer, or shorter?

ALL 3 PATHS ARE THE SAME LENGTH.

Chapter TWELVE . . .

RELATIONSHIPS THAT MAY NOT LEAD TO STRAIGHT LINE GRAPHS



$y = \frac{x^2}{4}$

x	y
+3	$+2\frac{1}{4}$
-3	$+2\frac{1}{4}$
0	0
+1	$+\frac{1}{4}$
-1	$+\frac{1}{4}$
+6	+9
-6	+9
+5	$+6\frac{1}{4}$
-5	$+6\frac{1}{4}$
+4	+4
-4	+4
$+6\frac{1}{2}$	$+10\frac{9}{16}$
$-2\frac{1}{2}$	$+1\frac{9}{16}$
$+\frac{1}{2}$	$+\frac{1}{16}$
$+5\frac{1}{2}$	$+7\frac{9}{16}$

Remember:

X	−	0	+
−	+	0	−
0	0	0	0
+	−	0	+

$y = \frac{12}{x}$

x	y
+4	+3
+3	+4
-4	-3
-3	-4
+2	+6
-2	-6
+6	+2
-6	-2
+5	$+2\frac{2}{5}$
-5	$-2\frac{2}{5}$
$+2\frac{1}{2}$	$+4\frac{4}{5}$
$-2\frac{1}{2}$	$-4\frac{4}{5}$
$+1\frac{1}{5}$	+10
$-1\frac{1}{5}$	-10
$+1\frac{5}{7}$	+7

$y = 2x - 2$

x	+5	-3	+1	+4	+3	-1	-4	+6	$-1\frac{1}{2}$	0	+2	-2
y	+8	-8	0	+6	+4	-4	-10	+10	-5	-2	+2	-6

89 y = x^2 / 4; y = 12 / x; y = 2x - 2

Tasks: Students complete the three tables based on the given formulas or rules and plot the related pairs of numbers on the chart.

Purpose: To help students realize that some rules or formulas lead to straight-line graphs and some lead to curves.

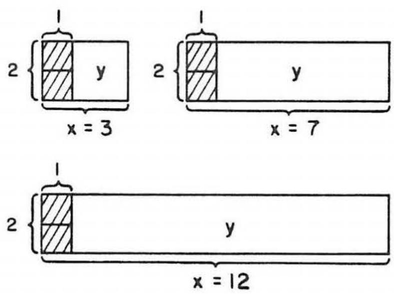
Unifying Ideas: Functions and relations; geometry.

The Lesson: Why do some relationships lead to straight lines and others to curves?

Let's take a look at the three equations given on pupil page 89.

First: $y=2x-2$

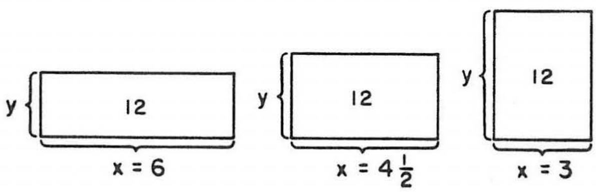
We might be thinking of a rectangle which is 2 inches wide and x inches long; 2 square inches are covered up. What is the area (y) of the uncovered part? The length varies as we let x assume different values.



x	y
3	4
7	12
12	22

Second: $y=\dfrac{12}{x}$

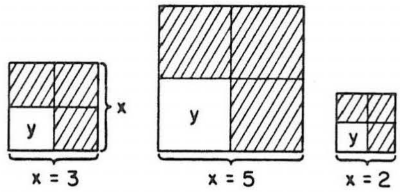
Suppose we know that a rectangle has an area of 12 square inches. As its length (x) varies, the height must also vary (inversely) if we are to maintain an area of 12 square inches.



x	y
6	2
$4\frac{1}{2}$	$2\frac{2}{3}$
3	4

Third: $y=\dfrac{x^2}{4}$

Consider a square divided into four smaller squares. What is the area of a small square (y) as the side of the larger square (x) varies?



x	y
3	$2\frac{1}{4}$
5	$6\frac{1}{4}$
2	1

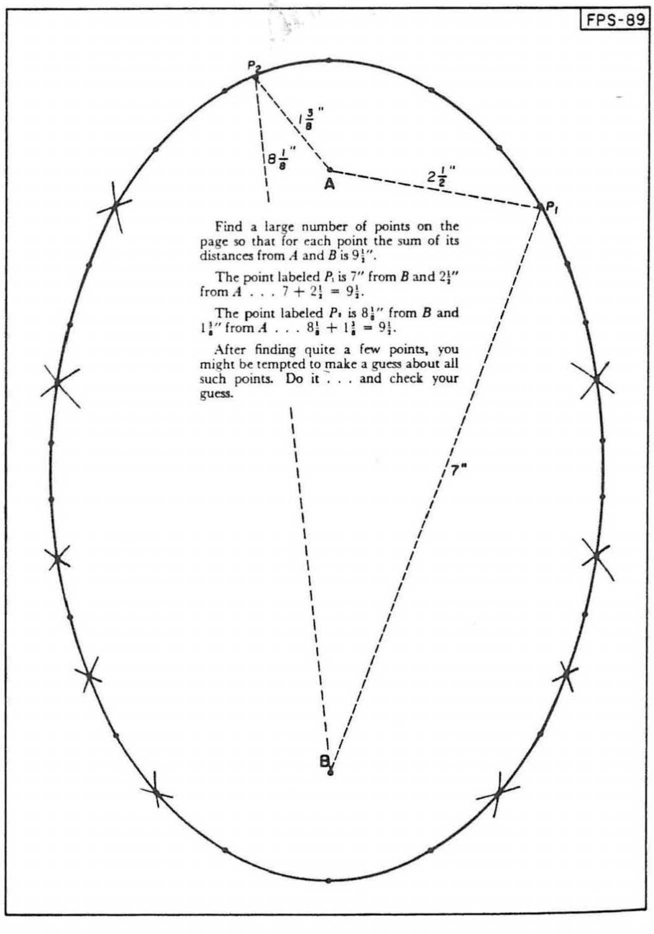
Why does a graph in the first problem lead us to a straight line and graphs in the other two lead to curves?

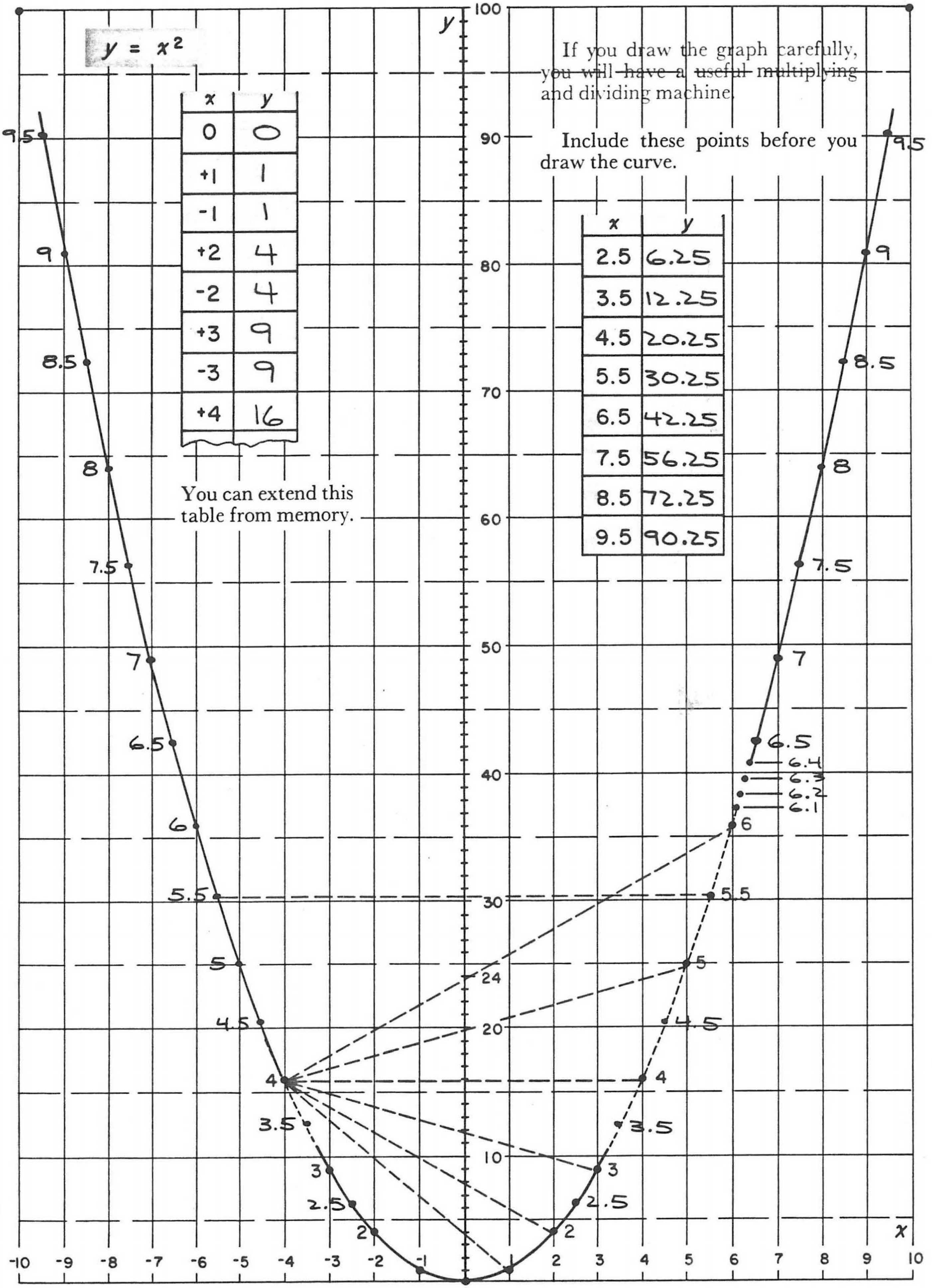
Experiment with such equations as:

$y = x$ $\quad$ $y = 2x$ $\quad$ $\underline{y = x^2}$ $\quad$ $\underline{y = \frac{30}{x}}$

$x + y = 12$ $\quad$ $x - y = 12$ $\quad$ $\underline{x \times y = 12}$

(Graphs of the underlined equations will not be straight lines.)





x	y
0	0
+1	1
-1	1
+2	4
-2	4
+3	9
-3	9
+4	16

You can extend this table from memory.

If you draw the graph carefully, you will have a useful multiplying and dividing machine.

Include these points before you draw the curve.

x	y
2.5	6.25
3.5	12.25
4.5	20.25
5.5	30.25
6.5	42.25
7.5	56.25
8.5	72.25
9.5	90.25

90-91 $y = x^2$ - a multiplying and dividing machine

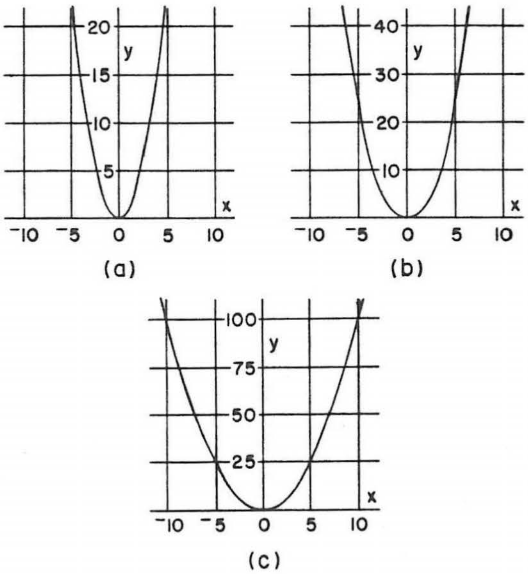
Tasks: Students complete tables based on the rule $y=x^2$ *and locate the corresponding points on the graph. They then draw the curve suggested by these points.*

Purpose: To help students realize that for graphing a rule like $y=x^2$ the selection of graphic scales is a matter of convenience.

Unifying Ideas: Multiplication and division; functions and relations; geometry.

The Lesson: You may already have considered the graph of $y=x^2$, but probably with vertical and horizontal scales marked off in equal subdivisions. What is the effect of using different scales? Why did we choose the scales on pupil page 90?

One approach to the first question is to consider the activity on practice sheet 76. The effect of different scales might squeeze the picture or stretch it out. Here are three different pairs of scales and the graph of $y=x^2$:



The choice of scales in *a* sends the graph off the paper before we have considered $x=5$. The choice in *b* limits us to values of x that are less than 7. Scale *c* spreads the graph over the whole grid, and we can consider all values for x between $^-10$ and $^+10$.

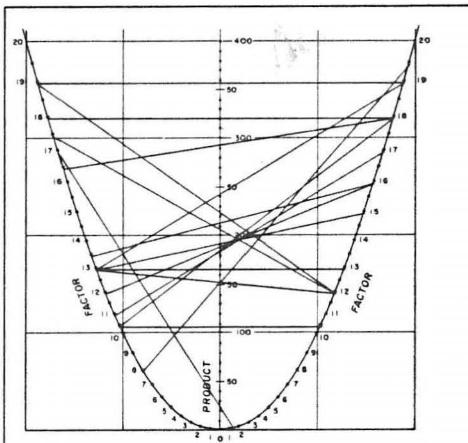
The relationship remains unchanged by the selection of scales, and our choice depends entirely on convenience. We simply wished to spread the graph out over the whole of page 90 and to be able to consider all values of x between $^-10$ and $^+10$.

Finally, after the graph was drawn, we wished to use it as a "multiplying machine" to find directly the product of any two numbers less than 10. If we use a straightedge to draw a line through a point on one branch of the curve and a point on the other branch, this line will also pass through a point on the y-scale whose distance from the origin is the product of the distances of the two points on the curve from the y-axis. The accuracy of the results depends on the accuracy of drawing the curve, of locating points on the curve, and of reading the scale.

Since a graph that helps find the product only of whole numbers smaller than 10 is of little use, we begin locating more points on the curve. (Pupil page 91 continues this activity.)

Practice sheet 90 is another graph of $y=x^2$ with different scales. This graph gives us a multiplication approximator for numbers smaller than 20. And it takes little practice to use the same device as a division approximator if divisor and quotient are less than 20. (Continued in the Notes on Pupil Page 91.)

FPS-90



When two factors are given, find one on each of the lines that curve up and out from 0. Connect these points with a straightedge. At what point does it seem to cut the center scale?

Where one factor is missing, lay a straightedge across the graph through the points indicating the given factor and the product. Notice where the straightedge cuts the other curved line.

If you work carefully, you can estimate the size of the missing factor or product within a small amount.

In the statement "12 × 15 = 180" we call 12 and 15 the "factors," and we call 180 the "product" of those factors. Is it true that, if any two of the three elements of such a statement are given, we can find the third?

one factor	15	9	12	17	18
another factor	10	12	16	17	15
product	150	108	192	289	270

The incomplete statement "162 ÷ 18 = ______" indicates that the product and one of the factors are given. To complete the statement, the missing factor must be found.

Find the missing factor in each statement:

12 x 12 = 144   306 ÷ 18 = 17

13 = 182 ÷ 14   13 x 8 = 104

Missing Terms	Determined by using the graph	Determined by computation
12 x 16 = 192	195	196
11 x 17 = 187	11	11
13 x 15 = 195	15	15
247 ÷ 19 = 13	13	13
324 ÷ 18 = 18	18	18
228 ÷ 19 = 12	228	228
10.5 x 18 = 189	190	189
12 x 17.5 = 210	210	210
216 ÷ 16 = 13.5	13.5	13.5
156 ÷ 8 = 19.5	156	156
10.5 x 10.5 = 110.25	110	110.25
361 ÷ 19 = 19	19	19
13 x 13 = 169	169	169
1.5 x 17 = 25.5	25	25.5
1.3 x 1.6 = 2.08	2	2.08
297 ÷ 16.5 = 18	18	18
158 ÷ 13 = $12\frac{2}{13}$	13	13

Perhaps you remember a shortcut for squaring any 2-digit number that ends in 5. If not, these examples and hints may save lots of time with 2.5^2, 3.5^2, etc.

I.

2.5	3.5	4.5	5.5
x 2.5	x 3.5	x 4.5	x 5.5
6.25	12.25	20.25	30.25

Use your graph on page 90 to approximate the following:

II.

	approx.
6 x 3.5	21.0
7.5 x 8	60
5 x 6.5	32.5
8.5 x 5	42.5

	approx.
9.5 x 7	66.5
8 x 6.5	52.0
5 x 8.5	42.5
7.5 x 9	67.5

Were your approximations close? If you notice the right-hand digits of the factors, you can be sure about the right-hand digit in the product. With this help, you can turn your approximations into exact answers.

How can you use the graph to give answers for the following?

III.

8 x 45	360
55 x 7	385
4 x 65	260

75 x 7	525
9 x 65	585
45 x 5	225

Suppose that we wish to improve our graph by locating more points on both halves of the curve. Let's begin by locating points whose x-coordinates are 6.1, 6.2, 6.3, and 6.4.

IV.

6.1 x 6.1 = 37.21
6.2 x 6.2 = 38.44
6.3 x 6.3 = 39.69
6.4 x 6.4 = 40.96

Perhaps you can develop some team work to mark off both halves of the curve in tenths.

If so, you have a graph that will help give you very close approximations for the following:

V.

6.2 x 7.5	46.5
9 x 63	567
45 x 64	2880
6.1 x 85	518.5
.55 x 63	34.65

8.1 x 79	639.9
57 x .48	27.36
$68\frac{1}{2}$ x $73\frac{1}{2}$	5034.75
8.9 x $95\frac{1}{2}$	849.95
8.6 x 8.6	73.96

As you have undoubtedly noticed, the graph is as useful in division as it is in multiplication.

Find the approximate quotients (or missing factors) in the following by using the graph:

VI.

70 ÷ 7.5	$9.33\frac{1}{3}$
43 ÷ 6.1	7.05
49 ÷ 8	6.125
$35\frac{3}{4}$ ÷ 5.5	6.5
$6\frac{3}{8}$ ÷ $8\frac{1}{2}$	$\frac{3}{4}$

56.7 ÷ 6.3	9
24.75 ÷ .45	55
$518\frac{1}{2}$ ÷ 85	6.1
8645 ÷ 91	95
59.5 ÷ .85	70

Using the graph, find approximate answers to the following:

VII.

	approx.
72^2	5184
4.4^2	19.36
$.83^2$	.6889
6.5^2	42.25

	approx.
53^2	2809
9.001^2	81.+
5.9^2	34.81
28^2	784

VIII. What Are My Rules?

$\sqrt{36} = 6$ $\sqrt{81} = 9$ $\sqrt{.16} = .4$

$\sqrt{100} = 10$ $\sqrt{.09} = .3$ $\sqrt{25} = 5$

$\sqrt{400} = 20$ $\sqrt{64} = 8$ $\sqrt{1.44} = 1.2$

Using the graph, find approximate answers for the following:

IX.

	approx.
$\sqrt{50}$	7.1
$\sqrt{90}$	9.5
$\sqrt{40}$	6.3

	approx.
$\sqrt{59}$	7.7
$\sqrt{.45}$	.67
$\sqrt{.75}$	.87

Tasks: *Students use the graph to find approximate answers to multiplication, division, squaring, and square root problems.*

Purpose: To help students learn to use charts in making approximations.

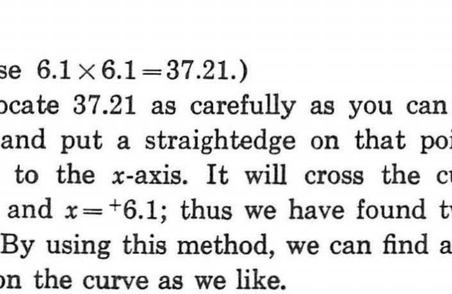
Unifying Ideas: Multiplication and division; functions and relations; geometry.

The Lesson: This flexibility of changing scales makes the same graph of $y=x^2$ a useful approximation of the multiplication tables. If each of our x-axis numbers were multiplied by 10, we need only read our vertical axis in 100's and find approximate products of all pairs of numbers less than 100. (See the graph and scales used on practice sheet 91.)

Since we can approximate the product of 8 and 4.5, we know that the product of 8 and 45 is approximately 10 times as much:

$$\begin{aligned} &\text{If} && 8\times 4.5=36\\ &\text{then} && 8\times 45=360,\\ &\text{and} && .8\times 45=36,\\ &\text{and} && .8\times 4.5=3.6, \text{ etc.}\end{aligned}$$

We can also locate more points on the curve in the following two ways: (1) by subdividing the x-axis scale, and (2) by a more accurate method which follows from the recognition that a horizontal line through a point on the curve crosses the y-axis at the square of the x-coordinate of that point.



(Because $6.1\times 6.1=37.21$.)

So, locate 37.21 as carefully as you can on the y-axis, and put a straightedge on that point and parallel to the x-axis. It will cross the curve at $x={}^-6.1$ and $x={}^+6.1$; thus we have found two new points. By using this method, we can find as many points on the curve as we like.

The square root sign, $\sqrt{}$, may be unfamiliar to some students. From the *What Are My Rules?* examples, its meaning should be quite clear. It can be understood as a direction to find the (nonnegative) number which, when squared, equals the number under the sign (called a *radical* sign). Thus, "$\sqrt{49}=x$" is a direction to find the (nonnegative) number which, when squared, equals 49.

It would be an excellent project for some students to make a very large graph of $y=x^2$ for demonstration purposes. They will be faced with substantial problems. Once they have selected the approximate size of the final graph, they must select appropriate scales (to utilize the entire graph and to provide for the uses of it after computation). They will need to decide on the degree of accuracy they can and wish to maintain. They must decide on the size of subdivisions they wish for each scale. This work can be most rewarding.

FPS-91

This is the same diagram that appeared on the previous page, but the scale has been changed. If both factors are less than 100, you can use it to estimate your answers.

The dashed line illustrates a division example:

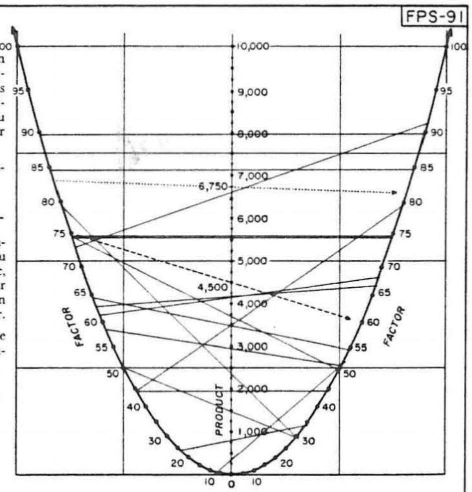
4,500 ÷ 75 = 60

You know that the answer is about 60. If you work out the arithmetic, you will find how close your estimate was. (Exact, in this case.) But try another.

The dotted line above the dashed line illustrates another division example:

6,750 ÷ 83 = $81\frac{27}{83}$

You know the answer is about 81.



	A	B	C	D	E	F	G	H
product	4,500	2,400	8,100	3,750	3,600	2,025	3,575	5,625
factor	75	80	90	75	45	45	65	75
factor	60	30	90	50	80	45	55	75

	estimate	result of computation
I	25	24
J	62	62
K	50	51
L	30	30
M	85	85
N	75	74
O	10	9
P	90	92
Q	60	63

Estimate the answer to each example below and record it on the left in the first column. Then find the exact answer and compare it with the estimate.

I. $\begin{array}{r} 24 \\ 35\overline{)840} \end{array}$ J. $\begin{array}{r} 62 \\ 68\overline{)4{,}216} \end{array}$ K. $\begin{array}{r} 51 \\ 59\overline{)3{,}009} \end{array}$

L. $\begin{array}{r} 30 \\ 51\overline{)1{,}530} \end{array}$ M. $\begin{array}{r} 85 \\ 85\overline{)7{,}225} \end{array}$ N. $\begin{array}{r} 74 \\ 74\overline{)5{,}476} \end{array}$

O. $\begin{array}{r} 9 \\ 52\overline{)468} \end{array}$ P. $\begin{array}{r} 92 \\ 73\overline{)6{,}716} \end{array}$ Q. $\begin{array}{r} 63 \\ 67\overline{)4{,}221} \end{array}$

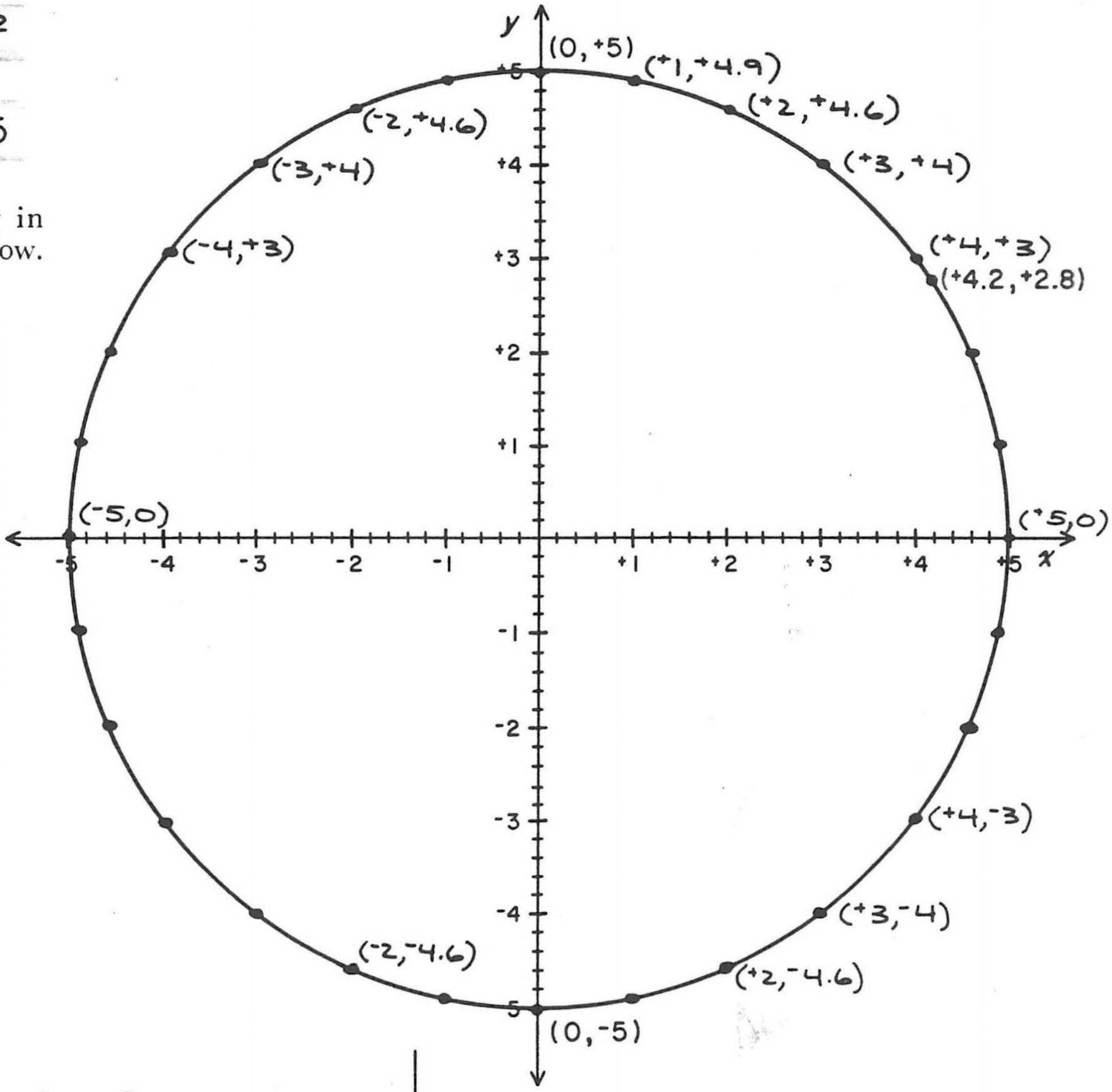
$y^2 = 25 - x^2$

or

$y^2 + x^2 = 25$

Before filling in the table, read below.

x	y
0	$^+5$, $^-5$
+3	$^+4$, $^-4$
-3	$^+4$, $^-4$
-4	$^+3$, $^-3$
+4	$^+3$, $^-3$
+5	0
-5	0
+2	$^+\sqrt{21}$, $^-\sqrt{21}$
-2	$^+\sqrt{21}$, $^-\sqrt{21}$



If $y^2 + x^2 = 25$ and $x = 0$ then $y^2 + 0 = 25$; so, $y^2 = 25$. But, $^+5 \times {}^+5 = 25$ and $^-5 \times {}^-5 = 25$. So, if $y^2 = 25$ then $y = {}^+5$ or $y = {}^-5$. Hence, when $x = 0$, $y = {}^+5$ or $y = {}^-5$. This gives us two points to graph: $(0, {}^+5)$ and $(0, {}^-5)$

When $x = {}^+3$, $x^2 = 9$. Then $y^2 + 9 = 25$. So, $y^2 = 16$ and $y =$ <u>+4</u> or $y =$ <u>−4</u>. This gives us two points: $({}^+3, {}^+4)$ and $({}^+3, {}^-4)$

When $x = {}^-3$, $x^2 = 9$. Again, $y^2 + 9 = 25$. So, $y^2 = 16$ and $y =$ <u>+4</u> or $y =$ <u>−4</u>. We have two more points: $({}^-3, {}^+4)$ and $({}^-3, {}^-4)$

When $x = {}^-4$, $x^2 =$ <u>16</u>. So, $y^2 +$ <u>16</u> $= 25$. Hence, $y^2 =$ <u>9</u> and $y =$ <u>+3</u> or $y =$ <u>−3</u>. Two more points: $({}^-4, {}^+3)$ and $({}^-4, {}^-3)$

When $x = {}^+4$, $y^2 =$ <u>9</u>. $({}^+4, {}^+3)$ and $({}^+4, {}^-3)$

When $x = {}^+5$ or ${}^-5$, $y^2 = 0$. $({}^+5, 0)$ and $({}^-5, 0)$

When $x = {}^+2$ or $x = {}^-2$, $x^2 = 4$ and $y^2 + 4 = 25$. So, $y^2 =$ <u>21</u>. Therefore, $y = {}^+\sqrt{21}$ or $y = {}^-\sqrt{21}$. This gives us four more points: $({}^+2, {}^+\sqrt{21})$, $({}^+2, {}^-\sqrt{21})$, $({}^-2, {}^+\sqrt{21})$ and $({}^-2, {}^-\sqrt{21})$

What number multiplied by itself gives 21 as the product?

$4 \times 4 = 16$ (too small)

$5 \times 5 = 25$ (too large)

It must be someplace between 4 and 5 or between $^-4$ and $^-5$. Let's try:

$4.4^2 =$ <u>19.36</u>

$4.5^2 =$ <u>20.25</u>

$4.6^2 =$ <u>21.16</u>

Which is closest for y if $y^2 = 21$?

92-93 From "y^2 + x^2 = 25" to "y^2 + x^2 = n^2" - the Pythagorean Theorem

Tasks: Students compute the missing items in the table based on the indicated rule by determining the values of y *when* x *is given. They then try to estimate the square root of 21 by trapping it.*

Purpose: To help students graph an equation.

Unifying Ideas: Structure; numeration; multiplication and division; functions and relations; geometry.

The Lesson: Notice that we have not avoided using the square root sign on pupil page 92. The idea of square root is an essential part of the development.

$$\text{If } y^2 = 16$$
$$\text{then } y = {}^{+}4 \text{ or } y = {}^{-}4.$$

But what do we do with the following?

$$y^2 = 21$$
$$y = \_\_\_$$

We use the common device of "trapping" the number we are looking for between two other numbers; we try to narrow the gap between these two numbers until we obtain a satisfactory approximation. If we remember the shortcut for squaring any number ending in a 5, we know that $45 \times 45 = 2{,}025$, so $4.5 \times 4.5 = 20.25$ . . . and that's .75 less than 21. It turns out that $4.6 \times 4.6 = 21.16$ . . . and that's .16 more than 21. So, 4.6 is the better approximation.

Suppose that we wanted a still closer approximation. We can extend the trial-and-error method and find:

$4.58 \times 4.58 = 20.9764$ (.0236 too small)
$4.59 \times 4.59 = 21.0681$ (.0681 too large)
etc.
4.583 is closer
4.5826 is closer
4.58257 is closer

(We can get as close as we please—but we can never find a decimal fraction whose square is 21. The only whole numbers whose square roots are rational numbers are the square whole numbers: 0, 1, 4, 9, 16, 25, etc. All other whole numbers have *irrational* numbers as square roots.)

This trapping method is essentially the same as the one we used to obtain approximations to π. We shall employ it again in the next chapter; thus π reappears as we search for a relationship between the radius of a circle and its area.

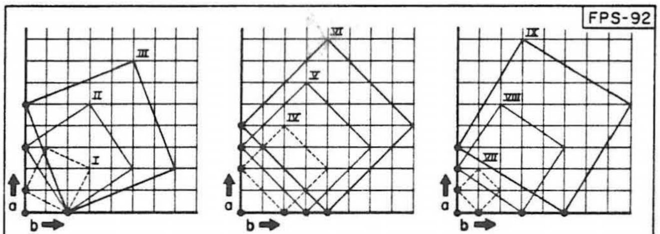
Practice sheet 92 lays an experimental basis for the Pythagorean Theorem, which reappears in the discussion on pupil page 93 and practice sheet 93.

Some students might like to look back at Chapter Four, "A Problem in Geometry," where we discussed Pick's Theorem. How many unit squares of area in I through VIII? All you need to do is (a) count the dots (crossing points) on the boundary and divide by 2, (b) add the number of dots inside the boundary, and (c) subtract 1 from the sum of (a) and (b). Thus, if d is the number of dots on the boundary and i is the number of dots on the inside then the formula for the area is:

$$\text{area} = \frac{d}{2} + i - 1$$

Pick's Theorem can be used to find the area of the squares pictured at the top of practice sheet 92. However, in this special case, we have a much quicker method—we use the Pythagorean Theorem to find the length of a side of the square.

FPS-92



How many tiles would be required to cover each of the three squares in each of the examples above?

For the last four examples in the table below, you may need to draw your own sketches — unless you've discovered a shortcut.

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII
a ↑ distance up	1	3	5	2	3	4	1	2	3	2	3	3	4
b → distance to right	2	2	2	2	3	4	1	3	5	5	2	3	3
tiles required to cover square	5	13	29	8	18	32	2	13	34	29	13	18	25

A game: "I've got three rules!"

"Call out any two numbers," Mrs. Boswell said. "I will apply each of my rules and write down my results. If you discover the rules then you know in advance what I am going to write."

Below are some of the results:

Your first number
Your second number
Mrs. Boswell writes

a	b	↓	a	b	↓	a	b	↓
3	4	9	3	4	16	3	4	25
2	7	4	2	7	49	2	7	53
5	5	25	5	5	25	5	5	50
8	6	64	8	6	36	8	6	100
9	1	81	9	1	1	9	1	82
10	5	100	10	5	25	10	5	125
$a, b \rightarrow a^2$			$a, b \rightarrow b^2$			$a, b \rightarrow a^2 + b^2$		
7	4	49	7	4	16	7	4	65
6	9	36	6	9	81	6	9	117
7	8	49	7	8	64	7	8	113
5	12	25	5	12	144	5	12	169

We are led to the conclusion that if $x = {}^+2$ then $y = {}^+\sqrt{21}$ or ${}^-\sqrt{21}$. This gives us the points $({}^+2, {}^+4.6)$ and $({}^+2, {}^-4.6)$.

And, if $x = {}^-2$ then $y = {}^+\sqrt{21}$ or ${}^-\sqrt{21}$. This gives us the points $({}^-2, {}^+\sqrt{21})$ and $({}^-2, {}^-\sqrt{21})$.

If, however, $y = {}^+2$ or $y = {}^-2$ then $y^2 = 4$ and

$$4 + x^2 = 25,$$
$$x^2 = 21,$$

and $x = {}^+\sqrt{21}$ or $x = {}^-\sqrt{21}$.

From this we can locate four points:

$({}^+4.6, {}^+2)$, $({}^-4.6, {}^+2)$,
$({}^+4.6, {}^-2)$, $({}^-4.6, {}^-2)$

Are you quite sure the graph will be a circle no matter how many more points we locate?

Let's see what happens if $x = {}^+1$ or ${}^-1$.

If $x = {}^+1$ or ${}^-1$ then $x^2 = 1$,

$$y^2 + 1 = 25,$$
$$y^2 = 24,$$
$$y = {}^+4.9 \text{ or } {}^-4.9.$$

What positive number multiplied by itself gives 24? It must be a little less than 5.

$4.8^2 = 23.04$ (too small)

$4.9^2 = 24.01$ (too large)

This provides enough information for four more points:

$({}^+1, {}^+4.9)$, $({}^+1, {}^-4.9)$,
$({}^-1, {}^+4.9)$, $({}^-1, {}^-4.9)$

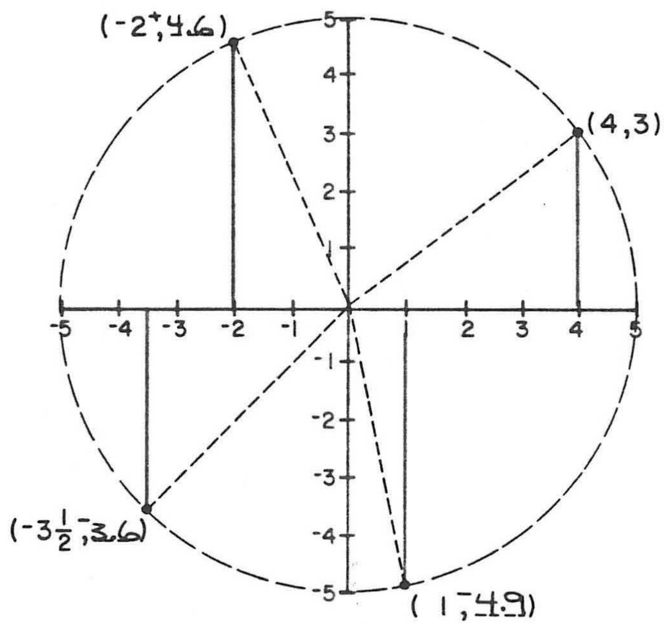
$({}^+4.2, {}^+2.8)$

We have selected a point on the curve. It seems to have a location of $x = {}^+4.2$ and $y = {}^+2.8$. (These are only approximations.) We expect to find that

${}^+2.8^2 + {}^+4.2^2$ is about 25,

and multiplying we have:

$7.84 + 17.64$ is 25.48 (about 25)



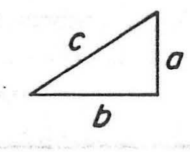
(To fill in the blanks, you need to compute only one bit of information. All the rest was done earlier.)

If from each point we locate, we draw a line perpendicular to the x-axis, a second line from that intersection to the center, and a third line from the point to the center, we have a triangle.

In each case, that triangle is a right triangle whose hypotenuse (or longest side) is a radius of our circle. Every radius is, of course, 5 units long.

Do you remember the Pythagorean Theorem?

In a right triangle, the square of the hypotenuse (c) equals the sum of the squares of the other two sides (a and b).



$$a^2 + b^2 = c^2$$

Compare: $y^2 + x^2 = 25$ and $a^2 + b^2 = c^2$

Can you imagine graphs for:

$y^2 + x^2 = 64$ $\quad y^2 + x^2 = 7$ $\quad y^2 + x^2 = 1$

or a graph for any equation of the form:

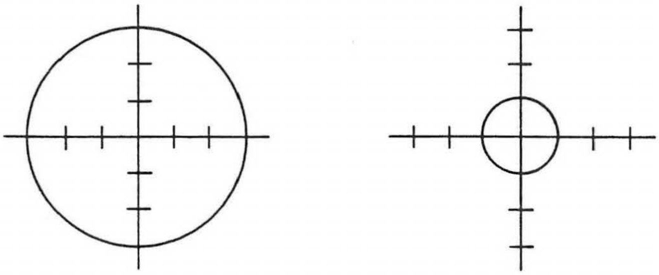
$y^2 + x^2 = n$?

Tasks: *Students continue finding values for either x or y in the equation on pupil page 92 when a value of the other is given. They then study the relationship of the Pythagorean Theorem to equations whose graphs are circles.*

Purpose: To help students graph an equation.

Unifying Ideas: Functions and relations; geometry.

The Lesson: What would the graphs of $x^2+y^2=9$ and $x^2+y^2=1$ look like?



$x^2 + y^2 = 9$ $\qquad\qquad$ $x^2 + y^2 = 1$

These graphs are circles with radii of $\sqrt{9}$ and $\sqrt{1}$, and with center at (0,0).

Students who are adept at graphing the equations they have already encountered can be asked to graph:

$y^2+(x+1)^2=25$	$(y+1)^2+x^2=25$
$(y+1)^2+(x+1)^2=25$	$y^2+(x-1)^2=25$
$(y-1)^2+x^2=25$	$(y-1)^2+(x+1)^2=25$
$y^2+(x+3)^2=25$	$(y+2)^2+x^2=25$
$(y+2)^2+(x+3)^2=25$	$(y-2)^2+(x+3)^2=25$

All will lead to circles with a radius of 5 units.

If the squared term involving x is $(x+1)^2$, the y-axis is, in effect, moved one unit to the right—or, the center of the circle will have an x-coordinate of $^-1$. If the squared term involving x were $(x-3)^2$, the y-axis would, in effect, be moved 3 units to the left and the center would have an x-coordinate of $^+3$. We obtain the y-coordinate of the center in a similar manner.

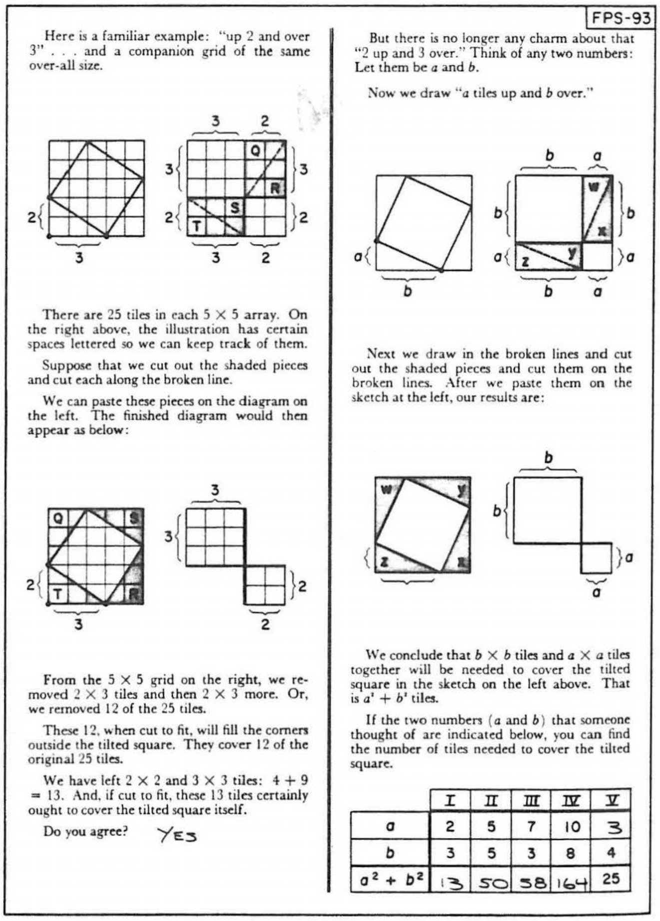
Thus, for the equations listed in the display above, the centers of the circles would have the following coordinates:

$(^-1,0)$	$(0,^-1)$
$(^-1,^-1)$	$(^+1,0)$
$(0,^+1)$	$(^-1,^+1)$
$(^-3,0)$	$(0,^-2)$
$(^-3,^-2)$	$(^-3,^+2)$

The number 25 appears in each equation and indicates that all the circles have radius 5 units. We are suggesting this activity for more independent workers.

A different way of reaching the Pythagorean Theorem in the right-hand column of practice sheet 93 is to notice that the side of the largest square is $(a+b)$ units long and its area is $(a+b)^2$ square units. Each of the corners is a triangle with base b units long and altitude a units. The area of each triangle is $\frac{1}{2}ab$ square units. The area A of the square is then obtained as follows:

$$\begin{aligned} A &= (a+b)^2-2ab \\ &= (a^2+2ab+b^2)-2ab \\ &= a^2+b^2 \end{aligned}$$


FPS-93

Here is a familiar example: "up 2 and over 3" . . . and a companion grid of the same over-all size.

There are 25 tiles in each 5 × 5 array. On the right above, the illustration has certain spaces lettered so we can keep track of them.

Suppose that we cut out the shaded pieces and cut each along the broken line.

We can paste these pieces on the diagram on the left. The finished diagram would then appear as below:

From the 5 × 5 grid on the right, we removed 2 × 3 tiles and then 2 × 3 more. Or, we removed 12 of the 25 tiles.

These 12, when cut to fit, will fill the corners outside the tilted square. They cover 12 of the original 25 tiles.

We have left 2 × 2 and 3 × 3 tiles: 4 + 9 = 13. And, if cut to fit, these 13 tiles certainly ought to cover the tilted square itself.

Do you agree? YES

But there is no longer any charm about that "2 up and 3 over." Think of any two numbers: Let them be *a* and *b*.

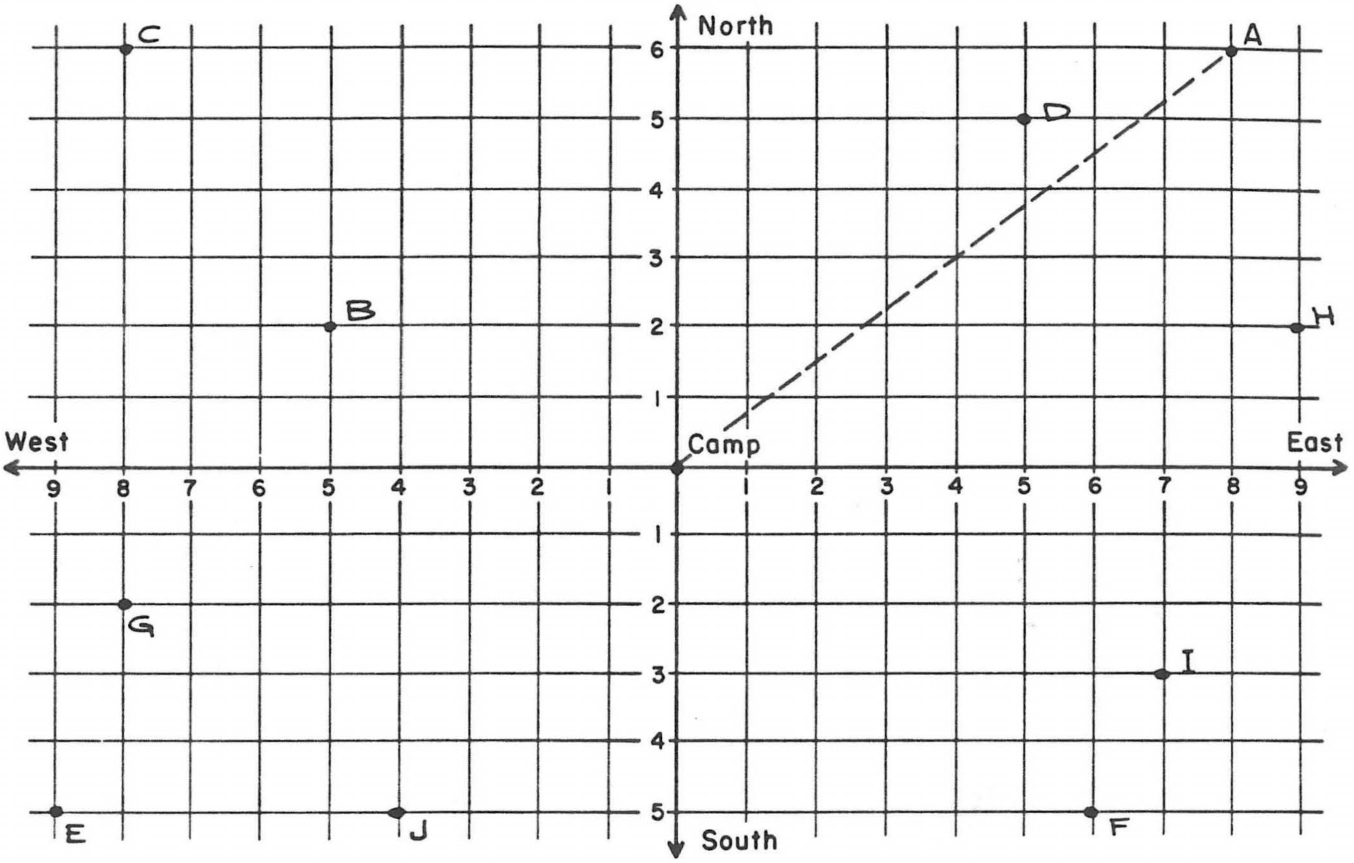
Now we draw "*a* tiles up and *b* over."

Next we draw in the broken lines and cut out the shaded pieces and cut them on the broken lines. After we paste them on the sketch at the left, our results are:

We conclude that $b \times b$ tiles and $a \times a$ tiles together will be needed to cover the tilted square in the sketch on the left above. That is $a^2 + b^2$ tiles.

If the two numbers (*a* and *b*) that someone thought of are indicated below, you can find the number of tiles needed to cover the tilted square.

	I	II	III	IV	V
a	2	5	7	10	3
b	3	5	3	8	4
a^2+b^2	13	50	58	164	25



A group of ten prospectors scattered from their camp. After a week's work, each reported his location with respect to the camp.

The radio operator at the camp made a map and marked the position of each man by placing a black dot and labeling it with the first letter of the prospector's name. The operator's notes indicated these positions:

Al: 8 miles east and 6 miles north

Ben: 5 miles west and 2 miles north

Carl: 8 miles west and 6 miles north

Dan: 5 miles east and 5 miles north

Ed: 9 miles west and 5 miles south

Frank: 6 miles east and 5 miles south

Gil: 8 miles west and 2 miles south

Harry: 9 miles east and 2 miles north

Ira: 7 miles east and 3 miles south

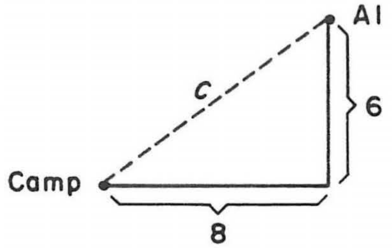
Jack: 4 miles west and 5 miles south

Locate each prospector on the map and label the point as the operator did.

The director of the prospecting expedition asked the operator, "How far is Al from camp?"

"Al is 8 miles east and 6 miles north. I'll tell you in a minute the shortest distance by plane."

He drew the following sketch:



$c^2 = 8^2 + 6^2 = 64 + \underline{36} = \underline{100}$

$c = \underline{10}$

"Al is $\underline{10}$ miles from camp," the radio operator advised.

"How far is Ben from Dan?"

"Well, Ben is 10 miles west of Dan and 3 miles south of him. I can tell you in a minute the distance between them — as a plane flies."

94-95 Applications of the Pythagorean Theorem; approximating square roots

Tasks: *Students represent the positions of ten men on a grid and then determine the distance of two of these men from other points on the grid.*

Purpose: To help students understand applications of the Pythagorean Theorem.

Unifying Ideas: Functions and relations; geometry.

The Lesson: Pupil pages and practice sheets 94 and 95 consider applications of the Pythagorean Theorem and of the "trapping" method of finding approximations of square roots.

If students have a good grasp of both ideas and do not need practice in computation then the work on these pages can be omitted or done only in part. You might also discuss the work but omit the computation.

The activity on pupil page 94 may be used to illustrate a very useful notion when we are using the same scale on each of the perpendicular axes: The distance of any point from (0,0) is the square root of the sum of the squares of the coordinates of the point. This result was suggested in the right-hand column of the last pupil page.

In the problem on pupil page 94, we can locate Al (point A) by noting that since he is 8 miles east and 6 miles north, he is at the point $(^+8,^+6)$. The distance between $(^+8,^+6)$ and (0,0) is

$$\sqrt{(^+8)^2+(^+6)^2}=\sqrt{64+36}=\sqrt{100}=10.$$

Thus Al is 10 miles from camp as the crow flies.

Ben is at $(^-5,^+2)$; so he is

$$\sqrt{(^-5)^2+(^+2)^2} \text{ or } \sqrt{29}$$

miles from camp. But how far apart are Ben and Al? Al is at $(^+8,^+6)$ and Ben is at $(^-5,^+2)$. If we subtract the distances measured on the x-axis (E and W) and those measured on the y-axis (N and S), we have the following:

(a)		(b)
$(^+8\,,\,^+6)$		$(^-5\,,\,^+2)$
$-(^-5\,,\,^+2)$	or	$-(^+8\,,\,^+6)$
$(^+13\,,\,^+4)$		$(^-13\,,\,^-4)$

We might interpret these results as:

(a) to go from Ben's position to Al's, you go 13 miles east (+13) and 4 miles north (+4), or

(b) to go from Al's position to Ben's you go 13 miles west (−13) and 4 miles south (−4).

You would also expect that the distance between Al and Ben is the same, whether measured from Al to Ben or from Ben to Al.

$$\sqrt{(^+13)^2+(^+4)^2} = \sqrt{169+16} = \sqrt{185}$$

$$\sqrt{(^-13)^2+(^-4)^2} = \sqrt{169+16} = \sqrt{185}$$

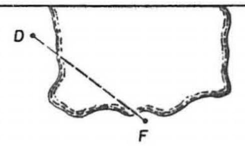
So we have a method for determining the distance between any two points once we know their coordinates.

Carl: $(^-8\,,\,^+6)$ and Ed: $(^-9\,,\,^-5)$

$(^-8\,,\,^+6)$		$(^-9\,,\,^-5)$
$-(^-9\,,\,^-5)$	or	$-(^-8\,,\,^+6)$
$(^+1\,,\,^+11)$		$(^-1\,,\,^-11)$

$$\sqrt{1^2+11^2} = \sqrt{1+121} = \sqrt{122}$$

FPS-94



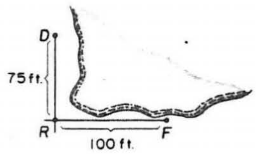
Dan and Fred wish to stretch a wire between their homes so that they can operate telephones they have made themselves.

How much wire will they need? They can't measure the distance directly because the wire will cross the lake (as shown in the diagram above) between Dan's house, D, and Fred's house, F.

They went about measuring the distance in this way.

They tied a string to the post in front of Dan's house, and another to a post in front of Fred's house.

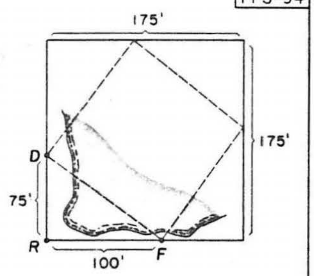
It required considerable experimentation but they finally got them to cross at point R (below) so that the two strings made a square corner. Like this:



The length of the string from F to R was measured — on dry land. It was 100 ft. long. It took 75 ft. of string to reach from D to R — on dry land.

But they wish to know how far it is from D to F — over the water.

So, they made a sketch very much like the sketches we have been studying.



How many square feet are enclosed by the larger square?

175 x 175 = 30,625 SQ. FT.

How many square feet are enclosed by each right-triangle corner?

$\frac{1}{2}$ x 75 x 100 = 3750 SQ. FT.

How many square feet are enclosed by four such corners?

4 x 3750 = 15,000 SQ. FT.

Then, how many square feet are enclosed by the tilted square (indicated by the broken line)?

30,625 − 15,000 = 15,625 SQ. FT

What is the length of the side of a square with an area of that many square feet?

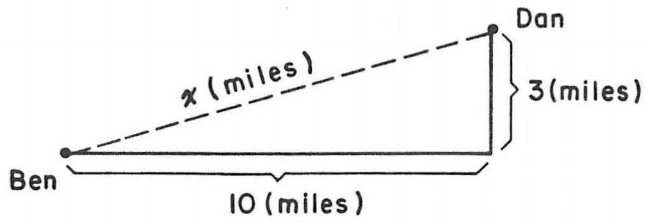
15,625 = 125 x 125

Hint:

100 x 100 = 10,000   120 x 120 = 14,400

110 x 110 = 12,100   130 x 130 = 16,900

So the number we are looking for must be more than 120 and less than 130.



$x^2 = 3^2 + \underline{10^2} = \underline{9} + \underline{100} = \underline{109}$

Now the operator has a problem. He must look for a number which, when multiplied by itself, or taken as a factor twice, gives 109.

$10 \times 10 = \underline{100}$ and $11 \times 11 = \underline{121}$

So he knows that the distance is more than <u>10</u> miles and less than <u>11</u> miles. He decides to try 10.5:

$10.5 \times 10.5 = \underline{110.25}$

The distance is <u>LESS</u> (*more* or *less*) than 10.5 miles. The operator makes the following calculations: ---------->

$10.3 \times 10.3 = \underline{106.09}$

$10.4 \times 10.4 = \underline{108.16}$

He knows that the distance between Ben and Dan is more than <u>10.4</u>, but less than <u>10.5</u> miles. So, he makes the following calculations:

$10.43 \times 10.43 = \underline{108.7849}$

$10.44 \times 10.44 = \underline{108.9936}$

$10.45 \times 10.45 = \underline{109.2025}$

$10.46 \times 10.46 = \underline{109.4116}$

"I have found," the operator reported, "that Ben and Dan are more than <u>10.44</u> miles apart, but less than <u>10.45</u> miles apart. Do you want a closer approximation?"

"No," was the answer, "that is close enough — to the nearest hundredth of a mile."

"Please give me a record showing how far each man is from camp. Also, I wish to know how far each man is from the other — to the nearest hundredth of a mile."

All the distances are recorded as numbers of miles. 10.44^+ means more than 10.44 miles, but less than 10.45 miles apart.

3.14

Al	0	13.34^+	16	4.12^+	20.24^+	11.18^+	17.88^+	4.12^+	9.05^+	16.27^+	10
Ben	13.34^+	0	5	10.44^+	8.06^+	13.03^+	5	14	13	7.07^+	5.38
Carl	16	5	0	13.03^+	11.04^+	17.80^+	8	17.46^+	17.49^+	11.70^+	10
Dan	4.12^+	10.44^+	13.03^+	0	17.20^+	10.04^+	14.76^+	5	8.24^+	13.45^+	7.07^+
Ed	20.24^+	8.06^+	11.04^+	17.20^+	0	15	3.16^+	19.31^+	16.12^+	5	10.29^+
Frank	11.18^+	13.03^+	17.80^+	10.04^+	15	0	14.31^+	7.61^+	2.23^+	10	7.81^+
Gil	17.88^+	5	8	14.76^+	3.16^+	14.31^+	0	17.46^+	15.03^+	5	8.24^+
Harry	4.12^+	14	17.46^+	5	19.31^+	7.61^+	17.46^+	0	5.38^+	14.76^+	9.21^+
Ira	9.05^+	13	17.49^+	8.24^+	16.12^+	2.23^+	15.03^+	5.38^+	0	11.18^+	7.46^+
Jack	16.27^+	7.07^+	11.70^+	13.45^+	5	10	5	14.76^+	11.18^+	0	6.40^+
---to--->	Al	Ben	Carl	Dan	Ed	Frank	Gil	Harry	Ira	Jack	CAMP

Tasks: Students determine the distances of each man from camp and from every other man.

Purpose: To help students understand applications of the Pythagorean Theorem and (on practice sheet 95) its converse.

Unifying Ideas: Functions and relations; geometry.

The Lesson: The top section of pupil page 95 shows one method of finding the square root of 109 to the nearest hundredth. Carrying out such computations for each entry in the chart at the bottom of the page would be an exhausting task and a waste of time.

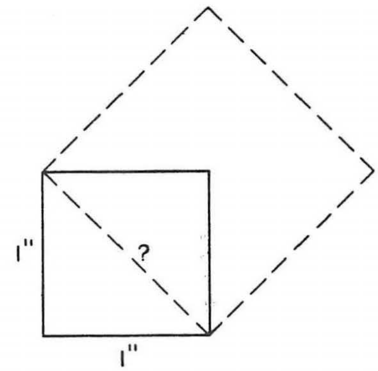
There are 110 entries of which only five are given. At first glance, the task of finding the other 105 entries may appear forbidding. However, if we examine the chart, we soon see how to cut down the work. The distance from Al to Al is, of course, 0 miles. How about the distance from Ben to Ben, from Carl to Carl, etc.? We can fill in the diagonal from the upper left-hand corner to the lower right-hand corner with zeroes. That leaves $105-9$, or 96, entries to be found.

We next note that the distance from Dan to Ben is the same as the distance from Ben to Dan—10.44+ miles. And the distance from Jack to Frank is the same as the distance from Frank to Jack—10 miles. And for every entry below the diagonal of zeroes there is an entry above that is the same. Thus our 96 computations are cut in half so that only 48 are left. If each student is assigned two or three computations the job can easily be done in a reasonable amount of time.

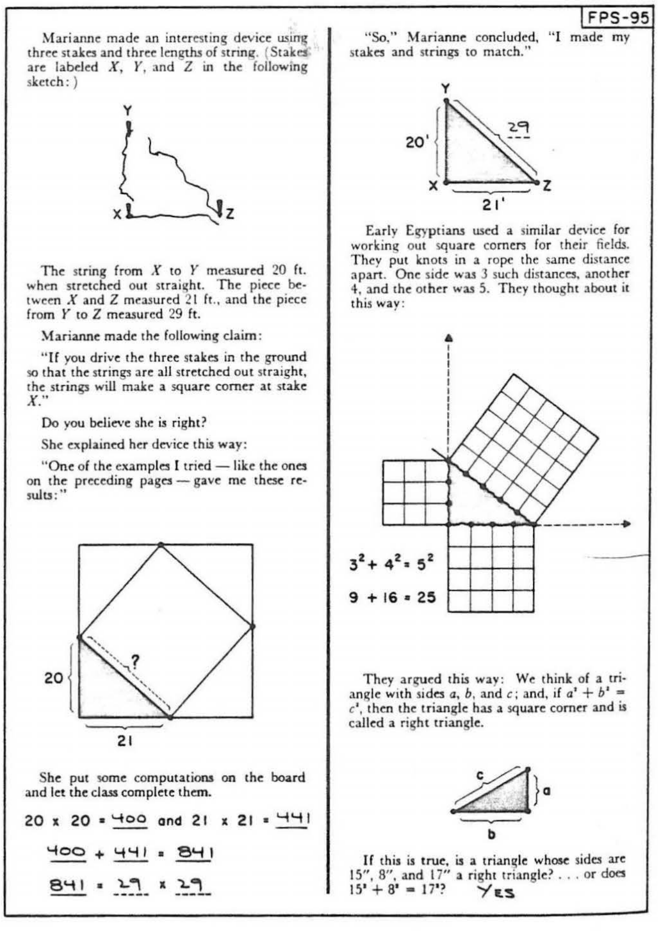
We do not intend to introduce in the text the traditional algorithm for finding square roots. However, the teacher who wishes to show this algorithm to the class should feel completely free to do so and to spend a little time talking about why it works. If students use the trapping method to find a few square roots as far as the second decimal place, they may see some significant parallels between the algorithm and the trapping method. However, the authors question the value of proficiency in using the algorithm. It is easy to forget, whereas there are other methods that do not depend on memory—such as the trapping method we have used or the more common method of using a table.

Perhaps a few words are in order to dispel any idea that the notation 10.44+ is more respectable than the notation $\sqrt{109}$. The first stands for an approximation, while the second stands for precisely the number we are talking about.

What is the length of the diagonal of a 1-inch square?



The square on the diagonal has an area of 1^2+1^2, or 2 square inches. The length of the diagonal is $\sqrt{2}$ inches. That's a precise answer, and it is much more useful in computations than 1.4+ or 1.41+ or any other approximation.



FPS-95

Marianne made an interesting device using three stakes and three lengths of string. (Stakes are labeled X, Y, and Z in the following sketch:)

The string from X to Y measured 20 ft. when stretched out straight. The piece between X and Z measured 21 ft., and the piece from Y to Z measured 29 ft.

Marianne made the following claim:

"If you drive the three stakes in the ground so that the strings are all stretched out straight, the strings will make a square corner at stake X."

Do you believe she is right?

She explained her device this way:

"One of the examples I tried — like the ones on the preceding pages — gave me these results:"

She put some computations on the board and let the class complete them.

20 x 20 = 400 and 21 x 21 = 441

400 + 441 = 841

841 = 29 x 29

"So," Marianne concluded, "I made my stakes and strings to match."

Early Egyptians used a similar device for working out square corners for their fields. They put knots in a rope the same distance apart. One side was 3 such distances, another 4, and the other was 5. They thought about it this way:

$3^2 + 4^2 = 5^2$

$9 + 16 = 25$

They argued this way: We think of a triangle with sides a, b, and c; and, if $a^2 + b^2 = c^2$, then the triangle has a square corner and is called a right triangle.

If this is true, is a triangle whose sides are 15", 8", and 17" a right triangle? . . . or does $15^2 + 8^2 = 17^2$? YES

Chapter THIRTEEN . . . THE AREA OF A CIRCLE

True or False?

(a) If two circles have the same radius then their areas are equal. T F

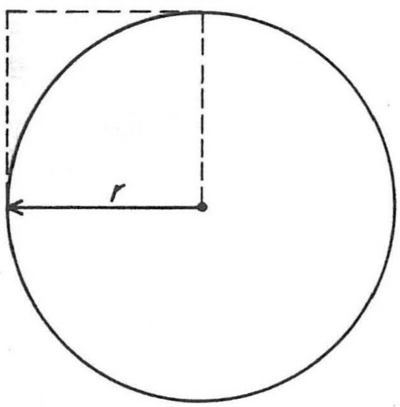
(b) If circle A has a larger radius than circle B then circle A has a larger area than circle B. T F

(c) If circle A has a smaller radius than circle B then circle A has a smaller area than circle B. T F

If you claim that all of these are true statements then you must believe that the area of a circle depends on its radius.

Since a radius is a segment, we cannot use it to measure area. Instead, we shall use the "square on the radius" as our unit of area.

We shall be looking for a ratio: the square on the radius is to the area of the circle as __?__ is to __?__.



$$\frac{r^2}{A} = \frac{?}{?}$$

How many tiles the size of r^2 ($r \times r$ or "r squared") would be needed to cover the region enclosed by the circle? 4

Could you place two whole tiles with size r^2 inside the circle so they would not overlap and no point on the tile would be outside the circle? No

How many such tiles would you need to be sure to cover all points in the region enclosed by the circle? 4

So, we see that the area of the circle is more than r^2 and less than $4r^2$.

That's not a very close approximation.

Suppose that we use tiles that are $\frac{1}{2}r$ by $\frac{1}{2}r$—or tiles with an area of $\frac{1}{4}r^2$?

Notice Diagram I on the next page. This is one way this might be done. The tiles inside the green line are clearly "inside" the circle. The tiles inside the blue line clearly cover all points in the region enclosed by the circle.

Put a dot in every tile that touches any point on the circle. Complete the blue line so it includes all tiles that touch the circle at any point.

How many tiles are there inside the blue line in Diagram I? 24

Now we know that the area is somewhere between 4 of these smaller tiles and 24 of them.

Too small: $4 \cdot \frac{1}{4} r^2$, or r^2

Too large: $24 \cdot \frac{1}{4} r^2$, or $6r^2$

This is a very poor approximation, but it is only the beginning.

Now, suppose that we use still smaller tiles. Let's make each tile $\frac{1}{4}r$ by $\frac{1}{4}r$ — or $\frac{1}{16}r^2$.

Follow the same procedure as before in Diagram II.

How many tiles are inside the green line, tiles that do not touch any point on the circle? 32

How many tiles are used to cover all points in the region enclosed by the circle? 68

Now we know that the area is somewhere between 32 of these still smaller tiles and 68 of them.

Too small: $32 \cdot \frac{1}{16} r^2 = 2r^2$

Too large: $68 \cdot \frac{1}{16} r^2 = 4\frac{1}{4}r^2$

96-99 Some experiments for determining a formula for the area of a circle

Tasks: Students try to trap the area of a circle by comparing the number of square tiles entirely within a circle with the number of squares which cover at least the region enclosed by the circle.

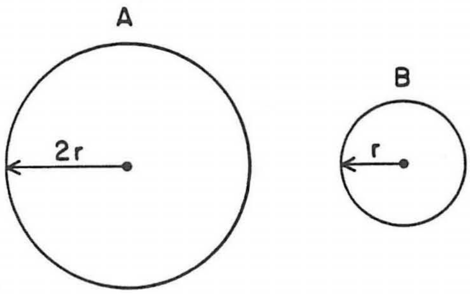
Purpose: To help students discover, by trapping, the relationship of the radius to the area of a circle.

Unifying Ideas: Functions and relations; geometry.

The Lesson: Pupil pages 96 through 99 lead students through another trapping experiment—searching for some way to measure the area of a circle.

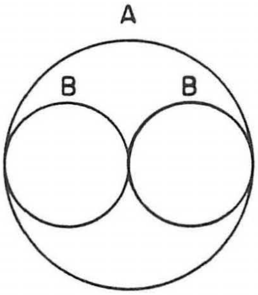
Again, intuition can only provide some valuable hints in suggesting a line of attack.

It is obvious that the longer the circumference of a circle, the larger the area of the circle. Does a circle with twice as large a circumference have twice as large an area? While our intuition might lead us to that conclusion, it is clearly not true. We know from earlier work that the circumference of a circle with twice the radius of another is twice the circumference of the smaller circle.



Circumference of A is $2\pi\ (2r)$, or $4\pi r$. Circumference of B is $2\pi r$, or $\frac{1}{2}\ (4\pi r)$.

But we can easily fit two of the smaller circles inside the larger circle with no overlap and lots of unused room:



To talk about area, we must define a "unit of area." For convenience, we would like a "square unit" of area. Two units suggest themselves—the square on the radius and the square on the diameter.

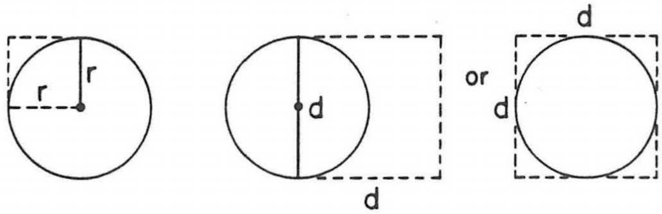
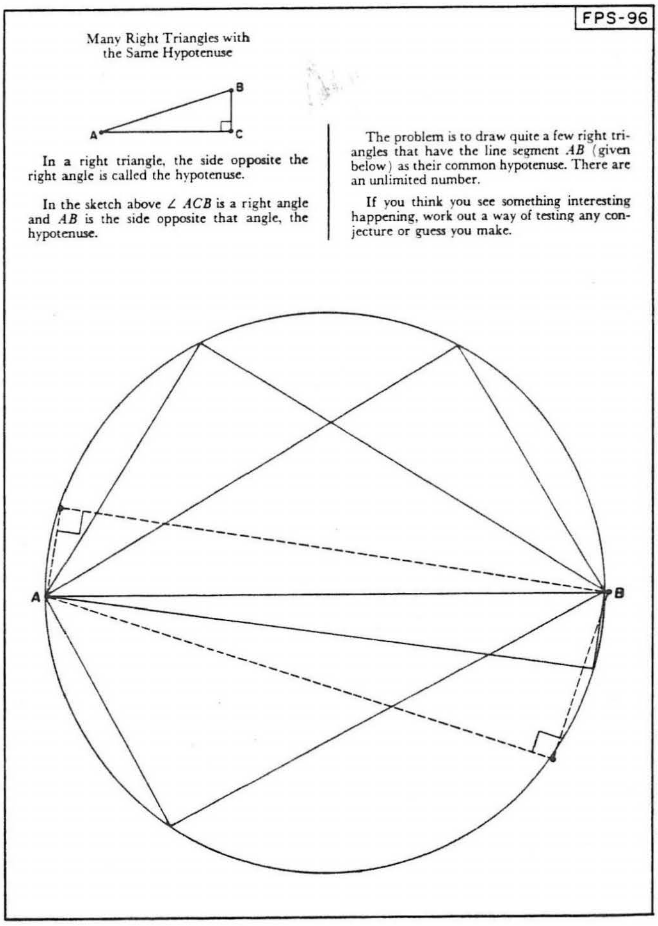


Fig. 1 Fig. 2 Fig. 3

Actually, either unit is appropriate. Figure 1 shows quite clearly that the area is less than $4r^2$. Figure 2 is rather clumsy, but the equivalent relationship in Figure 3 shows clearly that the area is less than d^2. But neither tells us anything more or less than the other, because $4r^2 = d^2$.

So our selection of r^2 rather than d^2 as a unit for our investigation is purely arbitrary.



FPS-96

Many Right Triangles with the Same Hypotenuse

In a right triangle, the side opposite the right angle is called the hypotenuse.

In the sketch above $\angle ACB$ is a right angle and AB is the side opposite that angle, the hypotenuse.

The problem is to draw quite a few right triangles that have the line segment AB (given below) as their common hypotenuse. There are an unlimited number.

If you think you see something interesting happening, work out a way of testing any conjecture or guess you make.

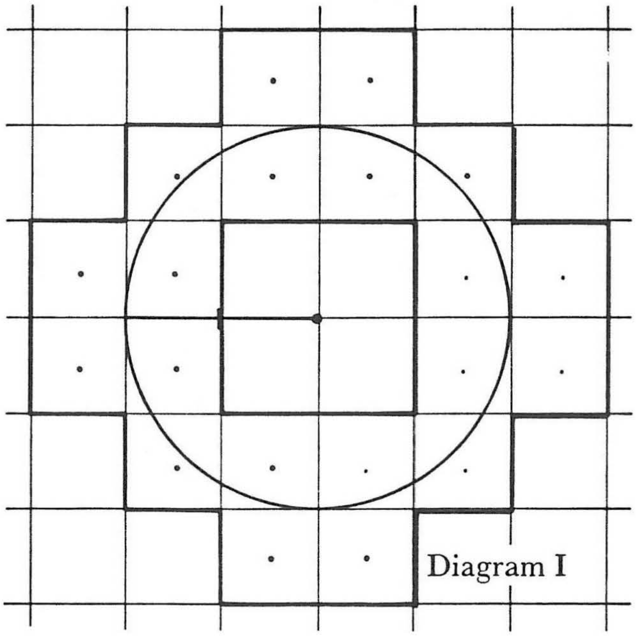


Diagram I

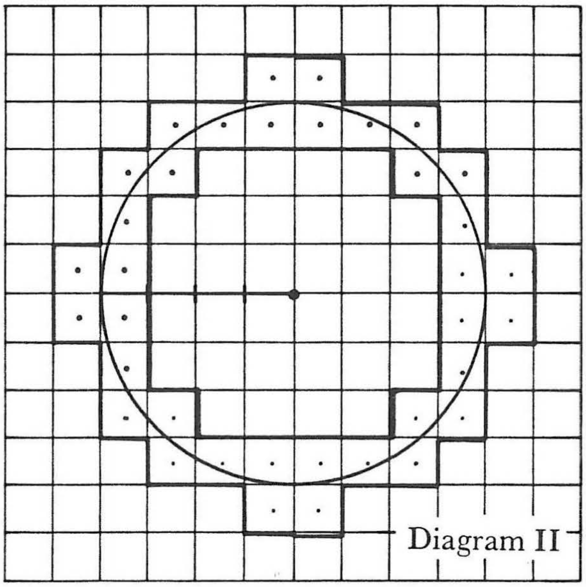


Diagram II

In Diagram III, we have tiles that are $\frac{1}{8}r$ on each side. So the tiles are

$$\frac{1}{8}r \cdot \frac{1}{8}r = \underline{\frac{1}{64}r^2}.$$

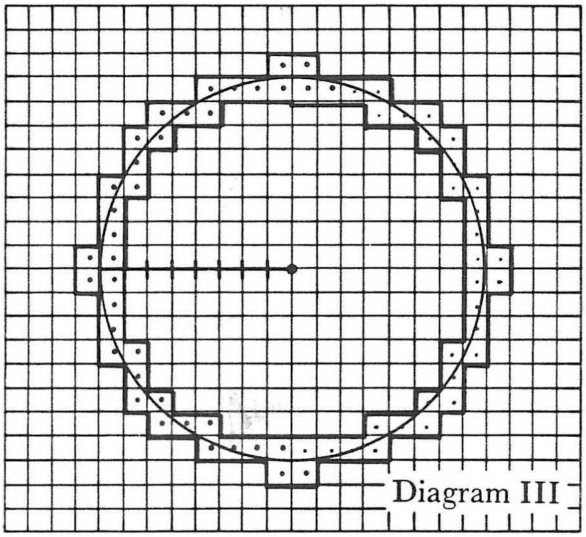


Diagram III

How many tiles are there inside the green line? 164

Inside the blue line? 232

Too small: $\underline{164} \cdot \frac{1}{64}r^2 = 2\frac{9}{16}r^2$

Too large: $\underline{232} \cdot \frac{1}{64}r^2 = 3\frac{5}{8}r^2$

Now we need a magnifying glass. Diagram IV is seen as if it were twice size. (Only one-fourth of the diagram is shown. How can you know what the rest of the diagram would show?)

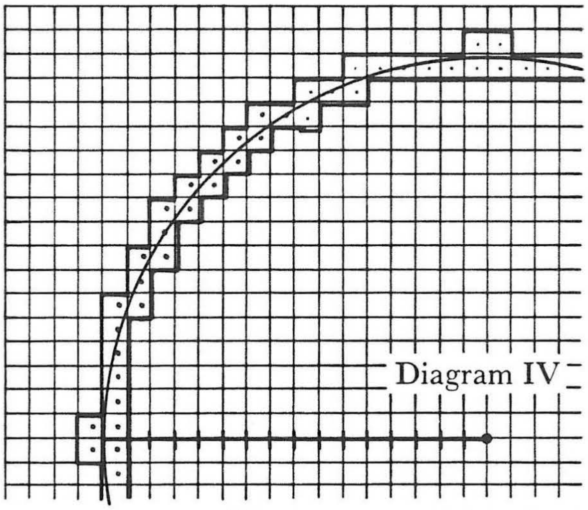


Diagram IV

(Under a magnifying glass)

The radius is now divided into 16 parts, so each tile is $\frac{1}{16}r \cdot \frac{1}{16}r$, or $\frac{1}{256}r^2$

How many tiles are there inside the green line? 732

How many are inside the blue line? 864

Too small: $\underline{732} \cdot \frac{1}{256}r^2 = 2\frac{55}{64}$

Too large: $\underline{864} \cdot \frac{1}{256}r^2 = 3\frac{3}{8}$

Diagram

	I	II	III	IV	V
Too small:	r^2	$2r^2$	$2\frac{9}{16}r^2$	$2\frac{55}{64}r^2$	$3.04r^2$
Too large:	$6r^2$	$4\frac{1}{4}r^2$	$3\frac{5}{8}r^2$	$3\frac{3}{8}r^2$	$3.24r^2$

(See the next page for Diagram V.)

***Tasks:** Students try to trap the area of a circle by comparing the number of square tiles entirely within a circle with the number of squares which cover at least the region enclosed by the circle.*

Purpose: To help students discover, by trapping, the relationship of the radius to the area of a circle.

Unifying Ideas: Functions and relations; geometry.

The Lesson: To push beyond the scant information that the area is less than $4r^2$, we subdivide our unit first into 4ths, into 16ths, into 64ths, and into 256ths on pupil page 97, and into 1,600ths on pupil page 98.

In each case, we wish to know how many we can fit inside without any overlapping and without touching the circle—to find an area which is *certainly* less than that of the circle.

Next, we wish to know how many we need to completely cover the area of the circle with non-overlapping subunits arranged so that no point of the enclosed region is uncovered.

If, in our experiments, there is any trace of doubt about whether a subunit is certainly inside or whether it is certainly necessary to cover the area, we will bend over backward to be sure. We won't count as "inside" any subunit that seems even to touch the circle, but we will count as "necessary to cover" any subunit that seems even to touch the circle.

This may seem a bit overcautious, but we wish to be certain that the area we are trying to find is trapped between the two areas—the one we know is certainly inside (and therefore smaller) and the one we know certainly covers more than the total area of the circle (and is therefore larger).

Notice that the subunits used in diagrams III and IV are the same size as printed on the page. However, the drawing of the circle is larger—as if under a magnifying glass. If diagram IV had been the same size as diagrams I, II, and III, the subunit chosen in IV would be $\frac{1}{4}$ as large as shown, because both dimensions would be cut in half. Clearly, the subdivisions in IV are squares of $\frac{1}{16}$ of the radius on a side—or $\frac{1}{256}$ of r^2.

Understanding this change of scale in the drawings is essential in progressing from diagram III to IV here and from IV to V on pupil page 98.

Further, we are considering only a portion of the drawing. Since a vertical line and a horizontal line, both through the center, divide the drawing into 4 congruent parts, we need only take our counts in one part and multiply them by 4 as counts for the whole drawing.

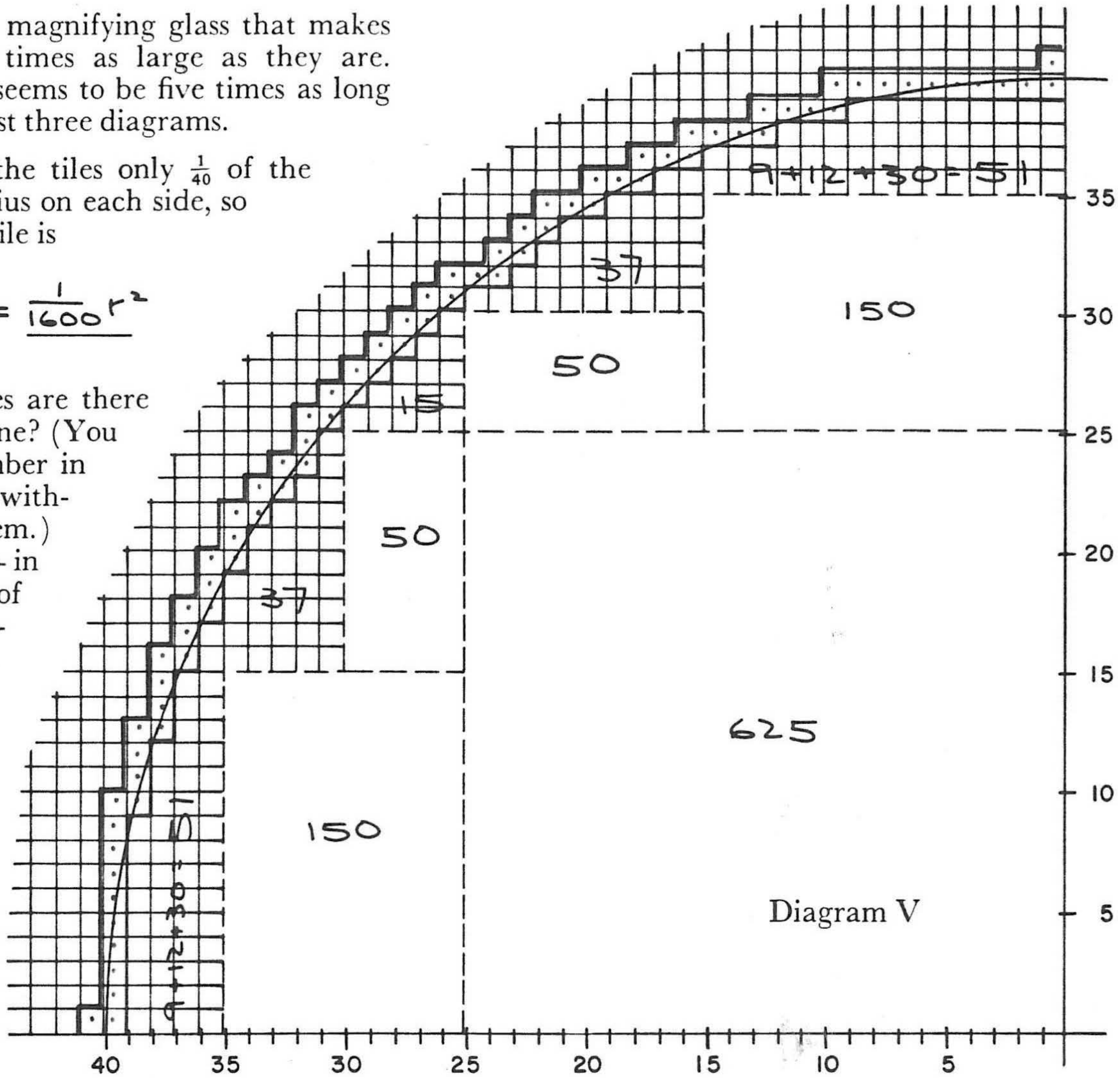
Perhaps at this point some of the students might like to guess about the number we are approaching. Some may already know that it is π again. Others, noticing the general direction the experiments are going, may guess that it will be about the same as the ratio of the circumference to the diameter.

Next we use a magnifying glass that makes things look five times as large as they are. Our radius now seems to be five times as long as it was in the first three diagrams.

We have made the tiles only $\frac{1}{40}$ of the length of the radius on each side, so the area of each tile is

$$\frac{1}{40}r \cdot \frac{1}{40}r = \frac{1}{1600}r^2$$

How many tiles are there inside the green line? (You can find the number in the large blocks without drawing them.) There are 1216 in this diagram of $\frac{1}{4}$ of a circle. In a diagram of the whole circle, there would be 4,864



How many more than that are there inside the blue line in this fourth of the total diagram? 81 more. In a similar diagram showing the whole circle, there would be four times as many, or a total of 324 more blue than green. So, the total number inside the blue line would be 5,188.

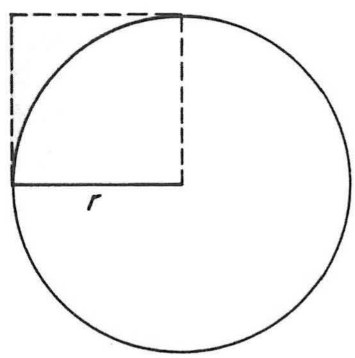
How many of these small tiles would be required to cover a square using the radius in Diagram V as the length of each side?

____

Too small: $4{,}864 \cdot \frac{1}{1600}r^2 = 3\frac{1}{25}\ r^2$

Too large: $5{,}188 \cdot \frac{1}{1600}r^2 = 3\frac{97}{400}\ r^2$

(Record these results in the chart on the previous page.)



The green block above suggests our idea of r^2. If we cut four of these blocks into 1,600 tiny squares each, we know we could put at least 4,864 of the tiny squares completely inside our circle. Also, we would not need more than 5,188 of the tiny squares to cover the region enclosed by the circle. Since each tiny square has an area of $r^2 \div 1600$, the area of the circle is somewhere between 3.04 r^2 and 3.24 r^2.

Tasks: *Students try to trap the area of a circle by comparing the number of square tiles entirely within a circle with the number of squares which cover at least the region enclosed by the circle.*

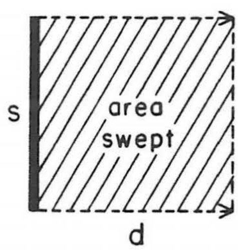
Purpose: To help students discover, by trapping, the relationship of the radius to the area of a circle.

Unifying Ideas: Functions and relations; geometry.

The Lesson: While the experiments and computations close in on π, consider discussing "sweep geometry," which you may attribute to Clyde or Harold. Clyde said he computed areas in terms of a sweeper and the distance swept. He said:

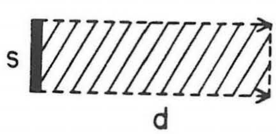
"Area = *average size of sweeper* times *average distance swept*."

"For example, consider a square:



"s is the length of the sweeper and d is the distance swept. The sweeper stays the same size (s) and all parts of the sweeper sweep the entire distance (d); so: $A = s \times d$.

"The same holds true of a rectangle:



$$A = s \times d$$

"However, sweeping a right triangle is a different matter:

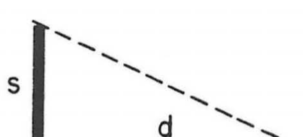


Fig. 1

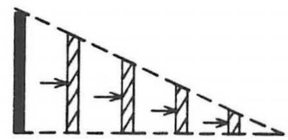
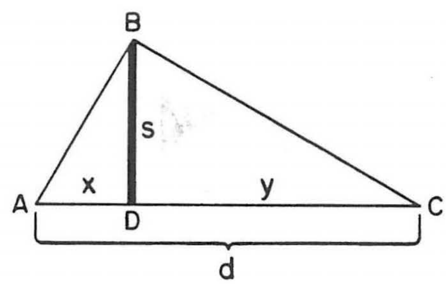


Fig. 2

"The sweeper (s) is full-size at the beginning and gradually shrinks (Fig. 2) to nothing as it sweeps." So Clyde says that the average size of the sweeper is $\frac{(s+0)}{2}$, or $\frac{1}{2}s$, and that it sweeps the full distance (d). Therefore,

$$A = \frac{1}{2} s \times d = \frac{sd}{2}.$$

"If the triangle is not a right triangle, it presents certain difficulties:



"As the sweeper (s) moves from A to D, it grows from 0 to s regularly." So Clyde says that its size is $\frac{1}{2}s$ and that the distance swept is x; therefore, the area of $\triangle ABD = \frac{1}{2}s \times x$.

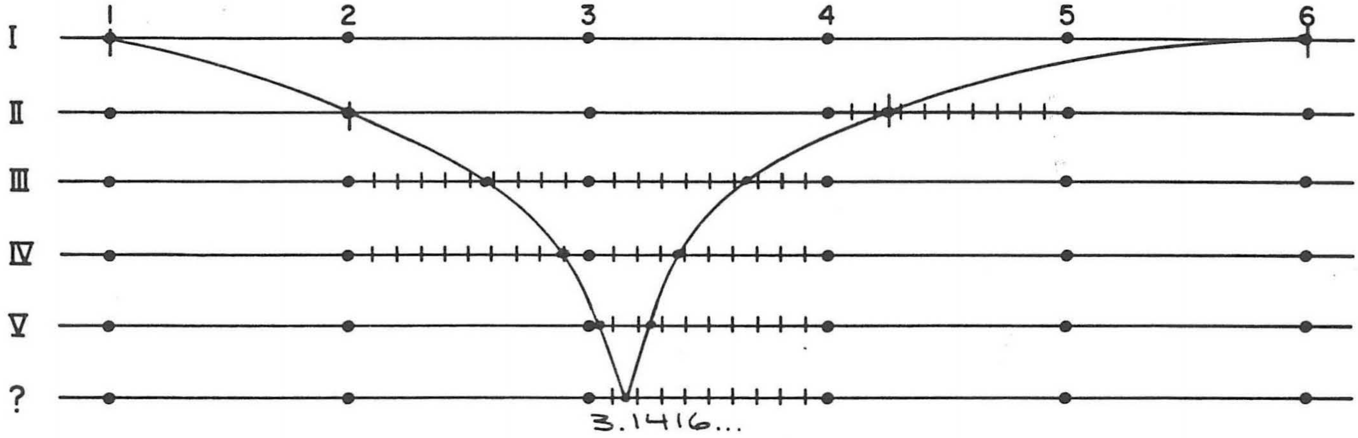
As the sweeper moves on across from D to C, it shrinks from s to 0—or has an average size of $\frac{1}{2}s$—and sweeps a distance of y; so the area of $\triangle BCD = \frac{1}{2}s \times y$. The area of ABC = the area of ABD plus the area of BDC. The area of $ABC = (\frac{1}{2}s \times x) + (\frac{1}{2}s \times y) = \frac{1}{2}s \times (x+y)$. Since $x+y=d$, the area of $\triangle ABC = \frac{1}{2}s \times d = sd/2$.

As Clyde develops his "sweep geometry" in this fashion, he considers the area of a circle. He has two approaches that are outlined briefly in the Notes on Pupil Page 99.

Express the results of the five studies in decimals and record below . . . (to the nearest hundredth)

	I	II	III	IV	V	?
Too small	$1.0r^2$	$2.00r^2$	$2.56r^2$	$2.87r^2$	$3.04r^2$	
Too large	$6.0r^2$	$4.25r^2$	$3.63r^2$	$3.36r^2$	$3.24r^2$	

What do you notice? Now, indicate these results on a series of number lines.



We could keep magnifying our diagrams and divide the radius into the smallest units we could handle.

In each experiment, the difference between the area that is clearly "too small" and the area that is "too large" would become less and less.

Could we design an experiment with such a large radius and such small divisions that we would reduce that difference to zero? No

As we kept recording our continuously magnified experiments on a number line, would they crowd in from both directions on a certain point? YES

We know that such a point must be located somewhere between the points labeled 3.1 and 3.2. What point is half-way between? 3.15

Our experiments have led us to believe that the point 3.14 is very close to the number we're looking for.

It should come as no surprise that more careful experiments will lead us closer and closer to a ratio we have encountered before: π — an irrational number, for which the approximations $3\frac{1}{7}$ and 3.1416 are commonly used.

We can note this relationship between the radius and the area of a circle as:

Area $= \pi r^2$

and the relationship between the circumference and radius as:

Circumference $= 2 \pi r$

If we were to express each in terms of the diameter, d, we could write:

Area $= \pi (\frac{1}{2}d)^2 = \frac{\pi d^2}{4}$ OR $\frac{1}{4}\pi d^2$

Circumference $= \pi d$

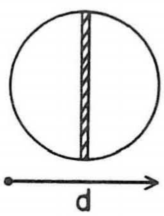
Notes on Pupil Page 99

Tasks: Students record the results of the investigations on pupil pages 96-98 in a chart and on number lines and consider what would happen if these records were extended. They compare their results with the information that the ratio of the area to the radius squared is approximately 3.14. They further try to express the relationship between the diameter of a circle and its area and circumference.

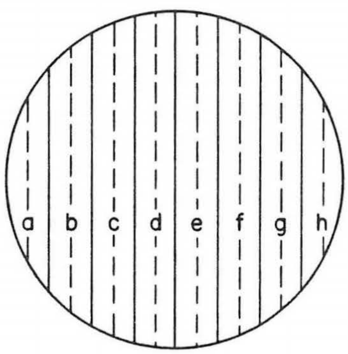
Purpose: To help students discover, by trapping, the relationship of the radius to the area of a circle.

Unifying Ideas: Functions and relations; geometry.

The Lesson: Clyde imagines a sweeper sweeping a circle. It grows from 0 (at the start) to a diameter (at the halfway point) and then shrinks to 0 as it sweeps the remainder:



But what is the average size of the sweeper? Clyde breaks the area swept into sections; he then measures the size of the sweeper when it is halfway across; finally he adds these measurements and computes an average:



a = about $\frac{7}{16}$ of a diameter

b = about $\frac{3}{4}$ of a diameter

c = about $\frac{15}{16}$ of a diameter

d = about 1 diameter

sum $3\frac{1}{8}$ diameter

e+f+g+h $3\frac{1}{8}$ diameter

$6\frac{1}{4}$

$$\frac{a+b+c+d+e+f+g+h}{8} = \frac{6\frac{1}{4}}{8} = \frac{25}{32} \text{ (of a diameter)}$$

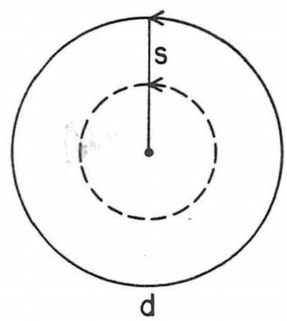
$$\text{Area} = \frac{25}{32} \text{ (of a diameter)} \times d \text{ (distance swept)}$$

Since the unit is a diameter squared, Clyde says the area is $^{25}/_{32}$ of the diameter squared.

If one wishes to convert this into units of r^2, he need only multiply by 4, since $d^2 = 4r^2$. So, Clyde says a good approximation is given by:

$$A = 4 \times \frac{25}{32} r^2 = \frac{25}{8} r^2 = 3\frac{1}{8} r^2$$

However, Clyde has improved on this clumsy method. He sees the area of a circle as the area swept by a radius. He argues that during the sweeping, the sweeper stays the same size, but one end travels further than the other; in fact, one end doesn't travel at all and the other travels the full circumference:



So, he says the average distance swept by the radius is $\frac{0 + c \text{ (circumference)}}{2}$ or $\frac{1}{2}c$, and the area is $\frac{1}{2}s \times r$, that is, "$\frac{1}{2}$ the circumference times the radius." He knows that the circumference is $2\pi r$; so,

$$\frac{1}{2}c = \frac{1}{2}(2\pi r), \text{ or } \pi r$$

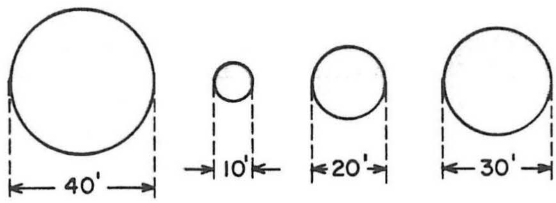
$$\text{and } \frac{1}{2}c \times r = \pi r \times r = \pi r^2.$$

A rather noteworthy result.

John Hayes, a gardener, said that he had in one day planted a round flower bed that was 40 feet across.

"That's nothing," Bill Jones said. "Today I planted three flower beds. One was 10 feet across; another was 20 feet across; and the third was 30 feet across."

John Hayes scratched his head. "Well, that's still not as much" . . . and an argument followed. Can you settle it?

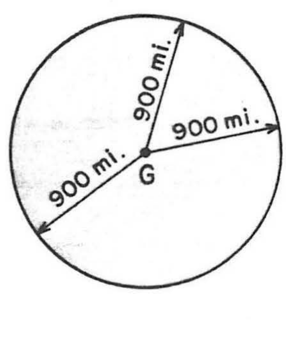


John Hayes planted 1,257 sq. ft. (APPROX.)

Bill Jones planted 1,100 sq. ft. (APPROX.)

(Use any of the approximations for π that we have discussed.)

When America's first astronaut, Col. John Glenn, was in orbit in 1962, the newspapers reported that he and his capsule were often about 100 miles above the earth, and that he could see about 900 miles in all directions. Of course, mountains would sometimes cut off his vision and the curvature of the earth was involved; but we can make a rough estimate of the size of the region he could see.



Col. Glenn could see about

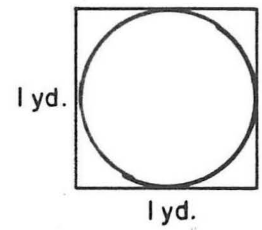
2,544,690 SQ. MILES OR ABOUT $2\frac{1}{2}$ MILLION SQ. MILES.

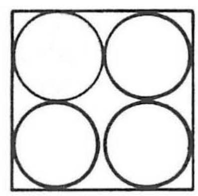
Mary said that if pizza pies were the same thickness, you would get the same amount in a pie that measured 14″ across as in four pies that measured 7″ across. Was she right?

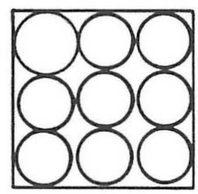
YES. ABOUT 154 SQ. IN. IN EACH.

Sarah had three pieces of cloth, each 1 yard square. From one, she wished to cut the largest possible circle. From the second, she wished to cut four circles the same size and as large as possible. From the third, she wished to cut nine circles the same size and as large as possible.

Finish the sketches below that suggest the situation.







How much of Sarah's 3 square yards was used for the circles?

.78 OR 21^{+} SQ. FEET.

In which case was there the most scrap cloth?

NONE. ALL ARE THE SAME.

In the first case, when Sarah cut out only one large circle, how much of the 1 square yard piece was scrap? ABOUT .21 OR .22 OR $\frac{3}{14}$ OF A SQ. YARD.

In which case was there the least scrap?

NONE. SAME IN ALL CASES.

How much scrap was there altogether?

.22 OR 6 SQ. FEET.

If Sarah had another piece of cloth 1 yard square and needed two circles the same size and as large as possible, what would the area of each circle be?

ABOUT $\frac{1}{4}$ SQ. YD. OR 2.4 SQ. FT.

How much would be scrap? 4.2 SQ. FT.

Explore the problem Sarah would face if she wished to cut three circles as large as possible and all the same size from a 1 yard square of cloth.

100-101 Putting the formula for the area of a circle to work

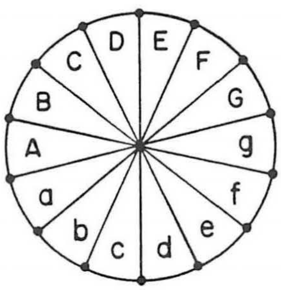
Tasks: Students solve five problems involving applications of finding areas of circles.

Purpose: To help students appreciate the usefulness of the relationship $A = \pi r^2$ to real-world problems.

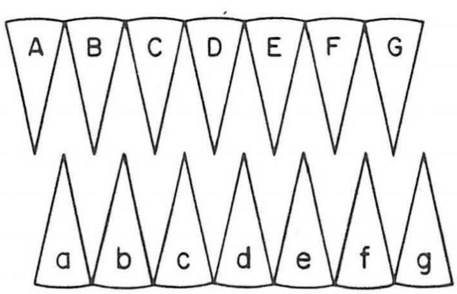
Unifying Ideas: Functions and relations; geometry.

The Lesson: There is yet another way of looking at the problem of finding the area of a circle.

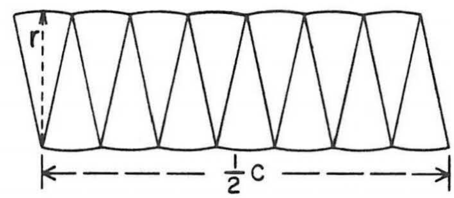
Suppose that we cut a circle into many pie-shaped pieces the same size, as shown in the diagram.



We stretch out half of them as an "upper set of teeth" and half as a "lower set of teeth":



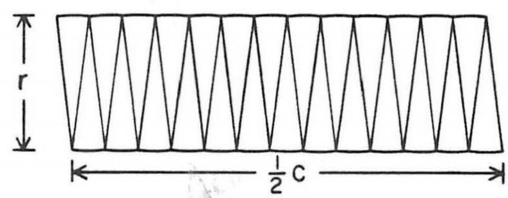
We push them together:



making a shape closely resembling a parallelogram. While the height varies because the top and bottom lines are actually composed of circular arcs, we can see that it approximates a radius, r, in length; and the length of the top and bottom is roughly the length of seven of those arcs, which is half the circumference.

So, we cannot be far wrong in saying that the area of this rearranged circle is about $\frac{1}{2}c \times r$, or $\frac{1}{2}(2\pi r) \times r$, or $\pi r \times r$, or πr^2.

As we cut the circle into smaller and smaller "teeth," the resulting shape resembles more closely a rectangle whose base is $\frac{1}{2}c$ and altitude is r:



The sequence of arcs more clearly resembles a straight line, and the variations in altitude become less.

Pupil page 100 includes several applications of finding areas. Note particularly Sarah's problem of cutting the largest circle or circles from a square yard of cloth. The results here are far from intuitive. Whether she cuts one, four, or nine, the amount of scrap is exactly the same. This is a surprise.

Also, Sarah would find that if she needed only three circles instead of four, she could increase the radius only about $\frac{1}{8}$—again, a surprising result.

Mr. Bowen bought 100 feet of fence to protect his garden. He wanted a garden that was rectangular in shape and as large as he could enclose with 100 feet of fencing.

He made his garden the shape of the green outline shown here — 30 feet long and 20 feet wide. Did he use all his fencing?

After he was finished, his neighbor said, "You could have made your garden larger if you had used other dimensions."

"But I had only 100 feet of fencing," Mr. Bowen answered, "and I wanted all the corners square."

"I still think I could enclose a larger area," the neighbor replied. Could he? Mr. Bowen made the following study:

length in feet of each of 2 opposite sides	30	35	40	45	25	23	21	27	28
length in feet of each of other 2 sides	20	15	10	5	25	27	29	23	22
square feet of area enclosed	600	525	400	225	625	621	609	621	616

He selected the three best examples and extended his study:

length in feet of each of one pair of sides	23	24	24.5	24.8	25	25.1	25.3	25.7	27
length in feet of each of the other pair	27	26	25.5	25.2	25	24.9	24.7	24.3	23
number of square feet of enclosed area	621	624	624.75	624.96	625	624.99	624.91	624.51	621

This result interested Mr. Bowen, and he explored the idea. Before beginning, he worked out some shorthand to help him simplify his work.

A is the number of square feet of area.
l is the number of feet in length.
w is the number of feet in width.

$$l \cdot w = A$$

"Suppose that I had 200 feet of fencing."

l	25	35	45	50	55	53	51	51.5
w	75	65	55	50	45	47	49	48.5
A	1,875	2,275	2,475	2,500	2,475	2,491	2,499	2,497.75

"Suppose that I had only 90 feet of fencing."

?

l	40	30	25	20	24	22	23	22.5
w	5	15	20	25	21	23	22	22.5
A	200	450	500	500	504	506	506	506.25

Mr. Bowen still was not thoroughly convinced. He tried similar studies, considering the problem each time with a different amount of fencing.

What do you think he decided when he considered having 121 feet? The largest enclosure would be 30.25 feet long and 30.25 feet wide. The area enclosed would thus be 915.06+ square feet.

Tasks: Students make studies to determine the rectangle with the largest area whose perimeter is 100 feet, 200 feet, 90 feet, and 121 feet.

Purpose: To help students discover that the rectangle with a given perimeter that encloses the largest area is a square.

Unifying Ideas: Functions and relations; geometry; measurement.

The Lesson: Mr. Bowen's problem is closely related to Sarah's problem of finding the maximum size of a specified number of circles in one square yard. He must find the largest rectangle whose perimeter is 100 feet. By considerable experimentation, he finds that it is a square, 25 feet on a side. Further experiments with other lengths, 200 feet and 90 feet, lead to the conclusion that regardless of what perimeter is specified, the rectangle enclosing the largest area is a square.

Suppose the limitation on shape were lifted—that any shape would do. What shape with a perimeter of 100 feet will enclose the largest area? Answer: a circle. The radius can be found:

$$\text{Circumference} = 100 = 2\pi r$$

$$r = \frac{100}{2\pi} = \frac{50}{\pi}$$

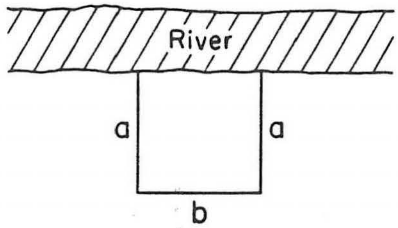
$$= 50 \div \frac{22}{7} = 15\frac{10}{11}$$

$$\text{Area} = \frac{1}{2}\,c \times r = 50 \times 15\frac{10}{11}$$

$$= 795\frac{5}{11}\ \text{sq. ft.}$$

This is a considerably larger area than a 25-foot square.

There is an interesting variation on Mr. Bowen's problem. Suppose a river runs across his land. He wants the 100 feet of fencing to enclose the largest rectangular area possible on three sides, using the river as the boundary on the fourth side:

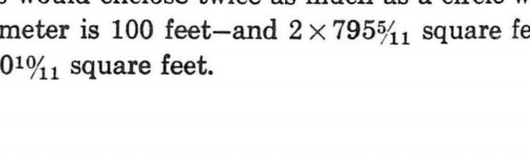


The area will be $a \times b$. We know that $2a + b = 100$. What shall we choose for a and b to enclose the largest area?

If we select a square then $a = b$. Since $2a + b = 100$, $a = 33\frac{1}{3}$ and $b = 33\frac{1}{3}$, and $a \times b = 1{,}111\frac{1}{9}$.

But he could do better—by enclosing half of a square: by making $b = 2a$. Then $a = 25$ and $b = 50$, enclosing an area of 1,250 square feet.

Again relax the requirement that the area be rectangular in shape. It comes as no surprise that he would use the fence in a semicircle, with the river as the straight side:

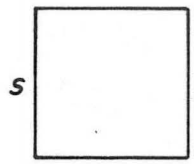


This would enclose twice as much as a circle whose perimeter is 100 feet—and $2 \times 795\frac{5}{11}$ square feet is $1{,}590\frac{10}{11}$ square feet.

XIV Problems About Area and Volume

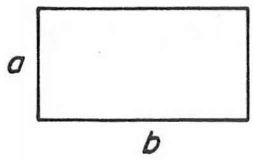
Letters in each sketch are known dimensions. Write a formula for finding the area of the green region in each sketch.

(1) a square



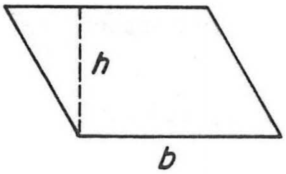
Area $= s^2$

(2) a rectangle



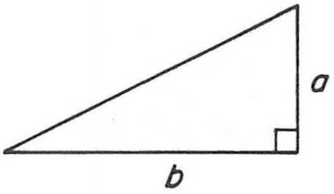
$A = ab$

(3) a parallelogram



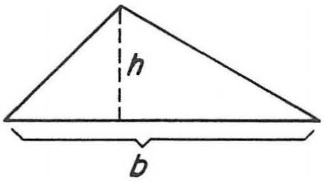
$A = bh$

(4) a right triangle



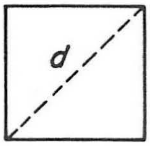
$A = \frac{1}{2}ab$

(5) any triangle



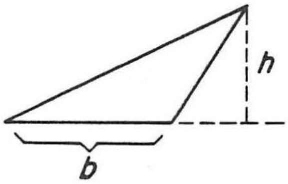
$A = \frac{1}{2}bh$

(6) a square



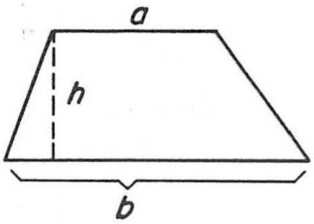
$A = \frac{1}{2}d^2$

(7) any triangle



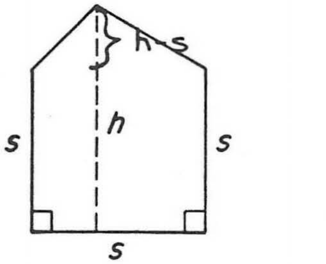
$A = \frac{1}{2}bh$

(8) a trapezoid



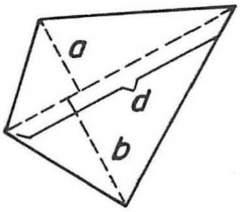
$A = \frac{1}{2}(a+b)h$

(9) a square with a triangle on top



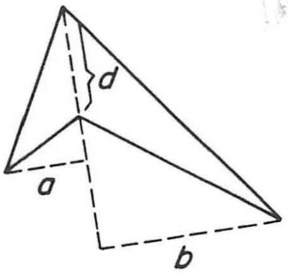
$A = s^2 + \frac{1}{2}s(h-s)$

(10) any quadrilateral (convex)



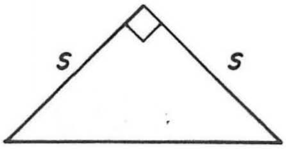
$A = \frac{1}{2}(a+b)d$

(11) any quadrilateral (concave)



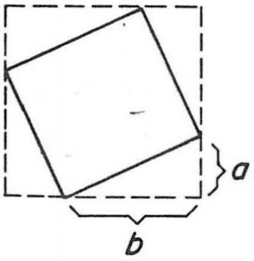
$A = \frac{1}{2}(a+b)d$

(12) a right isosceles triangle



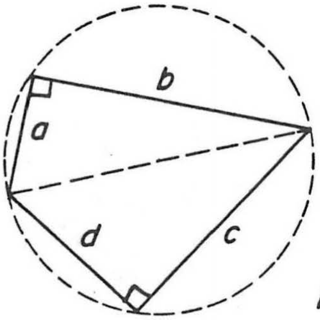
$A = \frac{1}{2}s^2$

(13) a square within a square



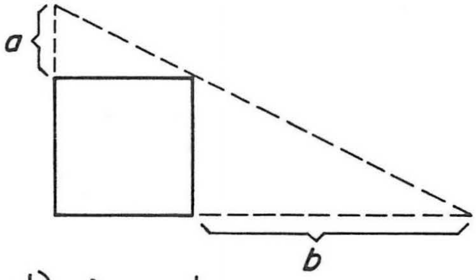
$A = a^2 + b^2$

(14) a quadrilateral in a circle — with two opposite vertices 180° apart



$A = \frac{1}{2}(ab+cd)$

(15) a square in a right triangle



$A = ab$

102-103 Finding areas of various polygonal regions

Tasks: Students give the mathematical shorthand or a "formula" for the areas of fifteen polygons.

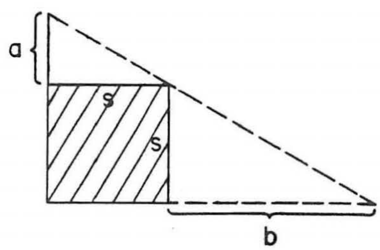
Purpose: To help students determine strategies for finding areas when certain dimensions are given.

Unifying Ideas: Addition and subtraction; multiplication and division; structure; geometry.

The Lesson: Pupil pages 102 and 103 present problems of finding areas of a large variety of polygons. No computations are required; rather, students are asked to indicate in mathematical shorthand the strategies they would use when certain dimensions are given. Each answer is, of course, a formula for finding an area. Consider having students work in several teams to determine these strategies or formulas.

All but three examples on pupil page 102 (13, 14, and 15) use two basic ideas: (1) that the area of a rectangle is "base × altitude" and (2) that a diagonal of a rectangle divides the rectangle into two parts of equal area. Elicit these two ideas with suitable examples on the chalkboard.

Example 5, and more particularly example 7, require some ingenuity in using these ideas:



We see $\triangle ACD$ as half of a rectangle whose area is $(b+x)\times h$; so the area of $\triangle ACD = \frac{1}{2}(b+x)\times h = \frac{1}{2}bh+\frac{1}{2}xh$. We see the area of $\triangle CBD$ is $\frac{1}{2}xh$. So:

$$\begin{aligned}\text{Area of }\triangle ABC &= \text{area of }\triangle ACD - \text{area of }\triangle CBD\\ &= (\tfrac{1}{2}bh+\tfrac{1}{2}xh)-\tfrac{1}{2}xh\\ &= \tfrac{1}{2}bh.\end{aligned}$$

Example 6 requires a different kind of ingenuity. Each half of the square can be seen as $\frac{1}{4}$ of a square whose sides are d, and $\frac{2}{4}\times d^2 = \frac{1}{2}d^2$.

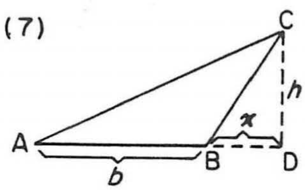
The last three examples bring in other ideas, none of which is new.

Example 13 is a straightforward application of the Pythagorean Theorem.

Example 14 depends on the fact that if a triangle inscribed in a circle has a diameter as one side then it is a right triangle.

Example 15 draws on the relationship between similar triangles:

(7)



The two shaded triangles are clearly similar right triangles; so,

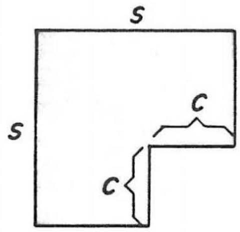
$$\frac{a}{s}=\frac{s}{b}$$
and $s^2 = ab$.

But s^2 is the area of the square; so $A = ab$.

It may be necessary to develop one or more of these strategies on the chalkboard for some students. Let them struggle with these problems, however, individually or in teams.

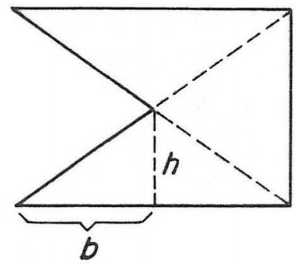
Letters in each sketch are known dimensions. Write a formula for finding the area of the green region in each sketch.

(16) a square with a square corner removed



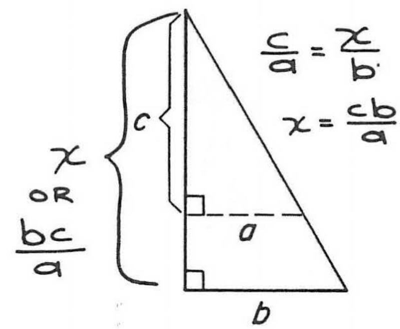
$A = S^2 - C^2$

(17) a rectangle with a wedge cut out



$A = 3bh$

(18) a small right triangle covering part of a large one



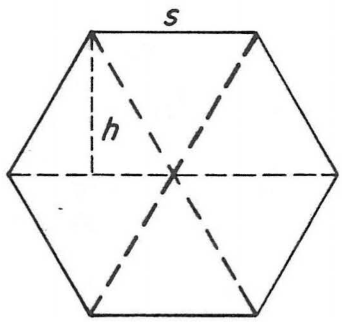
$\frac{c}{a} = \frac{x}{b}$

$x = \frac{cb}{a}$

$A = \frac{1}{2} b \times \frac{bc}{a}$

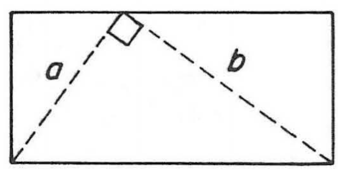
OR $\frac{b^2 c}{2a}$

(19) a regular hexagon



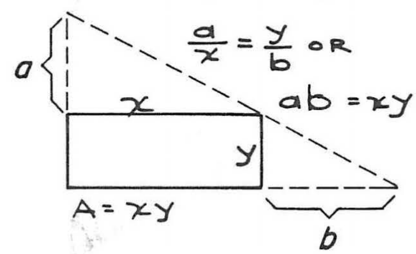
$A = 3sh$

(20) a right triangle in a rectangle



$A = ab$

(21) a rectangle in a right triangle



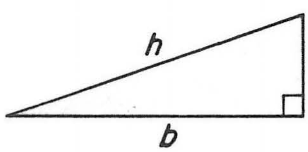
$\frac{a}{x} = \frac{y}{b}$ OR

$ab = xy$

$A = xy$

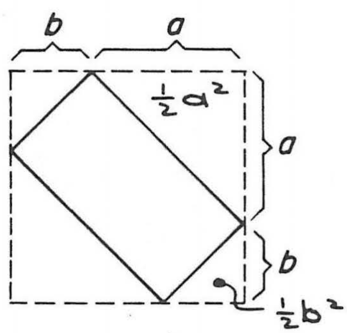
$A = ab$

(22) a right triangle



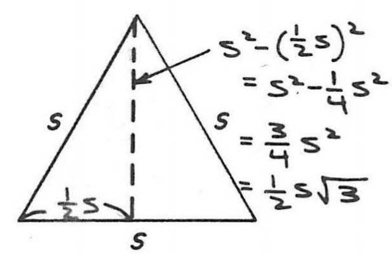
$A = \frac{1}{2} b \sqrt{h^2 - b^2}$

(23) a rectangle in a square



$A = 2ab$

(24) an equilateral triangle



$s^2 - (\frac{1}{2}s)^2$

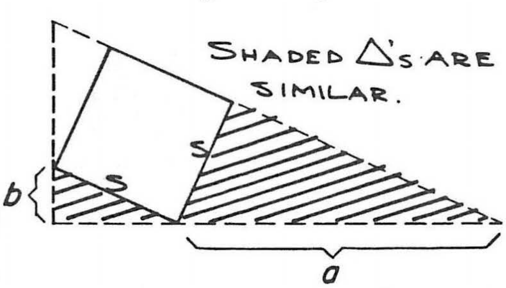
$= s^2 - \frac{1}{4}s^2$

$= \frac{3}{4}s^2$

$= \frac{1}{2}s\sqrt{3}$

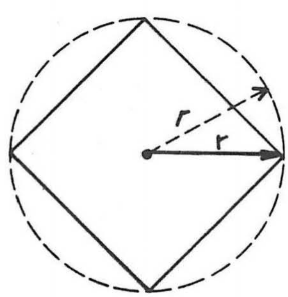
$A = \frac{1}{4} s^2 \sqrt{3}$

(25) a square in a right triangle



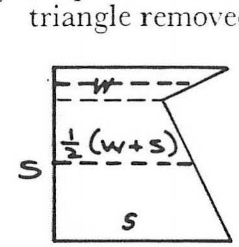
$A = ab$ $\quad \frac{a}{s} = \frac{s}{b}$ OR $s^2 = ab$

(26) a square in a circle with radius r



$A = 2r^2$

(27) a square with a right triangle removed



$A = S \times \frac{w+s}{2}$ OR $\frac{1}{2}s(s+w)$

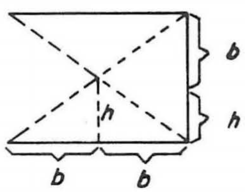
Tasks: Students give the mathematical shorthand or a "formula" for the areas of thirteen polygons.

Purpose: To help students determine strategies for finding areas when certain dimensions are given.

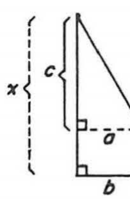
Unifying Ideas: Addition and subtraction; multiplication and division; structure; geometry.

The Lesson: Problems on pupil page 103 require, in general, more reasoning than those on the previous page.

Example 17 needs only a few additions to the sketch to help students realize that the area can be seen as the sum of the areas of six small congruent right triangles and that the area of each is $\frac{1}{2}bh$:



In example 18 the two triangles are similar; so:



$$\frac{c}{a} = \frac{x}{b}$$

$$ax = cb$$

$$x = \frac{cb}{a}$$

$$\text{Area} = \frac{1}{2}b \cdot x = \frac{1}{2}b \cdot \left(\frac{cb}{a}\right) = \frac{1}{2}\,\frac{b \cdot b \cdot c}{a} = \frac{b^2c}{2a}.$$

In example 19 the main diagonals of a regular hexagon divide it into six equilateral triangles with sides s. So, the area of each is $\frac{1}{2}sh$. Six of them have a total area of $6 \times \frac{1}{2}sh$, or $3sh$.

In example 20 the right triangle with area $\frac{1}{2}ab$ is half the area of the rectangle, and $2 \times \frac{1}{2}ab = ab$.

Example 21 is a generalization of example 15. The two smaller triangles are similar right triangles; So,

$$\frac{a}{x} = \frac{y}{b}$$

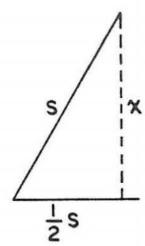
and $x \cdot y = a \cdot b$.

But $x \cdot y$ is the area of rectangle; so $A = ab$.

Example 22 utilizes the Pythagorean Theorem to find the altitude of the right triangle.

In example 23 the two larger triangles outside the rectangle are clearly $\frac{1}{2}a^2$ in area, and the two smaller ones are clearly $\frac{1}{2}b^2$ in area. Since there are two of each triangle, the sum of all four areas must be $a^2 + b^2$. The sides of the large square are $a + b$; so its area is $(a+b)^2$, and $(a+b)^2 = a^2 + 2ab + b^2$. Thus, the area of the rectangle is $(a+b)^2 - (a^2+b^2)$, or $(a^2+2ab+b^2)-(a^2+b^2)$, or $2ab$.

Example 24 has some difficult arithmetic based on using the Pythagorean Theorem to find the altitude of the triangle:



$$x^2 = s^2 - \left(\tfrac{1}{2}s\right)^2$$

$$= s^2 - \tfrac{1}{4}s^2$$

$$= \tfrac{3}{4}s^2 = \tfrac{1}{4}s^2 \cdot 3$$

$$x = \sqrt{\tfrac{1}{4}s^2 \cdot 3} = \tfrac{1}{2}s\sqrt{3}$$

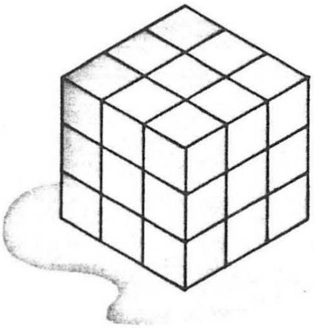
But x is the altitude of a triangle with base s; so,

$$\text{Area} = \tfrac{1}{2}s \times \left(\tfrac{1}{2}s\sqrt{3}\right) = \tfrac{1}{4}s^2\sqrt{3}.$$

(Do not expect all the class to follow the arguments and manipulations in this example.)

Surface and Volume of Solid Figures . . . requiring lots of imagination

a cube

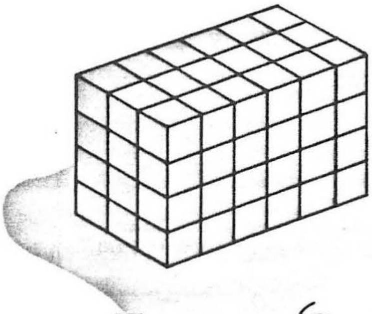


Here are unit cubes stacked in a pile: 3 layers, with 3 rows of 3 cubes in each row.

In such a stack there would be 27 unit cubes.

Using a face of a unit cube as a unit of area, each face of the stack above has 9 units of area. There are 4 sides, a top, and a bottom — or a total of 6 faces, and the total area of those faces is 54 units of area.

a rectangular solid



Here is a stack of unit cubes stacked in 4 layers with 3 rows of 6 cubes in each row.

In such a stack there would be 72 unit cubes.

There are 6 faces to the rectangular solid, and their combined area, or surface, is 108 units of area.

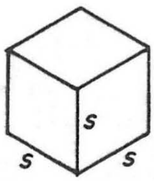
A problem for your imagination: How many unit cubes in the stack above could not be seen at all from any angle, no matter how you turned the stack? 8

Below, we shall think of the edge of a unit cube as a unit of length, the face of a unit cube as a unit of area, and a unit cube itself as a unit of volume.

Letters are known dimensions.

---

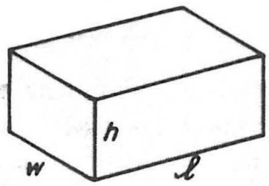
a cube



Volume = s^3

Surface = $6s^2$

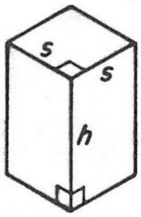
a rectangular solid



Volume = lwh

Surface = $2(hw+lw+hl)$

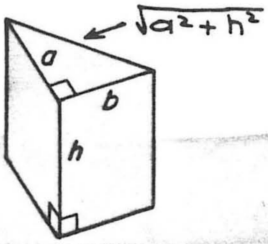
a right prism with a square base



Volume = s^2h

Surface = $2s^2+4sh$ OR $2s(s+2h)$

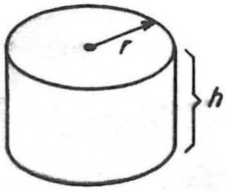
a right prism with a right triangle as base



Volume = $\frac{1}{2}abh$

Surface = $h(a+b+\sqrt{a^2+b^2})+ab$

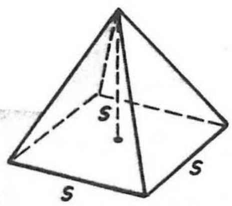
a right circular cylinder



Volume = $\pi r^2 h$

Surface = $2\pi r^2+2\pi rh$ OR $2\pi r(r+h)$

a regular pyramid with a square base



Base is s by s.
Height is also s.

Make a guess about its volume.

104-105 Finding the surface area and volume of various solids

Tasks: *Students write formulas for volumes and surfaces of solid figures. They also determine volumes and surfaces of rectangular solids by counting unit cubes and units of surface area.*

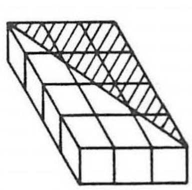
Purpose: To help students understand the idea of and determine the volume and surface of two right prisms, any rectangular solid, a right circular cylinder, and a square pyramid.

Unifying Ideas: Counting; addition and subtraction; multiplication and division; structure; geometry.

The Lesson: A large supply of small cubes, all the same size, would help greatly with most of the discussion on pupil pages 104 and 105. (We recommend following through the discussion on volume, and then returning to the idea of surface.) With such manipulative materials available, the idea of volume of a cube and of any rectangular solid should be quite evident.

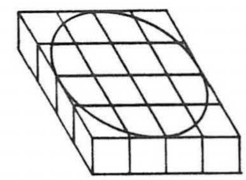
It would be useful to stress the fact that we can find the volume of one layer and then multiply by the number of layers. This is most important in anticipation of the transition from a rectangular solid to a right prism with a right triangular base. (A right prism is one whose lateral edges are perpendicular to a base.)

Now the bottom layer cannot be constructed out of unit cubes. However, a layer twice as large can be constructed:



If the legs of the right triangle are 4 units by 3 units then a base of 3×4 unit cubes, or 12 cubes, is clearly twice the size of the base layer we want—or 6 unit cubes. We can now multiply by the number of layers.

The base layer for the right circular cylinder is more difficult to see. However, we can construct a base layer of unit cubes that the base could be cut out of. If the radius is 2 then the smallest square we could use would be 4 by 4:



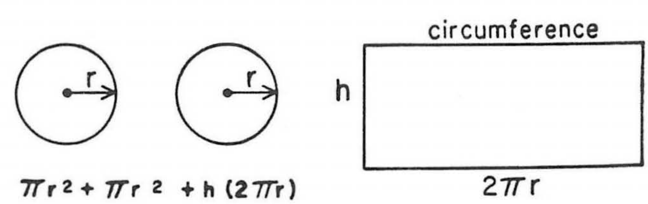
Now we can proceed with the notion that if there are πr^2 units in the circle, there would be πr^2 unit cubes in the base layer. We have only then to multiply by the number of layers.

Since we discuss a square pyramid on the next page, let the final example rest with a pure guess.

In returning to discuss the area of the surface in each case, the supply of small cubes will be most useful—until we reach the last two examples (the right prism with a right triangular base and the right circular cylinder).

There is no difficulty until we consider the largest rectangular face of the prism. We must use the Pythagorean Theorem to find one of its sides—$\sqrt{a^2+b^2}$.

Cutting out the top and bottom of a cylindrical box and then cutting the side from top to bottom quickly reveals the way to find the surface of the cylinder:



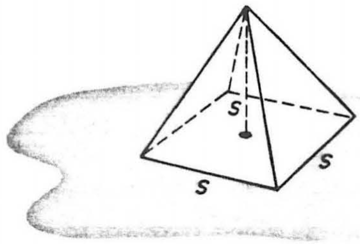
$\pi r^2 + \pi r^2 + h(2\pi r)$

or $2\pi r^2 + 2\pi r h$

or $2\pi r(r + h)$

We do not ask for the surface of the regular pyramid with altitude s and square base of side s. It is, however, $s^2(1+\sqrt{5})$.

Volume of a Regular Pyramid with a Square Base.

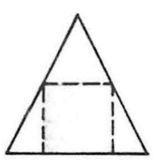


Imagine that this pyramid is made out of a cube that is s by s by s. The peak, or vertex, would be a point on the top of the cube.

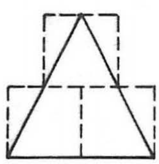
What is its volume?

Suppose that s *is 1.* All we know is that the volume is more than 0 and less than 1.

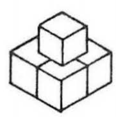
Suppose that s *is 2.* We could fit 1 full unit cube inside and we could have made the pyramid out of 1 unit cube placed on top of 4 unit cubes — or a total of 5 unit cubes. Here is a sketch to suggest these ideas:



1 fits inside

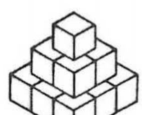


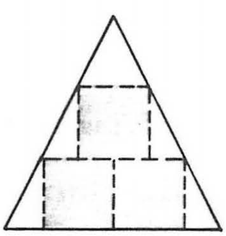
pyramid fits inside 5 cubes



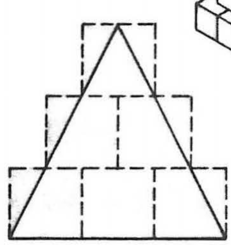
We know that if s = 2, the volume is more than __1__ and less than __5__.

Suppose that s *is 3.*





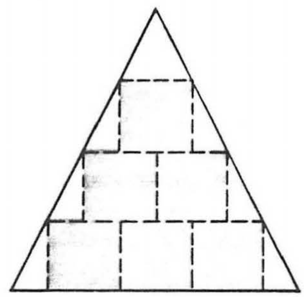
a stack of 4 + 1 fits inside



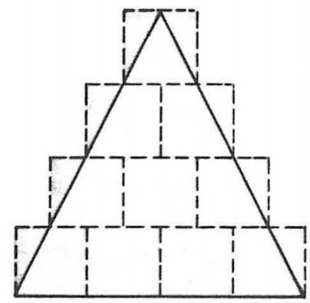
pyramid fits inside a stack of 9 + 4 + 1

We know that if s = 3, the volume is more than __5__ and less than __14__.

Suppose that s *is 4.*



a stack of __9__ + __4__ + __1__ fits inside



pyramid fits inside a stack of __16__ + __9__ + __4__ + __1__

We know that if s = 4, the volume is more than __14__ and less than __30__.

A pattern has emerged; so you could extend the experiments without any sketches.

Let's summarize our results thus far and extend the pattern a little.

if s is	volume is >	volume is <	average of > and <	s^3	$\frac{\text{avg.}}{s^3}$
1	0	1	$\frac{1}{2}$	1	$\frac{1}{2}$
2	1	5	3	8	$\frac{3}{8}$
3	5	14	$9\frac{1}{2}$	27	$\frac{19}{54}$
4	14	30	22	64	$\frac{11}{32}$
5	30	55	$42\frac{1}{2}$	125	$\frac{17}{50}$
6	55	91	73	216	$\frac{73}{216}$

The fractions in the column on the right are coming closer and closer to a very simple fraction. The last entry in that column is just $\frac{1}{216}$ more than __$\frac{1}{3}$__!

So, we can feel quite sure in guessing that the pyramid has __$\frac{1}{3}$__ the volume of the cube it is made from. The same relationship holds for any pyramid which is cut in this way from a rectangular solid. It also holds for any circular cone which is cut in this way from a right circular cylinder.

Notes on Pupil Page 105

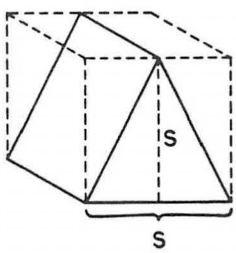
Tasks: Students explore the trapping technique for guessing the relationship of the volume of a certain regular pyramid with a square base to the volume of the cube from which the pyramid can be cut.

Purpose: To help students to discover the volume of a special pyramid.

Unifying Ideas: Counting; functions and relations; geometry.

The Lesson: Now let's go back to the volume of a regular pyramid with a square base that could be made out of a cube. Its altitude, of course, is the same as the sides of the square base.

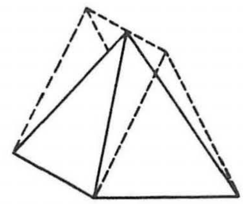
We might start to make a model of this figure by taking two slices off a cube with sides s and volume s^3:



We would have a wedge left. How much have we cut off? Quite clearly, we have already removed half of the cube, so the volume of the part left must be $\frac{1}{2}s^3$.

How many guessed that the volume of the pyramid would be $\frac{1}{2}s^3$? (In most groups, it is those who have no prior information.)

We have more to cut off.



How much have we cut off this time? How much of s^3 do we have left? Guess again!

Now to pupil page 105 proper. Again, we use the trapping technique in our search for a number to tell us how much of s^3 is left. We consider the number of cubes that could fit inside the pyramid (green) and the number of cubes that could completely enclose the pyramid (blue).

With drawings to help, we trap pyramids when $s=2$, $s=3$, and $s=4$. (We do not need to draw out the case $s=1$.) From those few results entered in a table, we can notice the pattern developing and extend the table. We find the average point between the "jaws" of our trap and note it. So,

$$\frac{1}{2},\quad \frac{3}{8},\quad \frac{19}{54},\quad \frac{11}{32},\quad \frac{17}{50},\quad \frac{73}{216}$$

$$\begin{array}{cccccc} -\frac{1}{6} & -\frac{1}{24} & -\frac{1}{54} & -\frac{1}{96} & -\frac{1}{150} & -\frac{1}{216} \\ \hline \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}$$

As the experiment continues, the average comes closer and closer to $\frac{1}{3}$. And we can say that we feel quite sure that the volume is about $\frac{1}{3}s^3$. In fact, it is exactly that.

In general, the volume of a pyramid or a cone is one-third the area of the base times the altitude.

XV A Problem About the Future

Arithmetic By Space Phone

Imagine that the telephone rings and you answer, "Hello."

"This is X-Nighyun in a spaceship near your Earth's moon. I'm from Mars. Alec Marson, my friend, once visited earth. He taught me to speak your language and told me to phone you if I ever got near enough to your planet. He said you would tell me about arithmetic. Do you know anything about arithmetic?"

What would be your answer?

"Well . . . well . . . Say, can you call me back tomorrow?"

"What's counting?"

"Well . . . well . . . Say, can you call me back tomorrow?"

"I certainly shall."

Puzzled, you say goodbye to X-Nighyun and hang up the phone. Questions buzz in your head. There's so much to explain, and so little to build on. How will you ever explain fractions, or even multiplication?

Plan of Attack

You've learned so much arithmetic over the years that you wonder how it can all be explained over the telephone even if you had a year.

So you decide on an unusual plan of attack. There is a way to teach X-Nighyun about counting, addition, and multiplication all in one easy lesson. When you are done, he will be able to count, add, and multiply — though he may not understand at all what he is doing. You might name this lesson:

Arithmetic Without Understanding

So, the next day you tell X-Nighyun to write down a whole string of symbols that are all different from each other.

"What kind of symbols?"

"I don't care — just make them all different."

"Done."

"Begun, you mean! When we run out of symbols you've written down, you'll have to invent some more.

What Are My Rules?

A.	I	II	III	IV
a	5	-5	$\frac{1}{2}$	7.3
$a + 7$	12	2	$7\frac{1}{2}$	14.3
$a - 4$	1	$^-9$	$^-3\frac{1}{2}$	3.3
$2a$	10	$^-10$	1	14.6
$2a + a$	15	$^-15$	$1\frac{1}{2}$	21.9
$2(a + a)$	20	$^-20$	2	29.2
a^2	25	25	$\frac{1}{4}$	53.29

B.	I	II	III	IV
x	7	$2\frac{1}{2}$	.8	0
$x + 3$	10	$5\frac{1}{2}$	3.8	3
$x - 3$	4	$^-\frac{1}{2}$	$^-2.2$	-3
$(x+3)(x-3)$	40	$^-2\frac{3}{4}$	$^-8.36$	$^-9$
x^2	49	$6\frac{1}{4}$	.64	0
$x^2 - 3^2$	40	$^-2\frac{3}{4}$	$^-8.36$	$^-9$
$15 - 3x$	-6	$7\frac{1}{2}$	12.6	15

106-109 Arithmetic by space phone (or arithmetic without understanding); What Are My Rules?

Tasks: *Students begin reading and studying the story of X-Nighyun. They also complete* What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: This last chapter introduces two different approaches to the study of algebra. On the top part of the pupil pages through 125 and all of 126-128, we have the story of X-Nighyun and the *generalization of arithmetic* approach. On the bottom half of pupil pages 106-125, we approach algebra through the problem of *developing a shorthand.* In the Notes on Pupil Pages, discussions of the latter are preceded by a blue line.

X-Nighyun and the Generalization of Arithmetic

Using pictures and sketches and manipulative materials has been indispensable in teaching students to understand arithmetic.

Suppose that you faced the task of teaching arithmetic right from the beginning without any such aids—over the space phone, in fact, with a young Martian called X-Nighyun on the other end of the phone. In developing such a fantastic situation, we shall simplify the problem by assuming that there is no language barrier and that the Martian pupil is a fast learner.

Perhaps you can discuss this situation with the class before considering the discussion reported in the top sections of pupil pages 106 through 128. What problems will you face? Where will you begin? What plan can you follow?

We have decided on a rather surprising way to initiate the conversation. It might be called "Arithmetic Without Understanding." (Perish the thought!) What possible purpose can such an approach serve?

Let's consider arithmetic as a simple-minded *game* in which play is rigidly controlled by a set of arbitrary rules. Suppose that you are reinventing the game and face the task of writing a set of rules. Perhaps the very best way would be to prepare two charts, called "addition" and "multiplication" charts, and explain the way in which they are to be used. This is precisely the approach we have chosen for our task in the pages that follow. You will notice that within a few half-pages, we are well into "arithmetic"—without understanding, of course.

Developing a Shorthand

The shorthand of algebra helps express ideas concisely. It cuts through a lot of verbiage and presents the essence of what we wish to record or communicate.

"I'm thinking of a number. If I add 7 to that number, the sum is 12. What number did I think of?" We boil this down and write:

$$a + 7 = 12$$
$$a = \underline{5}$$

Every student knows that the number thought of was 5. He needs no rules to substitute for thinking his way to the solution—that is, from the first equation to the second. He does not need to know that "you can subtract 7 from both sides of the equation."

We hope to give students many opportunities to think their way to solutions—before they have been given rules. They need this opportunity; they enjoy it—and, most important, they will discover the elementary rules on their own.

It is well to remember that every rule is something to remember that you could think out for yourself if you forgot it. It is entirely possible to wade quite far into algebra without having to remember anything new at all—and that is precisely what we are asking readers to do in these pages.

Rules should be summaries or conclusions that one remembers but that he once thought out for himself.

"Now, call the first symbol you wrote 'zero,' the next one 'one,' the next one 'two,' etc.

"Next, make two charts with many rows and columns. Above each column, write the symbols in order — zero, one, two, three, etc. — one to a column.

"At the left of the first column, write the symbols in order, beginning with zero, one to each row.

"Invent a symbol or mark for addition and another for multiplication. Put one of the marks on each table."

"But what is this addition and multiplication all about?"

"You don't need to know."

While you don't know what symbols X-Nighyun has invented, he has charts equivalent to these:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	4
2	2	3	4	5
3	3	4	5	6

X	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	4	6
3	0	3	6	9

"In the block in row 0 and column 0, write a 0.

"In the chart marked 'addition,' start filling in the blocks this way: As you move along a row or down a column away from the labeled end, write the next symbol from your list."

"Done. What about the other chart?"

"Select any block. It will be in a row and in a column. Now count the blocks which are *both* above the row *and* between that column and the left-hand edge."

A.	I	II	III	IV
r	2	10	0	2.5
$3r + 7$	13	37	7	$14\frac{1}{2}$
$3(r + 7)$	27	51	21	28.5
$20 - 4r$	12	$^-20$	20	10
$4(5 - r)$	12	$^-20$	20	10
$r^2 - r$	2	90	0	3.75
$r(r - 1)$	2	90	0	3.75
$3r^2$	12	300	0	18.75
$(3r)^2$	36	900	0	56.25

B.	I	II	III	IV
b	8	-5	1.2	0
$b - 1$	7	$^-6$	.2	$^-1$
$(b - 1)^2$	49	36	.04	1
$b^2 - 1^2$	63	24	.44	$^-1$
$(b + 1)(b - 1)$	63	24	.44	-1
$4b - b$	24	$^-15$	3.6	0
$b^2 - 2b$	48	35	$^-.96$	0
$b(b - 2)$	48	35	$^-.96$	0
$b(2 - b)$	$^-48$	$^-35$	.96	0

Tasks: *Students continue reading and studying the story of X-Nighyun. They also complete* What Are My Rules? *tables.*

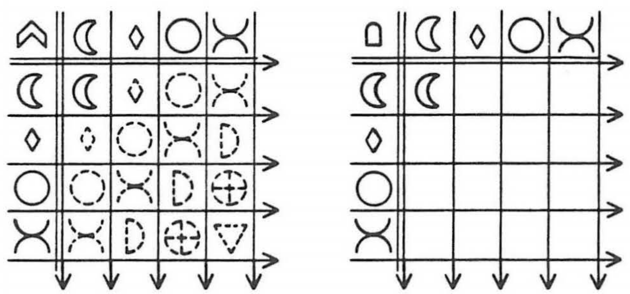
Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Suggest that the class invent a list of symbols (at least ten or twelve), as X-Nighyun was asked to do, as well as symbols for addition and multiplication. Suppose that their list starts with a quarter-moon:

.....for addition, and ....for multiplication

Using such symbols and following the rules as carefully as possible, fill in an addition and a multiplication chart:



The first entry inside the charts—upper left-hand block—is a quarter-moon called "zero."

Thus far, the charts are alike except for the symbols for addition and multiplication. The string of symbols on top and on the left could extend to as many as we have invented.

The directions for filling in the addition chart are given. We are ready to consider entries for the other chart.

You will notice that some rows of results have been tinted blue. This is to call attention to pairs of rows that show the same results.

The fourth and fifth rows of example *A* have the rule-indicators "$20-4r$" and "$4(5-r)$." We are tempted to write:

$$20 - 4r = 4(5 - r)$$

This is shorthand for the following argument:

"Think of a number. Subtract 4 times that number from 20. The answer is the first result. Then subtract the number you thought of from 5 and multiply the difference by 4. This answer is the second result. No matter what number you think of, the first and second results will be the same."

Is this a reliable conclusion? We find that it holds when r (the number thought of) is 2, 10, 0, and 2.5. Will it hold for negative numbers? Test, using $^-3$ and $^-\frac{1}{2}$:

$$20 - 4(^-3) = 4\left[5 - (^-3)\right]$$

$$\left.\begin{aligned} 20 - (^-12) &= 32 \\ 4(5+3) &= 32 \end{aligned}\right\}$$

and

$$20 - 4(^-\tfrac{1}{2}) = 4\left[5 - (^-\tfrac{1}{2})\right]$$

$$\left.\begin{aligned} 20 - (^-2) &= 22 \\ 4(5 + {^-\tfrac{1}{2}}) &= 22 \end{aligned}\right\}$$

Can anyone find a counterexample (or think of a number that won't work)?

Our results suggest other tentative conclusions:

$$r^2 - r = r(r-1)$$

$$b^2 - 1 = (b+1)(b-1)$$

$$b^2 - 2b = b(b-2)$$

and:

$$b(b-2) = {^-1}\left[b(2-b)\right]$$

Is each a reliable conclusion? Will each withstand every test that can be devised?

"Count?"

"Yes. For each such block, start with your symbol 'one' and write the list in order, one in each block. When you've filled them all, copy the last one in the block you first selected."

"What if there are no blocks *both* above the row *and* between the column and the left-hand edge?"

"Then write a 0 in that block."

X-Nighyun must have made a small chart — or he works with all hands at once — because he soon called out,

"Done!"

"Let's check your work. In the first row of your addition chart and in the first column, you should have written the string of symbols you invented and in the same order."

"Exactly."

"You can check each row in the multiplication chart in this way. The top or first row consists of zeroes. The next row is the string of symbols you invented and in the original order.

"For the next row, start with the 0 in your string and mark every other one starting 0, 2, 4, etc.

"For the next row, do the same but make each jump one symbol more than the jumps for the preceding row.

"Follow the same procedure for each succeeding row."

"Done."

"So you are ready to add and to multiply. "Three plus four" is an instruction that says: Find your symbol 3 at the top of a column in the addition chart and find the 4 at the end of a row. Find the symbol you wrote in the block in that row and column. It is the 7. So, we report that

$$3 + 4 = 7.$$

A.	I	II	III	IV
a	$2\frac{1}{2}$	.5 OR $\frac{1}{2}$	-8	0
$2a$	5	1	-16	0
$2a + 7$	12	8	-9	7
$4a + 14$	24	16	-18	14

B.	I	II	III	IV
c	$9\frac{1}{2}$	0	-5	37
$2c$	19	0	-10	74
$2c - 7$	12	-7	-17	67
$c - 3\frac{1}{2}$	6	$-3\frac{1}{2}$	$-8\frac{1}{2}$	$33\frac{1}{2}$

C.	I	II	III	IV
d	3	$1\frac{2}{3}$	$29\frac{1}{3}$	-1
$d + 4$	7	$5\frac{2}{3}$	$33\frac{1}{3}$	3
$3(d + 4)$	21	17	100	9
$3d + 12$	21	17	100	9

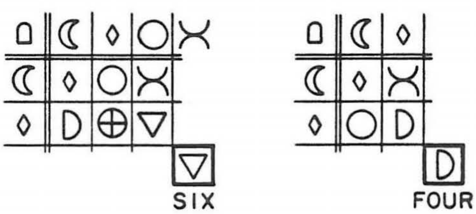
D.	I	II	III	IV
x	7	100	0	.05
$8 - x$	1	-92	8	7.95
$2(8 - x)$	2	-184	16	15.9
$16 - 2x$	2	-184	16	15.9

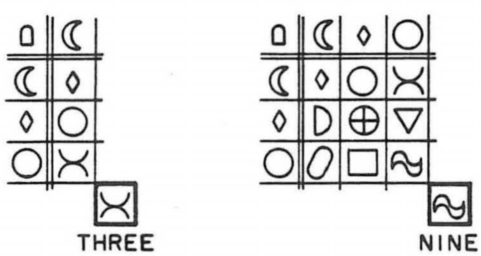
Tasks: *Students continue reading and studying the story of X-Nighyun. They also complete* What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

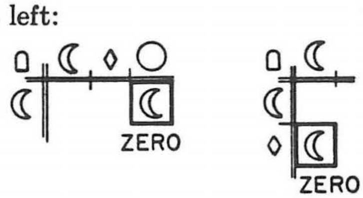
Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: The directions for finding entries in the multiplication chart are a bit tricky. "How many blocks above and to the left (inside the double line) of each block?" Begin with several examples and use the symbols that follow the quarter-moon:

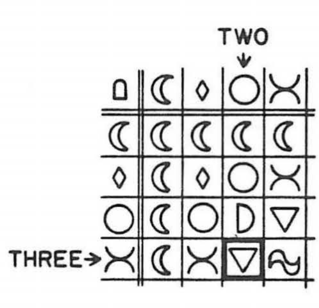




In the following, there are no blocks *both* above *and* to the left:



So, the completed chart would be the following:



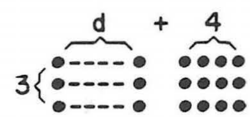
From the addition chart, we find that we would write:

In examples A and B we note that one of the blue-tinted rows shows results that are twice as large as the other:

$$(2a+7)\times 2=4a+14$$
$$(c-3\tfrac{1}{2})\times 2=2c-7$$

These tentative conclusions bear investigation.

A sketch might help visualize what is going on example C:



We can look at this as:

$$3(d+4)\text{; also, } 3d+3\times 4\text{, or } 3d+12$$

Example D might suggest this kind of activity: "I bought a dress. I gave the clerk \$8 and received x cents in change. I repeated the same purchase: $2(8-x)$. If, in the first place, I had bought 2 dresses and had given the clerk \$8 twice, or \$16, I would have received twice as much change, $2x$ cents, as I did when I bought one dress: $16-2x$."

Notice the similarity of pairs of rule-indicators:

$$4(s-r)=4\times 5-4\times r=20-4r \text{ (page 107)}$$
$$b(b-2)=b\times b-2\times b=b^2-2b \text{ (page 107)}$$
$$3(d+4)=3\times d+3\times 4=3d+12 \text{ (page 108)}$$
$$2(8-x)=2\times 8-2\times x=16-2x \text{ (page 108)}$$

One is tempted to make more general statements about a group of any three numbers a, b, and c you can think of:

$$a(b+c)=a\times b+a\times c=ab+ac$$
$$a(b-c)=a\times b-a\times c=ab-ac$$

Would these statements hold up when tested thoroughly? (They are, of course, pointing to a relationship often referred to as the distributive principle for multiplication over addition and over subtraction.)

"If I had said 3 times 4 or 3 multiplied by 4, you would use the multiplication chart and report your findings as:

$$3 \times 4 = 12$$

and that's all there is to addition and multiplication."

"You mean it's nothing more than looking something up in one of two charts?"

"Exactly!"

"But my charts aren't big enough to include all my symbols along the top and down the side."

"That's your problem. You know how to make your charts bigger if you need to."

"I guess you're right. But what's the point of playing such a game? Isn't there any use to arithmetic?"

"One thing at a time, please, one thing at a time. Now at least you know how to play the games of adding and multiplying.

"At least you know that no matter what pair of symbols you decide to add or multiply, there is a symbol that we call their sum in addition or their product in multiplication.

"If a and b are any two symbols you choose from the list, we know there must be symbols c and d which are their sum and product.

"We express that idea this way: For every a and every b, there are a c and a d such that

$$a + b = c \quad \text{and} \quad a \times b = d.$$

"Where we can say that, we say that we have defined addition and multiplication."

So much for arithmetic without understanding!

A.	I	II	III	IV
m	10	$\frac{1}{3}$	.1	$^{-}10$
$\frac{m}{2}$	5	$\frac{1}{6}$	.05	$^{-}5$
$\frac{m}{2} + 8$	13	$8\frac{1}{6}$	8.05	3
$3\left(\frac{m}{2} + 8\right)$	39	$24\frac{1}{2}$	24.15	9

B.	I	II	III	IV
r	6	$\frac{1}{2}$	1.2	$^{-}1$
$2r$	12	1	2.4	$^{-}2$
$\frac{2r}{3}$	4	$\frac{1}{3}$	.8	$^{-}\frac{2}{3}$
$\frac{2r}{3} + \frac{1}{2}$	$4\frac{1}{2}$	$\frac{5}{6}$	1.3	$^{-}\frac{1}{6}$

C.	I	II	III	IV
n	17	23	$^{-}5$	3
$n - 3$	14	20	$^{-}8$	0
$\frac{n-3}{2}$	7	10	$^{-}4$	0
$\frac{n}{2} - \frac{3}{2}$	7	10	-4	0

D.	I	II	III	IV
s	4	12	$^{-}2$	$2\frac{1}{2}$
$s + 5$	9	17	3	$7\frac{1}{2}$
$\frac{s+5}{3}$	3	$5\frac{2}{3}$	1	$2\frac{1}{2}$
$\frac{s}{3} + \frac{5}{3}$	3	$5\frac{2}{3}$	1	$2\frac{1}{2}$

Tasks: Students continue reading and studying the story of X-Nighyun. They also complete What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: What problems will X-Nighyun face in building larger addition and multiplication charts? He must extend his list of symbols. Why not treat the need for such additional symbols as providing an opportunity to dream up an entirely new design? (A persuasive reason for considering a system that uses new combinations of previously used symbols is strictly a practical one—the prospect of remembering so many different marks is not enticing.)

Our decimal system of numeration requires only ten distinct designs, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and then systematic combinations. We can easily tell what design comes after "237" in the natural order of the numbers—it's "238." The plan we use in making new combinations depends on our notion of "place value."

Suppose that X-Nighyun decided to build a system of symbols based on only his first four unique designs:

☾ ◊ ○ ⋈------

and to copy our idea of place value. His list would be something like this:

☾... ZERO	◊☾... FOUR	○☾... EIGHT
◊ ... ONE	◊◊ ... FIVE	○◊ ... NINE
○... TWO	◊○... SIX	○○... TEN
⋈.. THREE	◊⋈... SEVEN	○⋈.. ELEVEN

and would continue:

⋈☾..(12)	◊☾☾..(16)	◊◊☾..(20)
⋈◊..(13)	◊☾◊..(17)	◊◊◊..(21)
⋈○..(14)	◊☾○..(18)	◊◊○..(22)
⋈⋈..(15)	◊☾⋈..(19)	◊◊⋈..(23)

etc.

We would call this a place value system with a base of four.

Would such a decision on X-Nighyun's part lead to any confusion as we play Arithmetic Without Understanding by space phone? Not at all. We need never know whether he creates entirely new symbols as he extends his list; and, if he uses some plan to repeat parts of his symbols, we do not need to know what his plan is.

Thus, we've worked out basic rules so X-Nighyun can add or multiply simply by using his charts.

"Think of a number. Divide it by 2. If your result is 5 then you must have thought of 10." The following:

$$\text{(If) } \frac{m}{2} = 5$$
$$\text{(Then) } m = 10$$

is algebraic shorthand that captures the idea stated above in words. Some students may need to be reminded that we often have alternate ways of writing down the same idea; for example "$m \div 2$" would do just as well as "$m/2$".

A review of the agreements about parentheses may be in order:

1. Unless parentheses are used, we perform indicated multiplications and divisions first, and then additions and subtractions. And we work from left to right.

$$2+3\times 6 = 2+18 = 20$$

We perform the indicated multiplication first—as if we had written "$2+(3\times 6)$."

2. If parentheses are used, first carry out the work indicated inside the parentheses. We can use parentheses to indicate a different meaning for the above problem:

$$(2+3)\times 6 = 5\times 6 = 30$$

In the expression $3(m/2+8)$ we notice that inside the parentheses there are two operations indicated, division and addition. By agreement, we perform the division first. Thus, if m is 10, we write:

$$3(10/2+8) = 3(5+8) = 3\times 13 = 39$$

Patterns in the Charts

X-Nighyun reports the next day:

"I played your game with the charts. The charts interest me most. For every row in either one, there is a column exactly like it. Because of this, I could flip my chart around the diagonal from the "0,0" corner and the chart is unchanged (except my symbols get twisted on their sides). And this answers a question I had: When you select a pair of symbols, which one do you find above the columns and which at the end of the rows? It turns out that you get the same result one way as the other."

"Right!" you break in. "So we say that, for any pair of selections a and b,

$$a + b = b + a \quad \text{and} \quad a \times b = b \times a.$$

"We refer to these ideas as the commutative principle for addition and the commutative principle for multiplication."

"Big name for such a simple idea! But let me continue.

"There's a row and a column in each chart that is the same as a row and column in the other chart. Further, it is the row that lists all my symbols in order — zero, one, two, etc. For the addition chart, the zero row and the zero column; for the multiplication chart, they are the one row and the one column."

"Right again," you interrupt. "We say that 0 for addition and 1 for multiplication are 'identity elements' because, for every a,

$$a + 0 = a \quad \text{and} \quad a \times 1 = a.$$

"Identity elements, eh! More fancy language for a simple idea.

"I also notice that in the multiplication chart, the entire zero row and zero column are the same — all zeros — and zero appears nowhere else."

A.	I	II	III	IV	OR III
d	7	$^{-}3$	$\frac{1}{2}$	.4	$\frac{-1}{2}$
$2d$	14	$^{-}6$	1	.8	$^{-}1$
$d + 5$	12	2	$5\frac{1}{2}$	5.4	$^{-}4\frac{1}{2}$
$2(d + 5)$	24	4	11	10.8	9
$2d + 10$	24	4	11	10.8	9
d^2	49	9	$\frac{1}{4}$	.16	$\frac{1}{4}$
$3d + 2$	23	$^{-}7$	$3\frac{1}{2}$	3.2	$\frac{1}{2}$
$10 - d$	3	13	$9\frac{1}{2}$	9.6	$10\frac{1}{2}$
$-1d$	-7	3	$^{-}\frac{1}{2}$	$^{-}.4$	$\frac{1}{2}$

B.	I	II	III	IV
k	10	6	0	.5
$k + 6$	16	12	6	6.5
$k - 6$	4	0	$^{-}6$	-5.5
$(k+6)(k-6)$	64	0	$^{-}36$	$^{-}35.75$ OR $^{-}35\frac{3}{4}$
k^2	100	36	0	.25
$k^2 - 6^2$	64	0	-36	$^{-}35.75$
$3k - (k+2)$	18	10	$^{-}2$	$^{-}1$
$2k - 2$	18	10	$^{-}2$	$^{-}1$
$2(k - 1)$	18	10	$^{-}2$	$^{-}1$

110-113 Patterns in the charts: a+b = b+a and a*b = b*a; a+0 = a and a*1 = a

Tasks: Students continue reading and studying the story of X-Nighyun. They also complete What Are My Rules? *tables.*

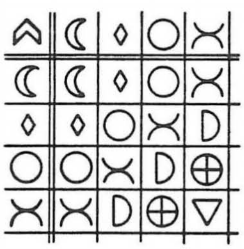
Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

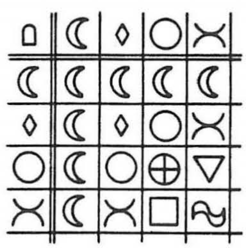
Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Here are the beginnings of two familiar tables—for addition and multiplication of whole numbers. Also, two tables that X-Nighyun might be looking at. What can you say about one of our familiar tables that is also true of X-Nighyun's table for the same operation?

+	0	1	2	3
0	0	1	2	3
1	1	2	3	4
2	2	3	4	5
3	3	4	5	6

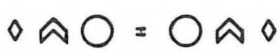
×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	4	6
3	0	3	6	9



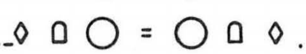


(If the class invented their own symbols, use the resulting tables in place of the last two, or compare all three pairs of tables.)

X-Nighyun notices a symmetry about the down-left diagonal in both his tables. Because of this symmetry, he finds that, for example,



and



This is, of course, true of our familiar tables.

In X-Nighyun's addition table, all entries in the down-left (or up-right) diagonals are the same. This is also true of our familiar tables.

He notices a row and a column in each that simply repeats the symbols above or to the left of the double line. The same is true of our familiar tables. In other words, if a is any number we think of, $a+0=a$ and $a\times1=a$. X-Nighyun would say that, for each a,

 and 

Look for other patterns in the tables that do not depend on the way the symbols are written.

Both examples A and B include an encounter with the distributive principle for multiplication over addition and over subtraction:

$$2(d+5)=2d+10$$
$$2(k-1)=2k-2$$

Example B provides a third instance of similar outcomes:

$$(x+3)(x-3) = x^2-3^2 \quad \text{(page 106)}$$
$$(b+1)(b-1) = b^2-1^2 \quad \text{(page 107)}$$
$$(k+6)(k-6) = k^2-6^2 \quad \text{(page 108)}$$

This leads to speculation that, for any pair of numbers r and t chosen,

$$(r+t)(r-t) = r^2-t^2.$$

Design tests that may shed some light on the reliability of this speculation. Consider pairs including one or two negative numbers, one or two fractions:

$$(^-5+7)(^-5-7) = (^-5)^2-7^2 = 25-49 = {}^-24$$

Check $(2) \times (^-12) = 24$

$$\left(\tfrac{1}{2}+\tfrac{2}{3}\right)\left(\tfrac{1}{2}-\tfrac{2}{3}\right) = \left(\tfrac{1}{2}\right)^2-\left(\tfrac{2}{3}\right)^2 = \tfrac{1}{4}-\tfrac{4}{9} = \tfrac{^-7}{36}$$

Check $\left(\frac{7}{6}\right)\times\left(\frac{^-1}{6}\right) = \frac{^-7}{36}$

This idea would be very useful as a shortcut to many computations—if it is reliable. We could find the product of 43 by 37 in our heads:

$$(40+3)(40-3) = 1600-9 = 1591$$

This is the kind of development that helps explain the part "rules" can play in algebra. If we can remember the pattern:

$$(a+b)(a-b) = a^2-b^2$$

then we have a shortcut available whenever an appropriate situation arises.

"We've noticed that, too. We say that, for any a and b,

$$a \times b = 0 \text{ if and only if } a = 0 \text{ or } b = 0.$$

"Nice way to put it! . . . and that's all the homework I did.

"Wait! I have a question. Can you only play your games with a pair of symbols? Seems to me that would get boring much too soon."

"Good question!

"Write down some of your symbols on separate pieces of paper. Label one a, another b, and another c.

"Find the sum of a and b; then find the sum of your result and your c selection."

"Done."

"Start over. Find the sum of b and c; then find the sum of a and your sum of b and c.

"Done! Say, is it an accident that I got the same final result in both cases?"

"Do you want me to tell you?"

"No. Give me a moment or two."

A moment or two later, X-Nighyun was back on the phone.

"It works every time! I tried using the same selections on the multiplying chart, and got the same results no matter which pair I selected first."

"We refer to this idea in this way: For every a, b, and c,

$$(a + b) + c = a + (b + c) \quad \text{and}$$

$$(a \times b) \times c = a \times (b \times c).$$

"We have fancy names for this idea, too: the associative principle for addition and the associative principle for multiplication."

"I'm amazed. All of this works out and you don't even know what my symbols look like."

A.

	I	II	III	IV
a	11	$\frac{2}{3}$	$^{-}4$	$^{-}11$
$2a$	22	$1\frac{1}{3}$	$^{-}8$	$^{-}22$
$2a + 8$	30	$9\frac{1}{3}$	0	$^{-}14$
$\frac{2a+8}{3}$	10	$3\frac{1}{9}$	0	$-4\frac{2}{3}$

B.

	I	II	III	IV
t	$5\frac{1}{2}$	8	.7	$^{-}2$
$t + 7$	$12\frac{1}{2}$	15	7.7	5
$2(t+7)$	25	30	15.4	10
$\frac{2(t+7)}{5}$	5	6	3.08	2

C.

	I	II	III	IV
x	4	9	$^{-}3$	$\frac{1}{2}$
$2x$	8	18	$^{-}6$	1
$-2x$	$^{-}8$	$^{-}18$	6	$^{-}1$
$15 - 2x$	7	-3	21	14

D.

	I	II	III	IV
y	10	$^{-}2$	0	$\frac{5}{6}$
$-3y$	$^{-}30$	6	0	$^{-}2\frac{1}{2}$
$-3y + 12$	$^{-}18$	18	12	$9\frac{1}{2}$
$\frac{-3y+12}{2}$	-9	9	6	$4\frac{3}{4}$

Tasks: *Students continue reading and studying the story of X-Nighyun. They also complete* What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: The top row and left-hand column of entries in the multiplication table contain nothing but zeros. That is, for each a, $a \times 0 = 0 = 0 \times a$. And we notice that zero will occur nowhere else in our table, no matter how far it is extended. Thus we know that, for each a and b, if $a \times b = 0$ then $a = 0$ or $b = 0$ (or both).

Thus far, we have limited addition and multiplication to cases in which we considered a pair of numbers. What could we mean by "$a+b+c$"? We consider two of the numbers and addition—which leads to a sum; then we add the sum and the remaining number. But which pair should we consider first? There are three basically different approaches we indicate by the use of parentheses:

$$(a + b) + c$$

$$a + (b + c)$$

$$(a + c) + b$$

Select any three numbers—such as 2, 8, and 5—and we find:

$$(2 + 8) + 5 = 10 + 5 = 15$$

$$2 + (8 + 5) = 2 + 13 = 15$$

$$(2 + 5) + 8 = 7 + 8 = 15$$

Try the same kind of experiment with the tables X-Nighyun might have designed. Then consider using the multiplication table:

$$(2 \times 8) \times 5 = 16 \times 5 = 80$$

$$2 \times (8 \times 5) = 2 \times 40 = 80$$

$$(2 \times 5) \times 8 = 10 \times 8 = 80$$

Will X-Nighyun find the same results when he uses his multiplication table?

Further experiments with four numbers, five numbers, etc., lead us to the conclusion that when addition is the operation, it doesn't make any difference in which order or with which groupings you add. Further, when multiplication is the operation, it still doesn't make a difference. These results are consequences of the commutative and associative principles for addition and for multiplication.

Those unfortunate people who are helpless because they "forgot the rules" when they encounter the following:

$$\frac{2(t + 7)}{5} = 6$$

$$t = \_\_\_\_$$

simply misunderstand what algebra is all about. Let's consider this example without resorting to any rules at all.

Clearly, $2(t+7)$ is some number. When that number is divided by 5, the result is 6. So, $2(t+7)$ must be 30, because 30 is the only number whose quotient by 5 is 6. Further, $t+7$ is some number. Twice that number is 30. Thus, $t+7$ must be 15. So, t must be 8.

$$\text{(if)} \quad \frac{2(t + 7)}{5} = 6$$

$$\text{(then)} \quad t = 8$$

The examples on pupil page 111 are designed to help students understand and develop facility with using this "thinking it through" procedure. This kind of background insures against the tyranny that results from dependence on rules. When one forgets a rule, his confidence need not be shaken. He may have to take a little longer route but, most important, he can still get there.

A Surprise for X-Nighyun

"So far, we have discussed ideas that apply as well to one chart as to another — except for the row and column of zeros in the multiplication table.

"Now I'm going to show you that the charts can be used together. Again, select some symbols and label them *a, b,* and *c*.

"First, add *a* and *b*. Multiply this sum by *c*. Note the product.

"Second, multiply *a* by *c*, then multiply *b* by *c*, and add the products.

"Surprised?"

"It can't be an accident that I got the same result in both examples! Let me change my selections . . . it works again. My guess is that you've got a fancy name for that one."

"You're right. We refer to the idea this way: for every *a*, *b*, and *c*,

$$(a+b)\times c=(a\times c)+(b\times c)$$

and we call it the distributive principle for multiplication over addition."

"The game is getting more interesting. That was a real surprise."

At this point you explain that the tables can be used in reverse as a "subtraction table" and a "division table."

"Pick a pair of symbols, and label with an *a* the one that comes further along in your string. Label the other with a *b*. Now, look for *a* in the blocks of the addition chart. You'll find a whole diagonal full of *a*'s. Pick the *a*-block that's in the column headed by a *b*. This block is in the row which you can label with a *c*. We record this as:

$$a-b=c$$

A.	I	II	III	IV
a	6	1	$\frac{1}{2}$	.3
$2a$	12	2	1	.6
$a+7$	13	8	$7\frac{1}{2}$	7.3
$2a+14$ or $2(a+7)$	26	16	15	14.6
a^2	36	1	$\frac{1}{4}$	.09
$3a^2$	108	3	$\frac{3}{4}$	.27
a^2-a or $a(a-1)$	30	0	$\frac{-1}{4}$	-.21
$25-a$	19	24	$24\frac{1}{2}$	24.7
$\frac{25-a}{2}$	$9\frac{1}{2}$	12	$12\frac{1}{4}$	12.35

B.	I	II	III	IV
y	5	9	1	1.2
y^2	25	81	1	1.44
y^2+y or $y(y+1)$	30	90	2	2.64
$y-10$	-5	-1	-9	-8.8
$5y^2$	125	405	5	7.2
$\frac{y^2}{2}$	$12\frac{1}{2}$	$40\frac{1}{2}$	$\frac{1}{2}$	.72
$\frac{5y^2}{2}$	$62\frac{1}{2}$	$202\frac{1}{2}$	$2\frac{1}{2}$	3.6
y^2+2y+1 or $y(y+2)+1$	36	100	4	4.84
$2y^2-y+5$ or $y(2y-1)+5$	50	158	6	6.68

Tasks: *Students continue reading and studying the story of X-Nighyun. They also complete* What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

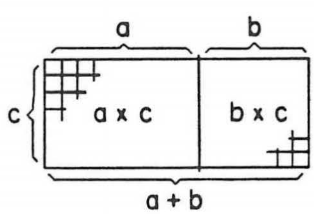
The Lesson: Noticing the relationships in the addition and multiplication tables does not prepare one to expect a close relationship between the tables—the notion of the distributivity of the multiplication operation over the addition operation. When the second factor is the multiplier, we notice the following pattern:

$$(a+b)\times c = a\times c + b\times c$$

When the first factor is the multiplier, we have:

$$c\times(a+b) = c\times a + c\times b$$

A sketch helps give insight into this principle:



If the letters hold places for numbers of units, we can find the total number of square units by seeing the problem in two different ways:

1. as c rows of $(a+b)$ units or $(a+b)$ columns of c units
2. as c rows of a units plus c rows of b units—or a columns of c units plus b columns of c units

Our tables thus far contain only symbols for positive whole numbers and zero. As we consider all numbers we know about, will the relationship hold? Are these statements true?

$$\left(\frac{1}{3}+\frac{2}{5}\right)\times\frac{1}{2} = \frac{1}{3}\times\frac{1}{2}+\frac{2}{5}\times\frac{1}{2}$$

$$({}^-4 + 9)\times{}^-3 = ({}^-4 \times {}^-3) + (9 \times {}^-3)$$

Can we expand on this idea when we use the operations of subtraction and division? Which of the following patterns produce statements true for all replacements of a, b, *and* c?

$$a(a-1) = a^2 - a \quad \text{(line 7 is A)}$$

$$y(y+1) = y^2 + y \quad \text{(line 3 is B)}$$

$$y(y+2)+1 = y^2 + 2y + 1 \quad \text{(line 8 is B)}$$

$$y(2y-1)+5 = 2y^2 - y + 5 \quad \text{(line 9 is B)}$$

All of these produce true statements only.

As students begin to write down the rule-indicators in the examples on pupil page 112, there will be a variety of appropriate designations. At the fourth line in example *A*, the rule might be expressed in words as "double a and add 14" or as "add 7 to a and double the result." In shorthand:

$$2a+14 \text{ or: } 2\times(a+7)$$

The distributive idea again, only slightly disguised:

$$2\times(a+7) = 2\times a + 2\times 7 = 2a+14$$

There are several other similar situations:

$$(a+b)\times c = a\times c + b\times c$$

$$(a-b)\times c = a\times c - b\times c$$

$$(a+b)\div c = a\div c + b\div c$$

$$(a-b)\div c = a\div c - b\div c$$

Other examples may lead to discussions of the distributive idea involving subtraction and division.

The rule-indicator for the bottom line of example *A* may be written in many ways:

$$(25-a)\div 2 \text{ or: } 25\div 2 - a\div 2 \text{ or: } \frac{25}{2}-\frac{a}{2}$$

$$\text{or: } (25-a)\times\frac{1}{2} \text{ or: } 25\times\frac{1}{2} - a\times\frac{1}{2}$$

Allow the greatest freedom possible in writing rule-indicators. The second line of example *A* calls for doubling:

$$2a \text{ or: } a+a \text{ or: } 2\times a \text{ or: } a\times 2\text{, etc.}$$

The only requirements are that the expression communicate a way to accomplish the desired results correctly and that it apply to all four instances.

We see that the a selection cannot come in X-Nighyun's string of symbols before his other (b) selection. And we agree with him that this requirement puts a serious limit on "sub-action."

We explain division in a similar fashion and again agree with him that there is a serious limit on "division."

$$a \div b = c$$

"And what's more," we point out, "we have to rule out dividing by 0. We shall discuss this later."

"So, whenever you make selections and add or multiply, you can then start with the sum or product and use either of the initial selections and subtract or divide. I'll put it this way: For every a, b, and c,

if $a + b = c$ then $c - b = a$ and $c - a = b$;

if $a \times b = c$ then $c \div b = a$ and $c \div a = b$."

"Very good . . . unless a or b is zero. In that case, you must modify the second part of your statement."

From Symbols to Numbers

"Let's talk about numbers."

"Not symbols?"

"No, numbers!"

"What are numbers?"

You knew this would happen; so you're prepared.

"Do you have hands?"

"Yes, I am using a hand to guide my space ship, another hand to hold the sandwich I'm eating, another hand to hold the book I'm reading, and my other hand is in my pocket. . . . Sure, I have four hands."

"That's four hands and four is a number."

"Do you mean that numbers are hands and hands are numbers?"

"No! No! No! No! . . . You see, I said 'no' four times." You tap the telephone with a pencil — tap, tap, tap, tap. "That's four noises. Boom! Boom! Boom! Boom! That's four booms . . . a boom for each of your hands

A.	I	II	III	IV
m	6	-1	0	9
$3m$	18	$^-3$	0	27
m^2	36	1	0	81
$m+17$	23	16	17	26

B.	I	II	III	IV
x	7	.7	$\frac{1}{3}$	4
$100-x$	93	99.3	$99\frac{2}{3}$	96
$50-2x$	36	48.6	$49\frac{1}{3}$	42
x^2-x OR $x(x-1)$	42	$^-.21$	$\frac{-2}{9}$	12

C.	I	II	III	IV
n	100	17	$\frac{1}{2}$	5
$2n-1$	199	33	0	9
$3n+1$	301	52	$2\frac{1}{2}$	16
$2n^2$	20,000	578	$\frac{1}{2}$	50

D.	I	II	III	IV
b	5	8	-2	$1\frac{1}{2}$
b^2-1 OR $(b+1)(b-1)$	24	63	3	$1\frac{1}{4}$
$10b$	50	80	-20	15
$\frac{b}{5}$	1	$1\frac{3}{5}$	$^-\frac{2}{5}$	$\frac{3}{10}$ OR .3

Tasks: Students continue reading and studying the story of X-Nighyun. They also complete What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: After introducing the notion of subtraction and division and pointing out that these "plays" are not always possible when you are limited to the tables that have been developed so far, the discussion moves from symbols to numbers.

Thus far, X-Nighyun has been playing a game in which all he did was follow a set of rules. True, he noticed and discussed patterns in the tables he made. But these were only patterns in tables of symbols, or marks on a piece of paper.

Perhaps it is most unfortunate that arithmetic can be played so easily in this way—as a manipulation of symbols (or numerals) that have little meaning beyond the rules for pushing them around.

We hope the foregoing fantasy and the developments to come will help to give students additional insight into the difference between symbols and numbers. Their appreciation of the distinction should not be judged by their verbalizations, but rather by their ability to use the game of arithmetic (as it has been outlined thus far to X-Nighyun) in coping with the real world of objects.

But how can one explain the idea of number over the space phone to a visitor from outer space, even if he is a fast-learning Martian, without pictures or manipulative materials? This is a question the class might well discuss before reading the approach outlined on pupil page 113 and following pages. There will probably be great similarity between the plans suggested by students and that outlined in the text.

At some point in the discussion it may be well to point out a fact that is often overlooked.

Consider example A, particularly lines 1 and 3. Most students will notice a rule connecting these entries and write it out as follows:

m	6	$^-1$	0	9
m^2	36	1	0	81

Of course, $m \times m$ is another way of indicating m^2. Or, one might hide this relationship in some complicated expressions, such as

$$(m+1)(m-1)+1$$

$$m(m+1)-m$$

$$(2m)^2 \div 4 \text{ ......etc.}$$

However, there is another rule-indicator that is sometimes not mentioned:

$(^-1m)^2$

m	6	$^-1$	0	9
$(^-1m)^2$				

$$(^-1\cdot 6)\cdot(^-1\cdot 6) = {^-6} \times {^-6} = 36$$

$$(^-1\cdot {^-1}) \times (^-1\cdot {^-1}) = {^+1} \times {^+1} = {^+1}$$

$$(^-1\cdot 0) \times (^-1\cdot 0) = 0 \times 0 = 0$$

$$(^-1\cdot 9) \times (^-1\cdot 9) = {^-9} \times {^-9} = 81$$

Thus, if an appropriate rule-indicator for example C, line 4, is $2n^2$ then $2(^-1n)^2$ is equally appropriate. Notice also that for example B, line 4, x^2-x and $^-1x(1-x)$ are equivalent.

. . . No, No, No, No . . . tap, tap, tap, tap, Boom, Boom, Boom, Boom . . . hand, hand, hand, hand . . . four hands, four booms, four taps, four no's. . . ."

"Aha! . . . and you gave me four examples. If there's enough for each hand and no more, that's four. Now I can tell you that I have four fingers on each hand."

"Another number is one . . . you have one hand in your pocket; Earth has one moon; our planet revolves around a single sun; you are one Martian."

"Herchoo," X-Nighyun explodes over the phone, "and I suppose that is one sneeze."

"Yes!"

"One 'yes.' "

This could go on for a long time, so you get on with your job.

"Tie a string around one finger of the hand you have in your pocket."

"One string around one finger."

"Now tie another string around another finger on that hand."

"Completed!"

"Two is another number . . . two pieces of string, two fingers with string on them . . . Boom, Boom. . . ."

"Say, then on that same hand I now have two fingers that do not have strings on them. Right?"

"Right. Three is another number. You have three hands that don't have any strings on any of the fingers. . . . Hand, hand, hand."

A.	I	II	III	IV
a	5	$\frac{1}{2}$	1.5	-3
b	8	$\frac{1}{4}$	8	10
$a + b$	13	$\frac{3}{4}$	9.5	7
$a + 2b$	21	1	17.5	17

B.	I	II	III	IV
w	23	.7	4	-4
x	9	.7	-4	4
$w + x + 5$	37	6.4	5	5
$w - x$	14	0	8	-8

C.	I	II	III	IV
c	3	76	7	$\frac{1}{2}$ OR .5
d	.7	24	12	$\frac{1}{2}$ OR .5
$c + d$	3.7	100	19	1
$c - d$	2.3	52	-5	0

D.	I	II	III	IV
y	3	9 OR 8	$\frac{1}{5}$	17 OR .3
z	7	8 9		.3 17
$y \times z$	21	72	$1\frac{1}{5}$	5.1
$y + z$	10	17	$6\frac{1}{5}$	17.3

Tasks: Students continue reading and studying the story of X-Nighyun. They also complete What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: The *What Are My Rules?* games (A, B, C, and D) change character. Those on preceding pages were versions of *I'm Thinking of a Number.* Now we consider versions of *I'm Thinking of a Pair of Numbers.*

If I bought a skirt and a dress, the total cost would be \$13. The skirt and 2 of the dresses would cost \$21. What is the price of the skirt? What is the price of the dress?

$$\text{If } \begin{cases} a + b = 13 \\ a + 2b = 21 \end{cases} \ldots$$

The extra dress added \$8 to the total purchase. If the dresses are \$8 each and a skirt and dress together are \$13 then a skirt must cost \$5:

$$\text{then } \begin{cases} a = 5 \\ b = 8 \end{cases}$$

Together (example C, III) Charlie and Dick had exactly a dollar. Charlie had 52¢ more than Dick. How much did each boy have?

$$\begin{cases} c + d = 100 \\ c = d + 52 \end{cases}$$

Suppose that we tackle this by trial and error. We'll start with two amounts, one 52 greater than the other—such as 62 and 10. Together that's only 72, so we need to use bigger numbers:

Charlie	Dick	Total	
62	10	72	try larger numbers
82	30	112	too large
75	23	98	⁻1 more once
76	24	100	

$c = 76$
$d = 24$

If one notices that the sum is 28 too small, he can add 14 to each and get the desired result directly.

$$\begin{array}{r} 62+10=\ 72 \\ 14+14=\ 28 \\ \hline 76+24=100 \end{array}$$

Urge students to make up a situation suggested by each of the examples.

It's 6.4 miles to Canton (example B, II). Part of the road is gravel, part is dirt, and 5 miles are paved. The length of the gravel stretch is the same as that of the dirt stretch. How many miles are gravel (w), and how many are dirt (x)?

$$w + x + 5 = 6.4$$
$$w - x = 0$$

Clearly 1.4 miles are not paved, and half of 1.4 is .7; so,

$$w = .7$$
$$\text{and } x = .7.$$

The area of a room is 72 square feet (example D, III). The length (y feet) plus the width (z feet) is 17 feet. How long is the room and how wide is it?

$$y \times z = 72$$
$$y + z = 17$$

One dimension is 8 feet and the other is 9 feet. In terms of the situation described, we should say the length is 9 feet and the width is 8 feet, since a room is usually longer than it is wide.

You may wish to write out some of these examples in the form in which they will usually be presented later. In the beginning include the "if" and "then"—and then agree the idea is implied even when the words are omitted.

$$w + x + 5 = 5$$
$$w - x = 8$$
$$w = \underline{4}$$
$$x = \underline{4}$$

And later:

$$c + d = 1$$
$$c - d = 0$$
$$c = \underline{\quad}$$
$$d = \underline{\quad}$$

Students should develop the practice of checking their answers by substituting in the "if" equations.

"Three, four, one, two — that's four numbers. Is that all the numbers there are . . . as many numbers as I have fingers on one hand?"

"No. There is a number for every symbol you invented, no matter how many you care to write. And that's only the beginning.

"Suppose that we use the word 'element' to refer to whatever it is we are talking about — taps, fingers, booms, hands, strings, labels, points, etc.

"We say that all collections or sets of elements that can be put in 1-to-1 correspondence with each other have the same number of elements, and we assign some symbol — mark or word — to indicate that number. Sometimes we call this word or mark a 'numeral.' "

"Aha!" X-Nighyun broke in. "What you started me off with was a game that you can play with numerals and tables. And now, you're going to teach me the game of numbers. Is that it?"

"You could put it that way."

"And I'll bet when I write out a play in the game of numerals, it will also make sense if I interpret it as a play in the game of numbers and things."

"You could put it that way."

"When two games are so much alike, you must have a lot of trouble telling one from the other."

"That's right."

Had you realized that you can play both these games — and an observer would find it most difficult to tell which game you were playing if all he saw were the results you wrote down?

A.	I	II	III	IV
m	-4	0	1	.25
n	4	6	1	.5
$m + n$	0	6	2	.75
$m - 2n$	-12	-12	-1	-.75

B.	I	II	III	IV
r	19	0	37	$5\frac{2}{3}$
s	4	10	0	$1\frac{1}{3}$
$(r - s) + 3$	18	-7	40	$7\frac{1}{3}$
$r + (s - 3)$	20	7	34	4

C.	I	II	III	IV
p	49	.5	$\frac{1}{2}$	0
q	100	.05	$\frac{1}{3}$	-7
$2p + q$	198	1.05	$1\frac{1}{3}$	-7
$p - q$	-51	.45	$\frac{1}{6}$	7

D.	I	II	III	IV	
a	9	$\frac{1}{2}$	.3	-4	-7
k	7	10	3	7	4
ak	63	5	.9	-28	
$k - a$	-2	$9\frac{1}{2}$	2.7	11	

Tasks: Students continue reading and studying the story of X-Nighyun. They also complete What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Unless a student has lost his way in arithmetic, the distinction between numeral and number serves no useful function. Some years ago the question "What is a fraction?" provoked a great deal of debate. Arguments were advanced that a fraction is "a rate pair," "a number," "a numeral," "a pair of numerals and a bar," "an ordered pair of numerals," "an ordered pair of numbers," etc. Few teachers found the discussion useful in helping them communicate with students.

When second-grade children write

$$\begin{array}{r} 19 \\ +\ 3 \\ \hline 112 \end{array}$$

to find the sum of 19 and 3, they have been subjected to the tyranny of rules—rules for manipulating numerals. When sixth-graders write:

$$\begin{array}{r} .7 \\ \times .3 \\ \hline 2.1 \end{array} \qquad \text{or: } \frac{3}{4} + \frac{1}{2} = \frac{3}{8}$$

they probably are manipulating symbols—with bad memories. Many students who get correct results may be playing arithmetic the same way, but with better memories. If a pupil has been taught to rely on his memory alone, he will probably play the game of arithmetic without understanding.

What we write down as we "do arithmetic" does not reveal whether we are thinking our way along or are simply relying on our memories—unless we make mistakes! It is from those mistakes that a teacher gets clues to whether pupils are manipulating numerals or thinking about numbers.

If we know the sum and difference of two numbers, have we enough information to find those numbers without fail?

(1) $a + b$	7	29	117	26	2	$\frac{5}{6}$		7.13
(2) $a - b$	3	21	83	0	$^-12$	$\frac{1}{6}$		7.07
(3) a								
(4) b								

As we test example after example, we become convinced that knowing the sum and difference is sufficient.

And some may suggest shortcuts for finding those numbers—methods to avoid trial and error. It would be helpful to notice that the entry in the third line (a) is half the sum of the entries in the top two lines; and that the entry in the fourth line is half the difference when the entry in the second line is subtracted from the entry in the first line:

$$\frac{7+3}{2} = 5 \qquad \frac{7-3}{2} = 2$$

$$\frac{29+21}{2} = 25 \qquad \frac{29-21}{2} = 4$$

$$\frac{2+(^-12)}{2} = {}^-5 \qquad \frac{2-(^-12)}{2} = 7$$

This may lead one to write the following patterns:

$$\frac{(a+b)+(a-b)}{2} = a$$

$$\frac{(a+b)-(a-b)}{2} = b$$

Let's plunge a little and manipulate these patterns:

$$\frac{(a+b)+(a-b)}{2} = \frac{(a+a)+(b-b)}{2} =$$

$$\frac{2a+0}{2} = \frac{2a}{2} = a$$

$$\frac{(a+b)-(a-b)}{2} = \frac{(a-a)+(b+b)}{2} =$$

$$\frac{0+2b}{2} = \frac{2b}{2} = b$$

Some may follow this kind of argument, convinced that knowing the sum and difference of two numbers is enough information to assure their finding those numbers. Others may not yet feel the power of such algebraic arguments.

Putting a Number Line to Use

Thus far, your discussions over the space phone have depended heavily on language. It would be most helpful to have something concrete to refer to — something other than hands and strings and booms — something that helps visualize ideas.

Let's try a number line.

"We shall begin today by drawing a straight line, and we'll think of what we draw as a part of a number line."

"What is a line, and what do you mean by straight?"

Harder than you thought, but not impossible. How would you explain?

(We assume that you have succeeded.)

"Mark a point on that line and label it with a 0. Mark another point on the line to the right of the first point and label it with a 1."

"Not so fast! I've got the first part — marking a point and labeling it 0. But which way is to the right and how far away shall I put the second mark?"

How can you explain "right" and "left" over the space phone? How can you describe a particular distance?

We suggest you avoid both these questions.

"Which way is to the right? Just take your pick, and remember which way you call 'to the right' and select any distance to the right you wish."

"Done."

"Put marks to the right and left of 0 so that each is the same distance from one of its neighbors as it is from the other.

"What do you mean by 'same distance'?"

If you only had a chalkboard and television rather than a space phone!

A.	I	II	III	IV
r	4	$\frac{1}{4}$	-2	1.7
s	6	3	-3	60
$2r+s$	14	$3\frac{1}{2}$	-7	63.4
$2(r+s)$	20	$6\frac{1}{2}$	-10	123.4
$2r+2s$	20	$6\frac{1}{2}$	-10	123.4
$r-2s$	-8	$-5\frac{3}{4}$	4	-118.3
$(r-2)s$	12	$-5\frac{1}{4}$	12	-18
$3s-r$	14	$8\frac{3}{4}$	-7	178.3
$3(s-r)$	6	$8\frac{1}{4}$	-3	174.9

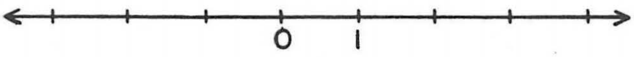
MANY POSSIBLE ANSWERS

B.	I	II	III	IV
b	6	.8	$\frac{1}{3}$	$\frac{1}{2}$
g	14	.3	5	$\frac{1}{2}$
$2bg$	168	.48	$3\frac{1}{3}$	$\frac{1}{2}$
$2b+g$	26	1.9	$5\frac{2}{3}$	$1\frac{1}{2}$
$2+bg$	86	2.24	$3\frac{2}{3}$	$2\frac{1}{4}$
$(2+b)g$	112	.84	$11\frac{2}{3}$	$1\frac{1}{4}$
$2b-g$	-2	1.3	$-4\frac{1}{3}$	$\frac{1}{2}$
$2(b-g)$	-16	1	$-9\frac{1}{3}$	0
$2b-2g$	-16	1	$-9\frac{1}{3}$	0

116-119 Putting a number line to use; What Are My Rules?

Here's one attempt at an answer to X-Nighyun's question: Place marks so that every chunk of line that would just include three marks on it would look exactly like every other such chunk. Would that do the job?

(We assume that you succeeded.)



"Our next job is to label all the points you marked to the right of the point labeled 0."

"Don't tell me. I've already invented labels I can use. But what about the points the other way from 0?"

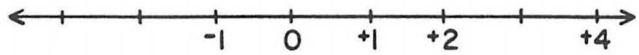
"We would say, 'to the left of zero.' We would use the same labels going in that direction."

"Then you must say 'two to the right' and 'two to the left,' etc."

"Yes; we do that by adding a mark to indicate direction. If we use a raised cross mark to indicate 'to the right' then we use a raised dash to indicate 'to the left' — and we would read these labels 'positive two' and 'negative two.' But don't ask me what these marks look like — you can invent your own."

"Done."

So, X-Nighyun must have a drawing that has the same characteristics as the following:



"Now I'm ready to assign you some homework."

"Good!"

"We have labeled only a few points on the line. Our task is to find labels for all points on the line."

"But there are an unlimited number."

"I didn't say it was easy."

A.

	I	II	III	IV
a	1	2	$^-1$	$\frac{1}{2}$
b	2	7	$^-2$	$\frac{1}{4}$
c	3	5	3	$\frac{1}{8}$
$a + b$	3	9	-3	$\frac{3}{4}$
$b + c$	5	12	1	$\frac{3}{8}$
$a + c$	4	7	2	$\frac{5}{8}$
$2a + b$	4	11	$^-4$	$1\frac{1}{4}$
$b + 2c$	8	17	4	$\frac{1}{2}$
$3a + 4c$	15	26	9	2

B.

	I	II	III	IV
r	4	$\frac{1}{2}$	2.5	$^-2$
s	$^-1$	$\frac{1}{3}$	1	$\frac{1}{2}$
t	5	$\frac{1}{4}$	.08	7
$r + s$	3	$\frac{5}{6}$	3.5	$^-1\frac{1}{2}$
$s + t$	4	$\frac{7}{12}$	1.08	$7\frac{1}{2}$
$r - t$	$^-1$	$\frac{1}{4}$	2.42	-9
$2(r - t)$	$^-2$	$\frac{1}{2}$	4.84	$^-18$
$3s + 3t$	12	$1\frac{3}{4}$	3.24	$^-22\frac{1}{2}$
$\frac{r + s}{2}$	$1\frac{1}{2}$	$\frac{5}{12}$	1.75	$^-\frac{3}{4}$

Tasks: Students continue reading and studying the story of X-Nighyun. They also complete What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Ask students to write true statements involving positive and negative numbers, limiting the operations to addition and subtraction. Then write the same statements changing each positive number to the corresponding negative number and vice versa. (But do not change the operation signs.)

$$^{+}2 + {}^{+}3 = {}^{+}5 \rightarrow {}^{-}2 + {}^{-}3 = {}^{-}5$$
$$^{+}7 - {}^{+}4 = {}^{+}3 \rightarrow {}^{-}7 - {}^{-}4 = {}^{-}3$$
$$^{+}4 - {}^{+}9 = {}^{-}5 \rightarrow {}^{-}4 - {}^{-}9 = {}^{+}5$$
$$^{+}4 + {}^{-}7 - {}^{+}8 = {}^{-}11 \rightarrow {}^{-}4 + {}^{+}7 - {}^{-}8 = {}^{+}11$$

Are the second statements true in each of these cases? (Yes.) Will this surprising result hold for all such statements? Consider a complicated one:

$$({}^{-}3 + {}^{+}9) - ({}^{+}13 - {}^{-}7) = \underline{{}^{-}14}$$
$$({}^{+}3 + {}^{-}9) - ({}^{-}13 - {}^{+}7) = \underline{{}^{+}14}$$

Let students test this idea. It will hold in every case. Does this shed a little light on the lack of concern about "to the right" and "to the left"?

Broaden the experiment, considering the operation of multiplication:

$$^{+}3 \times {}^{+}5 = \underline{{}^{+}15} \quad \text{and} \quad {}^{-}3 \times {}^{-}5 = \underline{{}^{+}15}$$
$$^{+}2 \times {}^{-}9 = \underline{{}^{-}18} \quad \text{and} \quad {}^{-}2 \times {}^{+}9 = \underline{{}^{-}18}$$
$$^{-}7 \times {}^{+}8 = \underline{{}^{-}56} \quad \text{and} \quad {}^{+}7 \times {}^{-}8 = \underline{{}^{-}56}$$
$$^{-}6 \times {}^{-}10 = \underline{{}^{+}60} \quad \text{and} \quad {}^{+}6 \times {}^{+}10 = \underline{{}^{+}60}$$

A very different result! The product is not affected at all. Will it hold for three factors?

$$^{+}2 \times {}^{-}3 \times {}^{-}4 = {}^{+}24 \quad \text{and} \quad {}^{-}2 \times {}^{+}3 \times {}^{+}4 = \underline{{}^{-}24}$$
$$^{+}3 \times {}^{+}2 \times {}^{+}5 = {}^{+}30 \quad \text{and} \quad {}^{-}3 \times {}^{-}2 \times {}^{-}5 = \underline{{}^{-}30}$$
$$^{-}4 \times {}^{-}5 \times {}^{-}3 = {}^{-}60 \quad \text{and} \quad {}^{+}4 \times {}^{+}5 \times {}^{+}3 = \underline{{}^{+}60}$$
$$^{-}2 \times {}^{+}7 \times {}^{+}3 = {}^{-}42 \quad \text{and} \quad {}^{+}2 \times {}^{-}7 \times {}^{-}3 = \underline{{}^{+}42}$$

Quite the opposite of results with two factors. Can you guess what happens with four factors?

"I'm thinking of three numbers: a, b, and c. They can be paired in essentially three different ways: a and b, b and c, a and c. If I tell you the sum of each pair, can you tell me the numbers I've thought of?" This extension of example A will help in investigating the question:

(1)	$a + b$	3	10	30	$1\frac{1}{2}$	26	2	$^{-}13$
(2)	$b + c$	6	22	41	$\frac{5}{6}$	17	5	4
(3)	$a + c$	7	14	45	$1\frac{1}{3}$	9	9	7
(4)	a	2	1	17	1	9	3	$^{-}5$
(5)	b	1	9	13	$\frac{1}{2}$	17	$^{-}1$	$^{-}8$
(6)	c	5	13	28	$\frac{1}{3}$	0	6	12

In each instance, trial and error will lead to a single combination of numbers a, b, and c that meet the conditions in lines 1, 2, and 3.

Some may begin looking for shortcuts. Consider lines 2 and 3:

$$b + c = 6$$
$$a + c = 7$$

By inspection, a must be larger than b—in fact, it must be 1 more; thus $a - b = 1$. Now compare this with line 1:

$$a + b = 3$$
$$a - b = 1$$

which is an example of what we explored on pupil page 115. We know that $a = \frac{1}{2}(3+1) = 2$ and $b = \frac{1}{2}(3-1) = 1$.

If students are ready for it, consider "subtracting" the equation in line 3 from that in line 2:

$$\begin{array}{r} b + c = 6 \\ -(a + c = 7) \\ \hline (b - a) + (c - c) = 6 - 7 \\ (b - a) + 0 \quad = {}^{-}1 \\ b - a = {}^{-}1 \end{array}$$

Investigation of shortcuts keeps leading deeper and deeper into algebra.

X-Nighyun Does His Homework

"I've got a system. I've got a system to name all the points on the line. It goes like this.

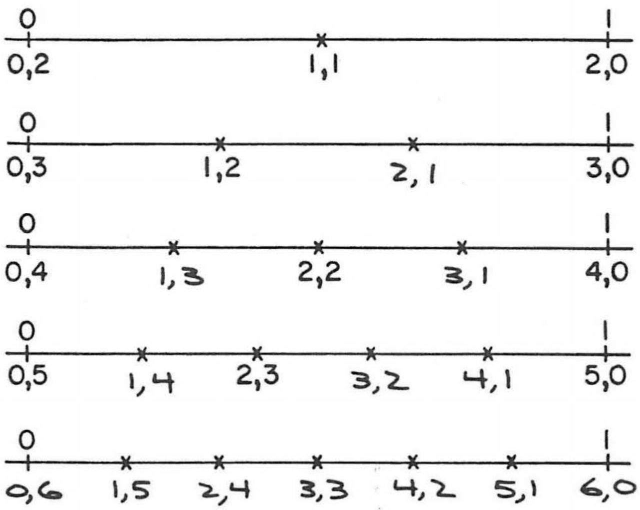
"On the piece of line from 0 to 1, you mark a point that divides the piece into two pieces of the same length. Or, you can mark two points on the piece of line from 0 to 1 to give you three pieces of the same length. Or, you can mark three points, etc.

"Each time, you can label the marked points by noting the number of pieces between 0 and the point and noting the number of parts between the point and 1.

"That will do it for a lot of points between 0 and 1."

"Slow down. Let me work out some examples."

Do you see X-Nighyun's system from the following examples? *(Please fill in the missing labels.)*



"I can divide the distance between 0 and 1 into as many parts of the same length as I like, and then count the parts on each side of my point.

A.	I	II	III	IV
x	5	7	0	3
y	1	4	8	-1
z	2	3	0	2
$x + y + 5$	11	16	13	7
$y + z - 2$	1	5	6	-1
$x + z$	7	10	0	5
$3x - y - 4$	10	13	$^-12$	6
$x - 2z$	1	1	0	-1
$2y + 3z$	8	17	16	4
$x + y + z$	8	14	8	4
$2x - y + 3z$	15	19	-8	13

MANY POSSIBLE ANSWERS.

B.	I	II	III	IV
k	3	$\frac{1}{3}$	$^-3$	7
m	2	$\frac{1}{2}$	$^-2$	0
b	7	$\frac{1}{6}$	$^-7$	$\frac{1}{2}$
$k + m + b$	12	1	-12	$7\frac{1}{2}$
$k + m - b$	-2	$\frac{2}{3}$	2	$6\frac{1}{2}$
$k - m + b$	8	0	$^-8$	$7\frac{1}{2}$
$k - m - b$	$^-6$	$-\frac{1}{3}$	+6	$6\frac{1}{2}$
$-1k + m + b$	6	$+\frac{1}{3}$	$^-6$	$-6\frac{1}{2}$
$-1k - m + b$	2	$-\frac{2}{3}$	-2	$-6\frac{1}{2}$
$-1k + m - b$	$^-8$	0	8	$-7\frac{1}{2}$
$-1k - m - b$	$^-12$	$^-1$	12	$-7\frac{1}{2}$

Tasks: Students continue reading and studying the story of X-Nighyun. They also complete What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: X-Nighyun's system for labeling an unlimited number of points between 0 and 1 will at first seem strange to many. In a sense, he carries out two operations to find a point and then labels it by describing those operations:

1. First he divides the distance into a certain number of parts of equal length.
2. Then he counts the parts on each side of a point and writes down the results, such as:

$$2, 3$$

From his notation you can see directly that he divided the line between 0 and 1 into five $(2+3)$ parts. His label is assigned to the point that has 2 lengths between it and 0, and 3 lengths between it and 1.

Thus his label provides what you need to know about the way he located a particular point.

Urge the class to discuss this system. Does it seem clumsy or crude? Does it provide more than one label for each point? (Yes.) Will this lead to confusion? Is it unnecessarily complicated? Can any two different points be labeled alike? What confusion would such a development lead to? (Fortunately, each label points to only one point.) How can you tell whether a point is more or less than halfway between 0 and 1?

You may want to ask a profound question at this point: Can X-Nighyun's system label all points on the line? He says that he can divide the line as finely as he wants; and intuition will lead most of the uninitiated to answer in the affirmative.

However, there are an unlimited number of points between 0 and 1 that cannot be labeled in this way. They will turn out to be those points we cannot label as rational numbers—such as $\sqrt{2}$, π, etc. (Each teacher must make his own decision about whether to raise this question at all and, if so, how far it ought to be pursued.)

In example *B*, notice the last eight rule-indicators in the first column rearranged as follows:

(3) $k + m + b = 12$	(10) $^{-}1k - m - b = {}^{-}12$
(4) $k + m - b = {}^{-}2$	(8) $^{-}1k - m + b = 2$
(5) $k - m + b = 8$	(9) $^{-}1k + m - b = {}^{-}8$
(7) $k - m - b = {}^{-}6$	(7) $^{-}1k + m + b = 6$

This reminds us of an earlier outcome, but the conditions here seem different—the operations are not the same in the compared rule-indicators.

In order to investigate this, we need to recall two general principles about directed numbers:

1. To subtract a directed number, just add its opposite instead.
2. To find the opposite of a directed number, just multiply it by $^{-}1$.

Let's look at rule-indicator 10:

$$^{-}1k - m - b$$

By principles 1 and 2, this is equivalent to:

$$^{-}1k + {}^{-}1m + {}^{-}1b$$

Now, by the distributive principle for multiplication over addition and principle 1,

$$^{-}1k + {}^{-}1m + {}^{-}1b = {}^{-}1(k + m + b)$$
$$= \text{the opposite of } (k + m + b).$$

We see that rule-indicator 10 must produce numbers that are the opposites of those produced by rule-indicator 3.

In similar fashion, we can show that the other paired rule-indicators must always produce pairs of opposites.

In example *B*, only three items of information are given to find the three numbers—*k*, *m*, and *b*. Would *any* three of the eight rule-indicators provide sufficient information? No, because if two of the three are related in the sense we have been investigating them then one of the two adds no clues, and we need three rule-indicators to provide three "independent" clues.

"Of course, I get several labels for each point, so I put an equal sign between them and say that one is as good as another. I also get new labels for 0 and 1."

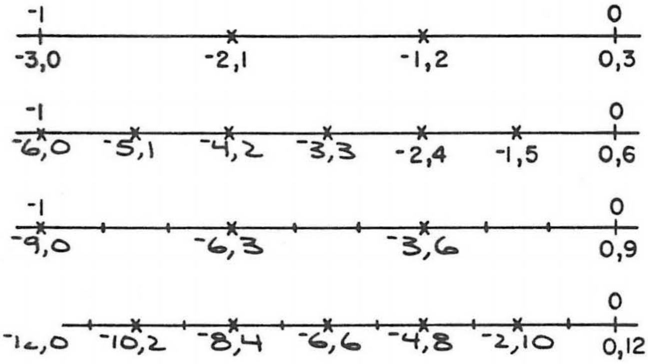
0 = 0,2 = 0,3 = 0,4 = 0,5

1 = 2,0 = 3,0 = 4,0 = 5,0

1,1 = 2,2 = 3,3

1,2 = 2,4 2,1 = 4,2

"On the left side of 0, I use the same kind of system. I note first the number of parts between 0 and the point, and then the number of parts between the point and $^{-}1$. I put a 'raised dash' before each label." *(Please label the cross marks.)*



-3,0 = -6,0 = $^{-}9,0$ = $^{-}12,0$ ETC.

-1,2 = $^{-}2,4$ = $^{-}3,6$ = $^{-}4,8$

-2,1 = $^{-}4,2$ = $^{-}6,3$ = $^{-}8,4$

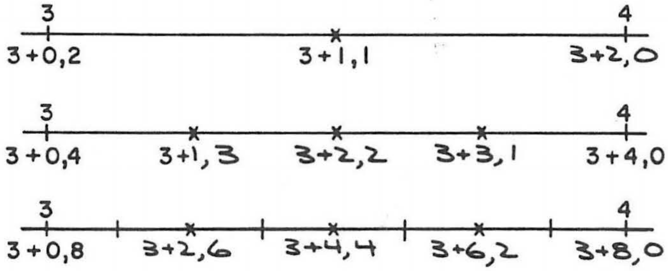
-3,3 = $^{-}6,6$ -5,1 = $^{-}10,2$

-1,5 = $^{-}2,10$ 0,3 = 0,6 = 0,9 ETC.

"Now, I am ready to consider points between 1 and 2, 2 and 3, 3 and 4, etc.

"For example, consider points that divide the piece between 3 and 4 into pieces of the same length. To label any of these points, I first write a 3, then a plus sign, and then a pair of numbers which, as before, tells the numbers of parts on each side of the point."

(Please label the cross marks.)



3 + 0,2 = 3+0,4 = 3+0,8 ETC.

3 + 1,1 = 3+2,2 = 3+4,4 ETC.

3 + 1,3 = 3+2,6 = 3+3,9 ETC.

3 + 3,1 = 3+6,2 = 3+12,4 ETC.

What do you think of X-Nighyun's system?

A.	I	II	III	IV
a	15	8	25	30
b	7	1	0	14
c	11	$^{-}1$	$^{-}25$	22
$2a+b+c$	48	16	25	96
$a+2b+c$	40	9	0	80
$a-b+2c$	30	5	-25	60

B.	I	II	III	IV
p	2	9	$^{-}2$	$\frac{1}{2}$
q	5	9	$^{-}5$	$\frac{3}{8}$
r	13	9	13	1
$2p+q+r$	22	36	4	$2\frac{3}{8}$
$p+2q-r$	-1	18	-25	$\frac{1}{4}$
$p-q+2r$	23	18	29	$2\frac{1}{8}$

Tasks: *Students continue reading and studying the story of X-Nighyun. They also complete* What Are My Rules? *tables.*

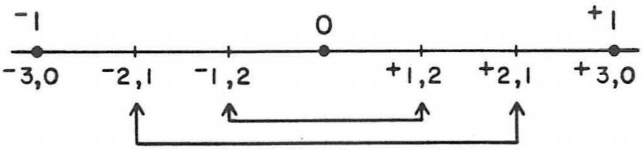
Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: X-Nighyun extends his labeling system to points between other points labeled with consecutive positive and negative integers.

He confronts us with a bit of confusion in the case of the negative integers. While the parts between the point and 0 are to the "right" of the point on the line, the number of such parts is written to the left of the separating comma. What does the class think of X-Nighyun's system at this point?

Consider these points and their labels:



Perhaps X-Nighyun's plan is not so confusing after all. Points equally distant from zero but in opposite directions from 0 are labeled alike except for the signs written to the left of the label.

The extension of the system to labeling points to the right of $^{+}1$ and the left of $^{-}1$ on the number line is straightforward.

X-Nighyun's labels for fractions or rational numbers are a method of describing how one would go about drawing a sketch or manipulating objects. He would indicate ¾ or ⅙ by writing:

$$\frac{3}{1} \text{ and: } \frac{1}{5}$$

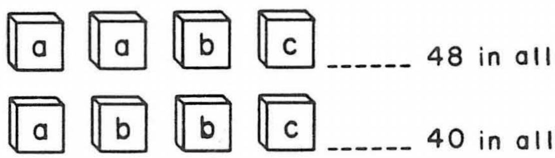
In other words, divide the sketch or object into 4 parts of equal length and point to 3 of them (leaving 1 part out); or divide it into 6 parts and point to 1 of them (leaving the other 5 out). To know the total number of parts, one need only add the two numbers indicated.

Now consider a familiar method we use for suggesting ¾ or ⅙:

In a sense, we divide a sketch into 4 parts and point to 3 of them (leaving 1 out); we divide it into 6 parts and point to 1 of them (leaving the other 5 out).

Thus, the difference is purely a difference in methods used to report the same activity, to record the same operations.

Consider a diagram to help students gain insight into case I in example *A:*



Suppose that we are talking about marbles—the same letters indicate the same number of marbles in certain boxes. How does the top group (48 marbles) differ from the second group (40)? A group of $a+b+c$ could be eliminated from each:

$$\boxed{a} = 48 - (a+b+c)$$
$$\boxed{b} = 40 - (a+b+c)$$

Is the number of marbles in box a larger or smaller than the number in box b, and by how much? Clearly,

$$a - b = 8.$$

As we consider the next line, we notice that

$$(a - b) + 2c = 30.$$
$$a - b = 8$$
$$2c = 30 - 8,$$
$$2c = 22,$$
$$c = 11.$$

(Of course, if we use boxes of marbles for problems involving negative numbers, we must be prepared to interpret a "negative marble." This is something that when combined with an ordinary marble produces no marbles!)

You Make a Request

"And now let me tell you about using these labels in your addition game and . . ."

"Wait a minute, friend. I have decided your set of symbols is just as good as mine, but I have a request. Since I'm very familiar with my symbols and you are just getting started, please use the same symbol system I do. It will be a little hard on you at first. However, the systems are basically the same."

"Fair enough."

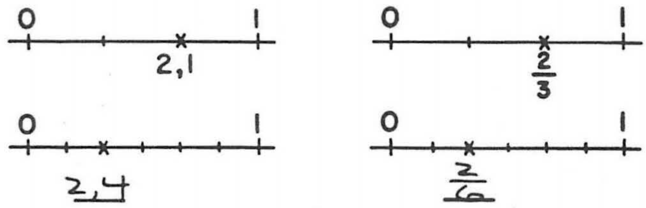
"I start out by dividing the part of the line between 0 and 1 into a certain number of parts of the same length and label each division point. But, I record what I see in a little different way.

"Beneath a short horizontal line I note the number of parts between 0 and 1. Above the line, I note the parts from 0 to the particular point."

"All right with me. I can easily change my symbols into yours. For example, my '2, 3' becomes your '5 under 2'."

"Or, 2 over 5 as we usually describe it . . . or, 'two-fifths' as we usually read it."

Here are some examples that show the problem one faces in converting from one of these systems to another:



More examples, but without sketches:

$4,3 \rightarrow \frac{4}{7}$	$0,6 \rightarrow \frac{0}{6}$	$2+1,3 \rightarrow 2\frac{1}{4}$
$2,9 \rightarrow \frac{2}{11}$	$5,0 \rightarrow \frac{5}{5}$	$5+7,1 \rightarrow 5\frac{7}{8}$
$5,13 \rightarrow \frac{5}{18}$	$7,7 \rightarrow \frac{7}{14}$	$4+4,4 \rightarrow 4\frac{4}{8}$

A.	I	II	III	IV
p	5	4	1	3
q	1	3	13	-1
r	3	2	9	-1
s	8	1	30	0
$p+q$	6	7	14	2
$r+q$	4	5	22	-2
$r+s$	11	3	39	-1
$p+s$	13	5	31	
$p+r$	8	6	10	2

B.	I	II	III	IV
g	10	2	23	1
h	7	8	18	$\frac{1}{2}$
m	5	1	30	$\frac{1}{4}$
k	1	11	9	$\frac{1}{8}$
$g-h$	3	-6	5	$\frac{1}{2}$
$m-h$	-2	-7	12	$\frac{-1}{4}$
$g+m$	15	3	53	$1\frac{1}{4}$
$h-k$	6	-3	9	$\frac{3}{8}$
$g+h+m+k$	23	22	80	$1\frac{7}{8}$

Tasks: *Students continue reading and studying the story of X-Nighyun. They also complete* What Are My Rules? *tables.*

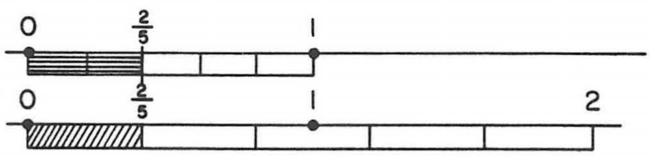
Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: When X-Nighyun agrees to adopt our symbols or labels, he says that his "2,3" becomes our "5 under 2." Why do you think he said "5 under 2" rather than "2 over 5" or "two fifths"?

One explanation is that he is thinking of the way he proceeds in his own system: first divide something into 5 parts, then point out a number of those parts.

If X-Nighyun were to advance such an argument, what could we say? Of course, we could explain that our fathers and forefathers have always read fractions in this way. That's a good reason, but X-Nighyun might be happier if we pointed out that one might read our fraction "2 over 5" as short for "take 2 objects and divide them into 5 equal parts—and we're referring to 1 of those parts."



In a sense, our instruction "divide by 5 and take 2" produces the same result as "take 2 and divide by 5" or, in short:

$$(a \div 5) \times 2 = (a \times 2) \div 5$$

and, more generally:

$$(a \div b) \times c = (a \times c) \div b$$

As X-Nighyun would test this statement, he would collect additional evidence of its reliability.

Suppose that X-Nighyun had preferred his system of labeling. Is there any strong argument to give him in support of our request?

One practical reason we might advance is that we have already worked out some simple rules by which we can carry out additions and multiplications rapidly. These are the "algorithms." Compare the following:

$$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6} \quad \text{and} \quad 1,2 \times 1,1 = 1,5$$

$$\frac{1}{3} + \frac{1}{2} = \frac{5}{6} \quad \text{and} \quad 1,2 + 1,1 = 5,1$$

Given time, could X-Nighyun work out shortcuts to adding and multiplying if he kept the records in his shorthand? Certainly. (Some students might like to try—and compare their results with similar work using our symbols.)

Let's think our way through a situation described in case I of example *A* with boxes of marbles:

$$\boxed{p} + \boxed{q} = 6 \quad \text{or} \quad \boxed{p} = 6 - \boxed{q}$$

$$\boxed{r} + \boxed{q} = 4 \quad \text{or} \quad \boxed{r} = 4 - \boxed{q}$$

Clearly, p is larger than r. In fact,

$$p - r = 2;$$

and the last line states that

$$p + r = 8;$$

so it must be the case that $p=5$ and $r=3$. From that point on, the situation is easy to analyze.

After such a discussion, consider showing the students a bit of shorthand that is a record of the same kind of argument:

$$\begin{array}{c} p + q = 6 \\ -(r + q = 4) \\ \hline (p + r) + (q - q) = 6 - 4 \end{array}$$

$$p - r = 2$$

and :

$$\begin{array}{c} p - r = 8 \\ +(p + r = 8) \\ \hline (p + p) + (r - r) = 2 + 8 \end{array}$$

$$2p = 10$$
$$p = 5$$

X-Nighyun Is Eager

"Done! I agree to use your symbols and I'll just file my equally good system.

"I wish to tell you about using these new symbols or labels in your addition game. First, we'll make a chart. Across the top we shall begin listing all possible labels."

"Wait a minute. You couldn't ever list all the labels I could make up for different points between 0 and 1."

"No, but I can make up a list which, if extended by my rules, would give you any label you can think of."

"How?"

"Instead of explaining my rules, it would be easier to start the list and let you see the pattern I'm following. It starts '0 over 1, 0 over 2, 1 over 1, 0 over 3, etc.'

$\frac{0}{1}$	$\frac{0}{2}$	$\frac{1}{1}$	$\frac{0}{3}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{0}{4}$	$\frac{1}{3}$
$\frac{2}{2}$	$\frac{3}{1}$	$\frac{0}{5}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{4}{1}$	$\frac{0}{6}$
$\frac{1}{5}$	$\frac{2}{4}$	$\frac{3}{3}$	$\frac{4}{2}$	$\frac{5}{1}$	$\frac{0}{7}$	$\frac{1}{6}$	$\frac{2}{5}$

"If I extended this list, I would include any label you could think up."

"Next, repeat the list down the side of the chart.

"Now, addition of labels means you look up one label in the top list, the other in the side list, and then find the entry in the block that's in that column and that row. Done!"

"But there aren't any entries in my chart."

As if he hadn't heard, X-Nighyun went on, "Then you make a similar chart that has different entries and call it the multiplication chart."

A.	I	II	III	IV
a	3	8	1	$^-3$
b	1	$^-2$	$\frac{1}{2}$	$^-1$
c	4	0	$\frac{1}{4}$	$^-4$
d	6	5	$\frac{3}{4}$	$^-6$
$a+b+c+d$	14	11	$2\frac{1}{2}$	-14
$a+b+c-d$	2	1	1	-2
$a+2b-c+d$	7	9	$2\frac{1}{2}$	-7
$a-b+c+3d$	24	25	3	-24

B.	I	II	III	IV
w	15	8	18	3
x	4	5	8	3
y	1	$^-2$	0	3
z	6	4	25	3
$(w+x)-(y+z)$	12	11	1	0
$w-(x+y+z)$	4	1	-15	-6
$(w-x)+(y-z)$	6	-3	-15	0
$w-(x-y-z)$	18	5	35	6

Tasks: *Students continue reading and studying the story of X-Nighyun. They also complete* What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: X-Nighyun claims he can begin a list of fractions that, if extended far enough, would include any fraction you can think of. He might also add that he could tell you the one that would precede it and the one that would follow it in the list.

Essentially, his order depends on grouping fractions according to the sum of the numerator and denominator of each, and ordering the groups according to the common sums. Finally, within each group the fractions are ordered according to the numerators.

Group I	Group II	Group III	Group IV
$\frac{0}{1}$	$\frac{0}{2}, \frac{1}{1}$	$\frac{0}{3}, \frac{1}{2}, \frac{2}{1}$	$\frac{0}{4}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}$

Group V	Group VI
$\frac{0}{5}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}$	$\frac{0}{6}, \frac{1}{5}, \frac{2}{4}, \frac{3}{3}, \frac{4}{2}, \frac{5}{1}$

Notice, in passing, that it is even possible for X-Nighyun to tell you where to find any particular entry in his list.

Each group contains the same number of entries as the number of the group. (Let's use Roman numerals to designate the group.) For example, $\frac{2}{7}$ is in group IX, and it is the 3rd $(2+1)$ one in the group. How many precede group IX? It is helpful to know the formula for finding the sum of the first n whole numbers:

$$1 + 2 + 3 + 4 \ldots\ldots + n. \quad \text{It is: } \frac{n(n+1)}{2}$$

$$1 + 2 + 3 \ldots\ldots + 8 = \frac{8(8+1)}{2} = 36$$

and $36+3=39$. So $\frac{2}{7}$ is the 39th entry in X-Nighyun's list. Further, it will be preceded by $\frac{1}{8}$ and followed by $\frac{3}{6}$. (X-Nighyun can also tell you which fraction corresponds to any given whole number, but this might take a bit of thinking.)

We can say that X-Nighyun has thus established a one-to-one relationship between whole numbers and fractions. He has demonstrated that "fractions are countable." And, since there is no limit to the counting numbers, there is clearly no limit to the number of fractions.

Consider another example, $\frac{7}{13}$. It is the 8th number $(7+1)$ in group XX.

$$1+2+3 \ldots +19 = 190$$
$$190 + 8 = 198$$

Thus, $\frac{7}{13}$ is the 198th entry in X-Nighyun's list.

Let's look at case I in example *A*:

$$\boxed{a+b+c} + d = 14$$
$$\boxed{a+b+c} - d = 2$$

If this were:

$$k+d=14$$
$$k-d=2$$

could you find *d*? By trial and error or by reasoning, we find that $d=6$ and $k=8$.

Notice that we have found that $d=6$. But we have also found out that $a+b+c=8$,

because $k=8$
and $a+b+c=k$.

Bit by bit, we are hoping students will develop insight into the fact that the manipulation of algebraic expressions is based on reasoning.

"But, again, I have no entries — only labels for the columns and the rows."

"Oh, make any entries you wish provided you follow a few simple rules:

(a) Every entry in both charts must be one of your labels;

(b) both charts must, when used, demonstrate the commutative and associative principles;

(c) each must have an identity element — and not the same one;

(d) and multiplication must distribute over addition.

"Of course, you are familiar with these rules because you explained them to me.

"Now as long as you don't violate these rules, make up any entries you like."

Then, as an afterthought, he added:

"And if an entry in the multiplication chart is a label for zero then the label at the top of that column or at the side of that row must also be a label for zero.

"Now it's your turn to do some homework!"

How should you begin?

+	$\frac{0}{1}$	$\frac{0}{2}$	$\frac{1}{1}$	$\frac{0}{3}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{0}{4}$
$\frac{0}{1}$	$\frac{0}{1}$	$\frac{0}{2}$	$\frac{1}{1}$	$\frac{0}{3}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{0}{4}$
$\frac{0}{2}$	$\frac{0}{2}$	$\frac{0}{2}$	$\frac{1}{1}$	$\frac{0}{6}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{0}{4}$
$\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{3}{1}$	$\frac{1}{1}$
$\frac{0}{3}$	$\frac{0}{3}$	$\frac{0}{6}$	$\frac{1}{1}$	$\frac{0}{3}$	$\frac{1}{2}$	$\frac{2}{1}$	$\frac{0}{12}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{1}$	$\frac{5}{2}$	$\frac{1}{2}$

A.	I	II	III	IV
x	3	$\frac{1}{2}$	$^-2.6$	10
y	1	5	.7	1.5
z	7	$\frac{2}{3}$	1.05	0
$x - y$	2	$^-4\frac{1}{2}$	-3.3	8.5
$y - z$	-6	$4\frac{1}{3}$	$^-.35$	1.5
$x + z$	10	$1\frac{1}{6}$	$^-.55$	10
$4y - 7$	$^-3$	13	$^-4.2$	$^-1$
$xy + z$	10	$3\frac{1}{6}$	$^-.77$	15
$x^2 + 2y + z$	18	$10\frac{11}{12}$	8.51	103

B.	I	II	III	IV
k	6	1	3	$\frac{1}{2}$
m	2	12	$^-3$	6
q	8	3	2	$\frac{2}{3}$
$2k - m$	10	$^-10$	9	-5
$3q - k$	18	8	3	$1\frac{1}{2}$
$q + \frac{m}{2}$	9	9	$\frac{1}{2}$	$3\frac{2}{3}$
kmq	96	36	$^-18$	2
$\frac{kq}{m}$	24	$\frac{1}{4}$	$^-2$	$\frac{1}{18}$
$\frac{k+m}{q}$	1	$4\frac{1}{3}$	0	$9\frac{3}{4}$

Tasks: *Students continue reading and studying the story of X-Nighyun. They also complete* What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: In the discussion on pupil pages 122 and 123, we are toying with a powerful mathematical idea. The only rules you can expect the students to devise that meet X-Nighyun's requirements will be the rules they already know.

If the group is eager, urge them to invent their own "rules" and test them against X-Nighyun's criteria.

Consider some obvious attempts in addition.

1. Add numerators and add denominators:

$$\frac{1}{2}+\frac{5}{7}=\frac{6}{9}$$

2. Add numerators and find the difference between denominators:

$$\frac{1}{2}+\frac{5}{7}=\frac{6}{5}$$

3. Cross-add numerators and denominators:

$$\frac{1}{2}+\frac{5}{7}=\frac{8}{7}\ \left(\text{or } \frac{7}{8}\right)$$

4. Add numerator and denominator of the first fraction for numerator and then add numerator and denominator of the second fraction for denominator, or vice versa:

$$\frac{1}{2}+\frac{5}{7}=\frac{3}{12}\ \left(\text{or } \frac{12}{3}\right)$$

5. Find the difference between the numerator of the first and the denominator of the second for the numerator of sum; and the difference between denominator of the first and the numerator of the second for the denominator of the sum:

$$\frac{1}{2}+\frac{5}{7}=\frac{6}{3}$$

Which of X-Nighyun's criteria does each proposal fail to meet?

Again, trial and error gives a powerful insight into the nature of some parts of mathematics! In such discussions and analyses, you will be breaking through the checkbook-balance notion that cannot see beyond arithmetic, and opening up a vista of mathematics that says: Here is a situation! What can you say about it?

I'm thinking of three numbers. How much must I tell you before you know my thoughts?

Each of the cases on pupil page 122 provides three clues. The hardest has an extra clue. Was this fourth clue necessary? If so, which of them could have been omitted? Again, let's use trial and error —and reason. Let's also capture our reasoning processes in some kind of shorthand (example *A*,I):

$$\begin{array}{r} y - z = {}^{-}6 \\ +(x + z = 10) \\ \hline (y + x)+(z - z) = 4 \end{array}$$

$$\begin{array}{r} x + y = 4 \\ +(\,x - y = 2\,)\ \text{(given)} \\ \hline (x + x)+(y - y) = 6 \end{array}$$

$$\begin{array}{r} 2x = 6 \\ x = 3 \end{array}$$

If $x=3$ and $x+y=4$ then $y=1$, etc., and the full solution follows.

Suppose that you know that

$$\begin{aligned} x - y &= 2, \\ y - z &= {}^{-}6, \\ x - z &= {}^{-}4. \end{aligned}$$

Are these clues sufficient? As in example *A*,I, "x may be 3, y may be 1, and z may be 7," but there are many other solutions as well, such as:

$$\begin{aligned} x &= 12 \\ y &= 10 \\ z &= 16 \end{aligned}$$

So, there are insufficient clues.

Able students might begin an investigation of ways to tell whether sufficient clues are given. Notice that from "$y-z={}^{-}6$" and "$x-z={}^{-}4$," we can find "$x-y=2$," which is the first clue given. With the second and third given, the first adds nothing— so we really have only two useful clues.

Just to show X-Nighyun how easily that can be done, you decide to add numerators and add denominators to find entries for the addition chart, and multiply numerators and multiply denominators for the entries in the multiplication chart. Since it pains you to write that $\frac{1}{2} + \frac{1}{3}$ equals $\frac{2}{5}$ — rather than $\frac{5}{6}$ — you decide to put a circle around any addition sign between these labels.

You don't even need to make charts because you can imagine them as you go. So, when you call him back you are ready to demonstrate your solution to his problem:

(a) Every pair of labels leads to an entry that is a label:

$$\frac{1}{3} \oplus \frac{2}{5} = \frac{3}{8} \qquad \frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$$

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

(b) The commutative and associative principles hold.

$$\frac{2}{3} \oplus \frac{1}{7} = \frac{1}{7} \oplus \frac{2}{3}$$

$$\frac{2}{3} \times \frac{1}{7} = \frac{1}{7} \times \frac{2}{3}$$

$$\left(\frac{1}{2} \oplus \frac{3}{4}\right) \oplus \frac{1}{3} = \frac{1}{2} \oplus \left(\frac{3}{4} \oplus \frac{1}{3}\right)$$

$$\left(\frac{1}{2} \times \frac{3}{4}\right) \times \frac{1}{3} = \frac{1}{2} \times \left(\frac{3}{4} \times \frac{1}{3}\right)$$

"That was an easy assignment," you tell X-Nighyun.

"Not as easy as you think, my friend. Your pattern for addition holds up for quite a ways . . . but show me your identity element. How about my rule (c)? Complete the following using your table:

$$\frac{1}{2} \oplus \underline{\quad} = \frac{1}{2} \qquad \frac{2}{5} \oplus \underline{\quad} = \frac{2}{5}.$$

You answer quickly that

$$\frac{1}{2} \oplus \frac{0}{0} = \frac{1}{2} \qquad \frac{2}{5} \oplus \frac{0}{0} = \frac{2}{5}.$$

"There is no such label as $\frac{0}{0}$, and it would not be an appropriate label on the number line you described to me.

A.	I	II	III	IV
$p + q$	21	10	60	0
$p - q$	9	-4	-8	.4
p	15	3	26	.2
q	6	7	34	-.2

B.	I	II	III	IV
$x + 2y$	14	27	36	42
$x - y$	5	-3	0	9
x	8	7	12	20
y	3	10	12	11

C.	I	II	III	IV
$a + 2b$	13	27	66	$1\frac{1}{6}$
$2a + 2b$	20	36	82	$1\frac{2}{3}$
a	7	9	16	$\frac{1}{2}$
b	3	9	25	$\frac{1}{3}$

D.	I	II	III	IV
$2r + t$	23	29	4	1.3
$r + 2t$	16	37	-4	1.1
r	10	7	4	.5
t	3	15	-4	.3

123 One attempt at adding and multiplying fractions; What Are My Rules?

Tasks: Students continue reading and studying the story of X-Nighyun. They also complete What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Which of X-Nighyun's four general rules would the following definition of addition violate?

"Add the numerator and denominator of the first fraction to find the numerator of the sum; add the numerator and denominator of the second fraction to find the denominator of the sum":

$$\frac{1}{3} \oplus \frac{2}{5} = \frac{1+3}{2+5} = \frac{4}{7}$$

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+b}{c+d}$$

The "sum" is itself a label—so it passes X-Nighyun's test *a*. Is the operation described commutative? Is it associative? Are these patterns for true statements?

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{c}{d} \oplus \frac{a}{b}$$

$$\left(\frac{a}{b} \oplus \frac{c}{d}\right) \oplus \frac{e}{f} = \frac{a}{b} \oplus \left(\frac{c}{d} \oplus \frac{e}{f}\right)$$

Clearly, the operation is not commutative:

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+b}{c+d} \quad \text{and} \quad \frac{c}{d} \oplus \frac{a}{b} = \frac{c+a}{d+b}$$

Let's consider the second question:

$$\left(\frac{a}{b} \oplus \frac{c}{d}\right) \oplus \frac{e}{f} = \frac{a+c}{b+d} \oplus \frac{e}{f} = \frac{a+e+b+c}{e+f}$$

and

$$\frac{a}{b} \oplus \left(\frac{c}{d} \oplus \frac{e}{f}\right) = \frac{a}{b} \oplus \left(\frac{c+d}{e+f}\right) = \frac{a+b}{c+d+e+f}$$

So it is not associative.

There is no identity element to make the following into a pattern of true statements:

$$\frac{a}{b} \oplus \_\_\_ = \frac{a}{b}$$

In fact, this attempt is a miserable failure.

The definition tried on pupil page 123 passes tests *a* and *b*, but it fails test *c*—there is no identity element. (Actually, this rule would also pass the last test if we think of multiplication in the usual way:

$$\left(\frac{a}{b} \oplus \frac{c}{d}\right) \times \frac{e}{f} = \frac{ae}{bf} \oplus \frac{ce}{df} = \frac{ae+ce}{bf+df}$$

$$= \left(\frac{a+c}{b+d}\right) \times \frac{e}{f} = \frac{ae+ce}{bf+df}$$

Here is an instance:

$$\left(\frac{1}{3} \oplus \frac{4}{9}\right) \times \frac{1}{2} = \frac{1 \times 1}{3 \times 2} \oplus \frac{4 \times 1}{9 \times 2} = \frac{1}{6} \oplus \frac{4}{18} =$$

$$\frac{1+4}{6+18} = \frac{5}{24} = \left(\frac{1+4}{3+9}\right) \times \frac{1}{2} = \frac{5}{12} \times \frac{1}{2} = \frac{5}{24}$$

This is truly a surprising outcome.)

Urge students to invent other rules for addition that can be put to X-Nighyun's tests *a*, *b*, *c*, and *d*. Can they find a set of rules that pass all the tests? Test *c* is perhaps the most difficult. It is, of course, satisfied by the usual rule:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

by replacing *c* with zero and stating that $d \neq 0$:

$$\frac{a}{b} + \frac{0}{d} = \frac{ad+(b \times 0)}{bd} = \frac{ad}{bd} = \frac{a}{b}$$

The four examples in the lower sections of pupil pages 123, 124, and 125 can be used to take an inventory of each student's grasp of some basic notions of algebra. Some may still have no other tools than trial and error. Please do not underrate a student's use of this technique; he must understand a basic idea of algebra to employ even this method and he will accumulate shortcuts later. Further, he has a method he can later fall back on whenever he forgets a rule or is uncertain about a rule.

On the other hand, students who have a flare for finding and using shortcuts ought to be encouraged in this direction, but should test their results.

The examples on pupil page 123 are quite straightforward and ought to present little or no trouble. Pupil pages 124 and 125 are more difficult.

"Try again and, I might add, your system would fail again in that it would not follow rule (d) — which you can check for yourself.

"Back to the asteroid pile, as we say on Mars."

"I'll talk to you tomorrow."

Another Attempt

We learned our lesson.

Rather than start all over, let's keep what we already have, trying variations.

Let's see what happens if we make up a new rule for $\oplus$ — you add numerators and multiply denominators. This would lead us to write:

$$\frac{1}{2} \oplus \frac{3}{5} = \frac{1+3}{2 \times 5} = \frac{4}{10}$$

$$\frac{3}{7} \oplus \frac{2}{3} = \frac{5}{21} \qquad \frac{7}{9} \oplus \frac{2}{3} = \frac{9}{27}$$

In general: $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b \times d}$

Does this pass X-Nighyun's rules (a) and (b)? YES

Is there a label that we can use as an identity element so we can complete the following?

$$\frac{1}{2} \oplus \frac{0}{1} = \frac{1}{2} \qquad \frac{2}{5} \oplus \frac{0}{1} = \frac{2}{5}$$

Finally, does the distributive principle hold? Consider the following pattern:

$$\left(\frac{a}{b} \oplus \frac{c}{d}\right) \times \frac{m}{n} = \frac{a \times m}{b \times n} \oplus \frac{c \times m}{d \times n}$$

Let's try one example: substitute 1 for a, 2 for b, 3 for c, 4 for d, 1 for m, and 3 for n.

$$\left(\frac{1}{2} \oplus \frac{3}{4}\right) \times \frac{1}{3} = \frac{1 \times 1}{2 \times 3} \oplus \frac{3 \times 1}{4 \times 3}$$

$$\left(\frac{4}{8}\right) \times \frac{1}{3} = \frac{1}{6} \oplus \frac{3}{12}$$

$$\frac{4}{24} = \frac{4}{72}$$ (a counter-example)

A most unhappy result!

A.	I	II	III	IV
$a - b + 3$	7	12	-13	$13\frac{3}{4}$
$a + b - 3$	3	44	13	$11\frac{1}{4}$
a	5	28	0	$12\frac{1}{2}$
b	1	19	16	$1\frac{3}{4}$

B.	I	II	III	IV
$2p + q$	23	97	30	$2\frac{5}{8}$
$p - q + 7$	14	18	-2	7
p	10	36	7	$\frac{7}{8}$
q	3	25	16	$\frac{7}{8}$

C.	I	II	III	IV
$xy - 1$	15	80	$1\frac{1}{2}$	.05
$x - y$	6	0	$4\frac{1}{2}$	.8
x	8	9	5	1.5
y	2	9	$\frac{1}{2}$	.7

D.	I	II	III	IV
$2ak$	182	25	4.2	$\frac{1}{6}$
$2a + k$	27	26	6.7	$\frac{11}{12}$
a	7	$\frac{1}{2}$	3	$\frac{1}{3}$
k	13	25	.7	$\frac{1}{4}$

124 Another attempt at adding and multiplying fractions; What Are My Rules?

Tasks: *Students continue reading and studying the story of X-Nighyun. They also complete* What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: On this page we consider another set of rules for $\oplus$: add numerators and multiply denominators. It passes X-Nighyun's test *a:*

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b \times d} \qquad (b \text{ and } d \neq 0)$$

Even though a or c or both a and c are zero, the result will be a label in X-Nighyun's list—since $b \times d$ cannot be zero.

Is the operation commutative?

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b \times d}$$

and

$$\frac{c}{d} + \frac{a}{b} = \frac{c+a}{d \times b} = \frac{a+c}{b \times d}$$

It is commutative!
(The operation sign $\oplus$ follows the rules we have agreed to test. The operation sign $+$ is only used between whole numbers and is interpreted in the usual way.)

Is the operation associative?

$$\left(\frac{a}{b} \oplus \frac{c}{d}\right) \oplus \frac{e}{f} = \frac{a+c}{bd} \oplus \frac{e}{f} = \frac{a+c+e}{bdf}$$

$$\frac{a}{b} \oplus \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \oplus \frac{c+e}{df} = \frac{a+c+e}{bdf}$$

It is associative!

We find it has an identity element, $0/1$:

$$\frac{a}{b} + \frac{0}{1} = \frac{a+0}{b \times 1} = \frac{a}{b}$$

Finally, assuming the usual rules for multiplying fractions, we test the distributive idea and find a counterexample immediately. Suppose that we look at this situation in a general way:

$$b \neq 0, \; d \neq 0, \; n \neq 0$$

$$\left(\frac{a}{b} \oplus \frac{c}{d}\right) \times \frac{m}{n} = \frac{a+c}{bd} \times \frac{m}{n} = \frac{am+cm}{bdn}$$

$$\left(\frac{a}{b} \oplus \frac{c}{d}\right) \times \frac{m}{n} = \frac{am}{bn} \oplus \frac{cm}{dn} = \frac{am+cm}{bdn^2}$$

We see that $bdn = bdn^2$ if and only if $n=1$.

Thus the rule under study holds up until X-Nighyun's test *d*. Then it fails for all but the special case in which $n=1$, in which case m/n is a whole number expressed in fractional form.

At this point, we must either reject this version of $\oplus$ or reject the usual version of $\times$ (which, incidentally, passes all four of X-Nighyun's tests) and find some new multiplication operation $\otimes$, which must pass tests *a*, *b*, and *c*, and provide that, for each a, c, m, $b \neq 0$, $d \neq 0$, and $n \neq 0$,

$$\left(\frac{a}{b} \oplus \frac{c}{d}\right) \otimes \frac{m}{n} = \left(\frac{a}{b} \otimes \frac{m}{n}\right) \oplus \left(\frac{c}{d} \otimes \frac{m}{n}\right).$$

Students who participate in this kind of discussion and experimentation are dealing with the fundamental idea of mathematical structure. They will find that they have a firm foundation for the mathematics to come—"traditional," "modern," "applied," or "pure."

On the other hand, those who are unable to participate fully ought not to be discouraged; they will not be expected to have digested these notions in a single encounter.

As with the example on pupil page 123, these might well be used as basis for taking a bit of inventory—in a most informal way. When questions arise, consider discussing them fully, rather than insisting on students' relying completely on their own resources. Often we learn most about the tools a student has acquired from discussing his questions with him.

Try as you will to work out some other way to follow X-Nighyun's rules, you will almost certainly be forced to a rule you already know:

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a \times d + b \times c}{b \times d}$$

Does our rule for multiplication meet all of X-Nighyun's requirements? YES

Do our rules for multiplication meet all of X-Nighyun's requirements? YES

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

So, this time when X-Nighyun calls, we are ready for him. We explain our rules for making entries in the two charts — the one marked "addition" and the other marked "multiplication."

"You may be surprised, but I have exactly the same rules you have. At least, they would lead to exactly the same entries in the two charts. So we are together.

"But I'm troubled that we have limited our discussion to the labels on the 'right' side of our number line — the ones you indicate with a 'raised cross' when you wish to distinguish them from the others on the left, the ones you indicate with a 'raised dash'."

"You mean labels for what we refer to as positive and negative numbers?"

"Yes. But I'm not sure I know any reason to keep talking about labels and numbers. You can be sure that I shall know when you are talking about labels and when you are talking about numbers. If you are worried in any particular case then let me know — and if I have questions, I'll ask them!"

A.	I	II	III	IV
$2(a+b)$	22	32	150	0
$a + 2b$	14	31	125	-4
a	8	1	25	4
b	3	15	50	-4

B.	I	II	III	IV
$(x-y)3$	36	-18	21	$\frac{3}{4}$
$x + y$	18	8	61	$\frac{3}{4}$
x	15	1	34	$\frac{1}{2}$
y	3	7	27	$\frac{1}{4}$

C.	I	II	III	IV
$4m - n$	3	18	18	1
$n - m$	3	-3	3	$\frac{1}{2}$
m	2	5	7	$\frac{1}{2}$
n	5	2	10	1

D.	I	II	III	IV
$3r - 2t + 7$	10	26	14	37
$r + 2t - 3$	14	14	18	7
r	5	9	7	10
t	6	4	7	0

125 Finally: a/b + c/d = (a*d + b*c) / b*d and a/b x c/d = a*c / b*d; What Are My Rules?

Tasks: *Students continue reading and studying the story of X-Nighyun. They also complete* What Are My Rules? *tables.*

Purpose: To help students better understand the structure of arithmetic. To help them understand how to use the shorthand of algebra.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations.

The Lesson: Almost in desperation, one falls back on his experience with fractions to come up with rules that pass X-Nighyun's basic tests.

But let's not assume they will meet his requirements. Let's investigate.

We shall indicate our definitions in this way:

Rules for ADDITION and MULTIPLICATION

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

$$(b \neq 0 \text{ and } d \neq 0)$$

A casual look at the definitions or rules for these operations indicates that both the sum and the product

$$\frac{ad + bc}{bd} \quad \text{and} \quad \frac{ac}{bd}$$

would occur in X-Nighyun's list. They pass test *a*.

Are both operations commutative?

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{c}{d} + \frac{a}{b} = \frac{cb + ad}{db} = \frac{ad + bc}{bd}$$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \quad \text{and} \quad \frac{c}{d} \times \frac{a}{b} = \frac{ca}{db} = \frac{ac}{bd}$$

Both are commutative.

Are they associative?

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{ad + bc}{bd} + \frac{e}{f}$$

$$= \frac{(ad + bc)f + (bd)e}{bdf}$$

$$(1) \qquad = \frac{adf + bcf + bde}{bdf}$$

$$\frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} + \frac{cf + de}{df}$$

$$= \frac{a(df) + b(cf + de)}{bdf}$$

$$(2) \qquad = \frac{adf + bcf + bde}{bdf}$$

Since expressions 1 and 2 are equivalent, we must conclude that our familiar rule for addition of fractions makes addition associative.

What about multiplication?

$$\frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right) = \frac{a}{b} \times \frac{ce}{df} = \frac{ace}{bdf}$$

$$\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{ac}{bd} \times \frac{e}{f} = \frac{ace}{bdf}$$

That was short—and encouraging.

Each has an identity element listed by X-Nighyun.

$$\frac{a}{b} + \frac{0}{c} = \frac{ac + b \cdot 0}{bc} = \frac{ac}{bc} = \frac{a}{b}$$

$$\frac{a}{b} + \frac{c}{c} = \frac{ac}{bc} = \frac{a}{b}$$

Success again.

Now, the final test! Is multiplication distributive over addition?

$$\left(\frac{a}{b} + \frac{c}{d}\right) \times \frac{m}{n} = \frac{am}{bn} + \frac{cm}{dn} = \frac{amdm + bncm}{bndn}$$

$$\left(\frac{a}{b} + \frac{c}{d}\right) \times \frac{m}{n} = \frac{ad + bc}{bd} \times \frac{m}{n}$$

$$= \frac{adm + bcm}{bdn}$$

Does

$$\frac{amdn + bncm}{bndn} = \frac{adm + bcm}{bdn} \,?$$

Let's rearrange and factor:

$$\frac{(adm + bcm)\cancel{n}}{bnd \times \cancel{n}} = \frac{adm + bcm}{bdn}$$

and, dividing the numerator and denominator of the expression on the left by *n*, it is easy to see that it is equivalent to the one on the right!

Success! We've met all X-Nighyun's tests. In the process, of course, we have had a little refresher course in manipulating fractions.

(See the general comments on the following pages.)

"Quite a lecture! But you're probably right. Before continuing, here is a little test of your ability to use our Earth arithmetic."

Here are your questions for X-Nighyun.

1. "Rockton and Newton are 945 miles apart by railroad. A slow freight train — 'S' for short — leaves Rockton and travels 45 miles per hour. How long will it take for S to reach Newton?"

Answer: 21 HOURS

2. "A fast freight train — 'F' for short — leaves from Newton and travels at the rate of 60 miles per hour. How long will it take to reach Rockton?"

Answer: $15\frac{3}{4}$ HRS. OR 15 HRS. & 45 MIN.

3. "If S left Rockton at 1 P.M., what time did it arrive at Newton?"

"I suppose it arrived at 22 P.M., whatever 'P.M.' means."

Always more explaining to do. Could you make X-Nighyun understand our system of talking about time?

4. "All right, I'll change my question. If S left from Rockton and F from Newton at the same time, how much earlier would F reach Rockton than S reached Newton?"

Answer: $5\frac{1}{4}$ HRS.

5. "How far would S be from Newton when F reached Rockton?"

Answer: $236\frac{1}{4}$ MILES ($5\frac{1}{4} \times 45$)

"S left from Rockton and F left from Newton at 1 P.M. on the same day. Now I have several questions to ask:

6. "How far apart would they be at 2 P.M.?"

Answer: 840 MILES APART.

7. "How far apart at 5 P.M.?"

Answer: 525 MILES APART.

8. "How far apart at 7 P.M.?"

Answer: 315 MILES

9. "How far apart at 1 A.M.?"

Answer: 315 MILES

10. "At what time did they meet?"

Answer: 10 P.M. (AFTER 9 HRS.)

11. "How far apart at 4 A.M.?"

Answer: 630 MILES

12. "How much further has F to go?"

Answer: 45 MILES

13. "How much further has S to go?"

Answer: 270 MILES

14. "When S reached Junction City, 50% of its fuel had been used up. When it got to Newton, there was still 20% left. About how far is Junction City from Rockton?"

Answer: ABOUT 590 MILES

15. "Of course, you could only make a reasonable estimate. Actually, Junction City is 245 miles further from Rockton than from Newton. Now can you tell me how far it is from Junction City to Newton."

Answer: 350 MILES

16. "Now, here is a brainbuster! S left Rockton at 1 P.M. and traveled at the rate of 45 miles per hour. S met F at Junction City; and F had traveled at the rate of 60 miles per hour. At what time did F leave Newton?"

Answer: AT ABOUT 8:23 P.M.

"That sure was a brainbuster. I can work more easily with glooms per glud. How about you?"

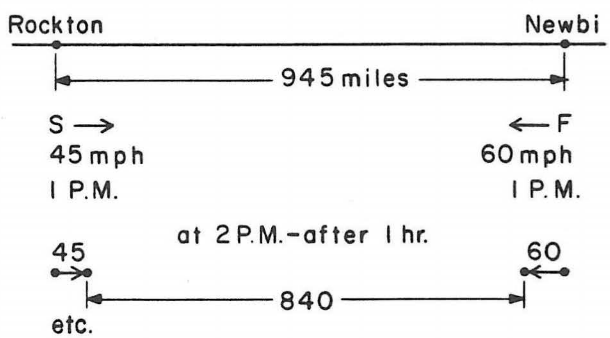
126 Some story problems for X-Nighyun

Tasks: Students compute answers to the questions posed to X-Nighyun.

Purpose: To give students experience in solving relative motions problems.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: Pupil page 126 can be used as another kind of inventory that requires much more than computation. You might suggest the students read ahead several questions and then prepare a sketch to help. This one might be useful:



General Comments

The space phone dialog between X-Nighyun and his friend on Earth could go on, but we must leave that for further invention.

Perhaps the most important task ahead is to reconsider the rules we have adopted for working with positive and negative numbers and zero. Do our rules meet the basic tests of X-Nighyun in a similar context? We can begin by asking:

Is the sum of any two such numbers also a positive or negative number or zero? Is the same true of the product of any such pair?

If a and b are positive or negative numbers or zero, can we say that there exists a c within this same list that we can point to as a sum and a d we can point to as the product? Yes. So we can say that we have a "definition" or a "set of rules to follow" for both addition and multiplication.

For any pair of numbers a and b from our list, is it the case that

$$a + b = b + a \quad \text{and} \quad a \times b = b \times a\ ?$$

We can try a few to be sure we all understand what we mean:

$$^-2 + {}^+7 = {}^+7 + {}^-2 \qquad {}^-2 \times {}^+7 = {}^+7 \times {}^-2$$
$$^+5 = {}^+5 \qquad {}^-14 = {}^-14$$

If we are satisfied that our general statement is true then we say that our definitions for addition and multiplication are such that those operations are *commutative.*

For any group of three numbers a, b, and c, from our list, is it the case that

$$(a + b) + c = a + (b + c)$$
$$\text{and}$$
$$(a \times b) \times c = a \times (b \times c)$$

Again, we try a few examples just to be sure we understand what we are saying:

$$(^-3 + {}^+2) + {}^-5 = {}^-3 + (^+2 + {}^-5)$$
$$(^-1) + (^-5) = {}^-3 + {}^-3$$
$$^-6 = {}^-6$$
etc.

$$(^-3 \times {}^+2) \times {}^-5 = (^-3 \times {}^+2) \times {}^-5$$
$$^-6 \times {}^-5 = {}^-3 \times {}^-10$$
$$^+30 = {}^+30$$
etc.

If we agree that our general statement is true, we can say that our definitions for addition and multiplication are such that those operations are *associative.*

X-Nighyun lumped these two notions together—commutativity and associativity—in his test b. He would probably add that because of both properties there is a general "Any-Order, Any-Grouping Rule" for addition and a parallel rule for multiplication. That is, the order in which you add or the grouping of the numbers to be added does not affect the sum, and similarly for multiplication.

Further, for any number a, there are numbers b and c in the list such that

$$a + b = a \quad \text{and} \quad a \times c = a.$$

These numbers are 0 and 1. For example:

$$^-3 + 0 = {}^-3 \qquad {}^-7 \times 1 = {}^-7$$
$$^+9 + 0 = {}^+9 \qquad {}^+13 \times 1 = {}^+13$$

X-Nighyun's test c is passed.

This brings us to the distributive idea discussed on the next page of notes.

You shouldn't be surprised, one day, if X-Nighyun started asking some strange questions. Like these:

"Now that I understand something about your arithmetic, I'm going to find out something about how clever you are.

"If I made a trip to Astra A from here near your planet's moon, and then returned, it would be 750,000,000 glooms. How far is it from here to Astra A?"

"But how can I tell when I don't know what a gloom is?"

"If you told me that a round trip between New York and Chicago is 1680 miles, I could tell you that those cities are 840 miles apart . . . and I have no idea about the length of your mile."

"Oh, you want to know how many glooms it is from you to Astra A?"

"Of course."

"The distance must be about

375,000,000 GLOOMS"

"I once made a trip starting from Astra A and going to Astra C, stopping at Astra B on the way. The total trip was 204,000 glooms. It is twice as far from Astra A to Astra B as it is from Astra B to Astra C. How far is it from Astra A to Astra B?"

Answer: 136,000 GLOOMS

"I would have saved 17,500 glooms by going straight from Astra A to Astra C. What is the shortest distance from Astra A to Astra C?"

Answer: 186,500 GLOOMS

"How far is it from here to Astra B if I stop at Astra A on the way?"

Answer: 375,136,000 GLOOMS

"If my speed is 50,000 glooms per glud, how many gluds will it take for me to get to Astra A from here?"

Answer: 7,500 GLUDS

"If I returned immediately at half that speed, how long would I have been gone?"

Answer: 22,500 GLUDS

"On your moon, I weigh 18.5 chorns when I have my space suit on. Now, 40% of that weight is my suit. How much do I weigh?"

Answer: 11.1 CHORNS

"When I am in my suit and carrying another suit, how much would the total weight be?"

Answer: 25.9 CHORNS

"A shipment of these space suits weighed 555 chorns. How many space suits were in the shipment?" (75 × 7.4 = 555)

Answer: 75 SPACE SUITS

"At 20,000 glooms per glud, how long would it take me to go from Astra C to Astra B?"

Answer: 3.4 GLUDS

"At the same speed, how long from Astra B to Astra A?"

Answer: 6.8 GLUDS

"At the same speed, how long from Astra A to my present position near your moon?"

Answer: 18,750 GLUDS

"Now it's your turn!"

Tasks: *Students compute answers to the questions posed by X-Nighyun.*

Purpose: To give students experience in solving relative motions problems.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: Students may be surprised that they, like X-Nighyun, can answer many questions about situations described in most unfamiliar units of measure.

General Comments

In considering positive and negative numbers and 0, we want the distributive idea to hold. Because of this, we are required to include in our definition of multiplication the rule that the product of two negative numbers is a positive number. Here is an example to show why.

It is quite easy for us to interpret the idea of $^-3\times{}^+2$ or $^+2\times{}^-3$ as "a trip of negative 3 taken twice"—and consider it "a trip of negative 6." And, since we want to preserve the commutative principle, we make our definition such that

$$^-3\times{}^+2={}^+2\times{}^-3={}^-6.$$

Further, our definitions of addition and multiplication lead us to say that:

$$^-3+{}^+3=0 \quad \text{and} \quad 0\times{}^-7=0.$$

This leads us to say that

$$(^-3+{}^+3)\times{}^-7=0\times{}^-7=0.$$

But if the distributive idea holds,

$$\underline{(^-3\times{}^-7)}+(^+3\times{}^-7)=0,$$
$$\underline{(^-3\times{}^-7)}+{}^-21=0.$$

Now we should also like there to be exactly one number whose sum with $^-21$ is 0. So the underlined expression must stand for $^+21$, since

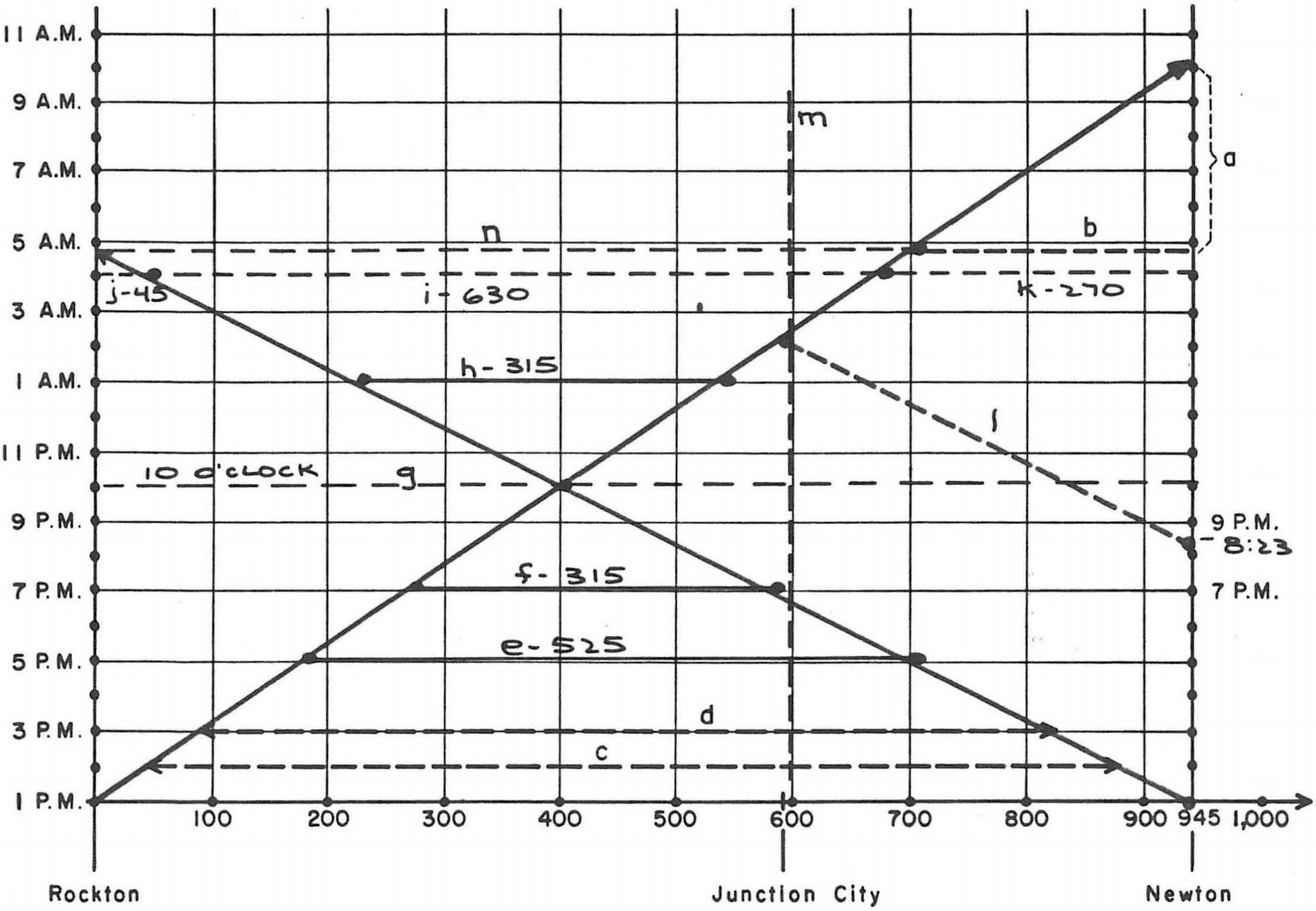
$$^+21+{}^-21=0.$$

So $^-3\times{}^-7={}^+21$.

A Few Comments on Inverses

Again, time and space must leave important aspects of the dialog with X-Nighyun to go unreported here. For example, we should like to be able to discuss the facts that: (1) for all numbers a and b that we have considered, there is one—and only one—number x such that $a+x=b$; we call this number $b-a$; and (2) for all numbers a and b that we have considered (provided that $b\neq0$), there is one—and only one—number x such that $ax=b$, and we call that number b/a.

This would all be a rather stiff way to talk over the space phone about subtraction as the "inverse" of addition and division as the "inverse" of multiplication.



By interplanetary mail, X-Nighyun sent a note along with the sketch above.

While you were asking all those questions about the two freight trains, I used the arithmetic you taught me to find the answers. But I also made a diagram to help me out.

The diagram contains much of the information you gave and much of the information you made me figure out.

Another important use for the diagram is that it keeps me from making any unreasonable answers. I checked the answer to every problem — except one — by estimating the answer with the diagram above. The exception was that one about fuel.

I've marked parts of the sketch that I used to help with your third, fourth, and fifth questions.

I needed to answer the first two questions before I could draw the sketch.

I'm going to let you decide how I used this diagram to find out whether any answers were unreasonable.

All I did to test my answer to your brain-buster was to draw the broken blue line.

Thanks for your explanations about arithmetic. I know there is more, but I wish to spend some time putting together the pieces you gave me. The patterns help me see what's going on.

Sincerely,

X-Nighyun

Can you discover ways in which X-Nighyun used his sketch?

Tasks: Students discuss ways to interpret X-Nighyun's sketch.

Purpose: To explore graphic solutions to the problems posed on pupil page 126.

Unifying Ideas: Addition and subtraction; multiplication and division; functions and relations; measurement.

The Lesson: In review, we can summarize a plan of attack for explaining arithmetic over the space phone in the most general terms. We might outline the plan as shown in the chart below.

We have at no time in the discussion referred to these statements as "axioms" to be used in building a logically consistent structure. We are convinced that elementary-school students do not look at arithmetic as evolving in that way; and we do not think any attempt to talk about such an endeavor will help students overcome the obstacles to understanding left by traditional approaches to arithmetic.

Rather, we have tried our best to help students think of arithmetic as finding ways to record their observations about significant developments in the real world—with emphasis on finding increasingly simple ways of finding increasingly difficult results.

Finally, in this last chapter we have taken a backward look to indicate one way in which the previous findings of students can be stated in general terms.

ADDITION	MULTIPLICATION
1. Define Addition For all numbers a and b, there is one and only one number x called the sum: $a + b = x$	**2. Define Multiplication** For all numbers a and b, there is one and only one number x called the product: $a + b = b + a$.
3. Explain the Commutative Idea for Addition For all numbers a and b, $(a + b) + c = a + (b + c)$.	**4. Explain the Commutative Idea for Multiplication** For all numbers a and b, $(a + b) \times c = a \times c + b \times c$
5. Explain the Associative Idea for Addition For all numbers a, b, and c, $a - b = x$	**6. Explain the Associative Idea for Multiplication** For all numbers a, b, and c, $a + 0 = a$.
7. Explain the Distributive Idea For all numbers a, b, and c, $a \times b = x$	
8. Define Subtraction For all numbers a and b, there is one and only one number x called their difference: $a \times b = b \times a$.	**9. Define Division** For all numbers a and b, provided $b \neq 0$, there is one and only one number called their quotient: $(a \times b) \times c = a \times (b \times c)$.
10. Identity Element for Addition For every number a, $a \div b = x$	**11. Identity Element for Multiplication** For every number a, $a \times 1 = a$.
	12. Point out that, for all numbers a and b, $a \times b = 0$ if and only if $a = 0$ or $b = 0$.

APPENDIX

THE EXPANDING LANGUAGE OF MATHEMATICS

Many unfamiliar expressions are heard in current discussions of mathematics. Some teachers are interested; others are disturbed; some are confused. The purpose of this appendix is to illustrate the extremely simple ideas indicated by some of the new words and expressions.

Because the concepts dealt with are not at all new, many teachers will prefer to discuss them in the language they have always used. They can do this without any qualms of conscience. All teachers are urged to use new words and expressions *only after* they have a firm comprehension of the ideas, have familiarized themselves with the language, and can use it comfortably. The ideas are important: the language is useful only if it helps present the ideas more clearly.

1. Sets

Mathematicians often borrow words from the general language and give them a specific mathematical meaning. They usually prefer this to inventing words of their own. Such a word is *set*. We are familiar with such expressions as:

"*Set* it on the table."
"The sun will *set* at 5:30 P.M."
"That stage *set* is ingenious."
"I'm going to *set* my hair."
"Let's play a *set* of tennis."
"The cement will *set* by morning."

None of these has much, if anything, to do with the meaning mathematicians have given the word *set*. Their meaning has more in common with:

"How many pieces in the *set* of dishes?"
"Breakfast *sets* for $24.99."
"He has a *set* of ivory chessmen."
"I got a gold-plated pen and pencil *set*."

They looked at the word *set* as it might be used in place of *group*, *collection*, *combination*, *aggregate*, *family*, etc., and then proceeded to give that general idea a precise mathematical meaning.

Sets in Mathematics. Many uses of the word *set* in mathematics have a rather familiar ring. Here are a few in which the word might be replaced by *group* or *collection* as those words are used in everyday discourse.

The set of all counting numbers between 10 and 15 has 11, 12, 13, and 14 as its members.

Mathematicians use braces to suggest a set, so they might indicate this particular set of numbers by:

$$\{11, 12, 13, 14\}$$

A mathematician might wish to direct your attention to all the counting numbers and zero that are multiples of 7, in which case he would write:

$$\{0, 7, 14, 21, 28, \ldots\}$$

The three dots are a notation for "and so on in a like manner."

The members of a set need not be numbers. We might wish to consider, for example:

> The set of all cities in the United States with populations in excess of one million.
>
> The set of all objects in my purse.
>
> The set of redheaded boys.
>
> The set of all triangles.
>
> The set of all letters in the alphabet that are never used as vowels.

2. Union of Sets

One way of combining sets is called "union," for which the symbol $\cup$ is most commonly used.

(1) $\{A, B\} \cup \{A, C, E\} = \{A, B, C, E\}$
What letters occur either in the set with members A and B or in the set with members A, C, and E? Answer: the letters A, B, C, and E.

(2) $\{1, 5, 4\} \cup \{4, 6, 1\} = \{1, 4, 5, 6\}$ (or $\{4, 6, 5, 1\}$, etc.)

(3) $\{\text{Al, Ed}\} \cup \{\text{Carl, Ed}\} = \{\text{Al, Ed, Carl}\}$

(4) $\{m, n\} \cup \{r, s, t\} = \{m, n, r, s, t\}$

(5) $\{7, 2, 9\} \cup \{9, 2\} = \{7, 2, 9\}$

(6) $\{a, b, c\} \cup \{c, b, a\} = \{a, b, c\}$

(7) $\{b, x, y\} \cup \{x\} = \{\_\_, \_\_, \_\_\}$

(8) $\{\text{Jane}\} \cup \{\text{Jane, Mary}\} = \{\_\_, \_\_\}$

(9) $\{P, Q\} \cup \{R, S, T\} = \{\_\_, \_\_, \_\_, \_\_, \_\_\}$

(10) $\{(3, 4), (7, 9)\} \cup \{(7, 9), (4, 3)\} = \{(3, 4), (7, 9), (4, 3)\}$

(11) $\{(7, 0)\} \cup \{(0, 7)\} = \{(7, 0), (0, 7)\}$

(12) $\{(7, 0)\} \cup \{(7, 0)\} = \{(7, 0)\}$

(13) $\{(1, 8)\} \cup \{(8, 1)\} = \{(\ ,\ ), (\ ,\ )\}$

(14) $\{(0, 0), (0, 1), (1, 1)\} \cup \{(1, 0), (0, 1)\} = \{(\ ,\ ), (\ ,\ ), (\ ,\ ), (\ ,\ )\}$

(15) $\{(a, b), (b, a)\} \cup \{(b, b), (a, b)\} = \{\_\_\_\_\_\_\_\_\_\_\}$

(16) $\{r, s, t\} \cup \{s, t, a, r\} = \{\_\_\_\_\_\_\_\_\_\_\}$
(17) $\{17, 23, 47\} \cup \{32, 47, 71\} = \{\_\_, \_\_, \_\_, \_\_, \_\_\}$
(18) $\{(1, 1), (2, 2)\} \cup \{(2, 1), (1, 1)\} = \{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\}$

3. Intersection of Sets

Another way of combining sets is called "intersection," for which the symbol $\cap$ is most commonly used.

(1) $\{R, S, T\} \cap \{T, O\} = \{T\}$
What letters occur both in the set with members R, S, and T and in the set with members T and O? Answer: Only the letter T.
(2) $\{a, b, c, d\} \cap \{c, a, e\} = \{a, c\}$
(3) $\{5, 7, 3\} \cap \{57, 73, 7\} = \{7\}$
(4) $\{a, b, d\} \cap \{b, e, a, d\} = \{a, b, d\}$ (or $\{b, a, d\}$, etc.)
(5) $\{3, 6, 9, 12, 18\} \cap \{4, 8, 12, 16\} = \{12\}$
(6) $\{(0, 0), (1, 5), (2, 10)\} \cap \{(5, 1), (1, 5)\} = \{(1, 5)\}$
(7) {Bill, Sam, Tom} $\cap$ {Tom, Sam, Ed} = {Sam, Tom}
(8) {Bill, Dick, Tom} $\cap$ {Carl, Dick} = {________}
(9) $\{(7, 1), (5, 3), (0, 8)\} \cap \{(1, 7), (0, 8)\} = \{(\ ,\ )\}$
(10) $\{8, 4, 9, 1\} \cap \{4, 1, 9, 8\} = \{\_\_\_\_\_\_\_\_\}$
(11) $\{(28, 3), (23, 8)\} \cap \{(23, 8), (28, 3)\} = \{\_\_\_\_\_\_\_\_\_\_\}$
(12) $\{(0, 1), (1, 0), (1, 2)\} \cap \{(2, 1), (1, 0), (0, 1)\} =$
$\{\_\_\_\_\_\_\_\_\_\_\}$
(13) $\{0, 2, 4, 6, 8, 10, 12\} \cap \{0, 3, 6, 9, 12, 15\} = \{\_\_\_\_\_\_\_\_\_\_\}$
(14) $\{1, 3, 5, 7, 9, 11\} \cap \{2, 4, 6, 8, 10, 12\} = \{\_\_\_\_\_\_\_\_\_\_\}$
In example 14 there are no members common to both sets, so the intersection in this case is called "the empty set."

4. The Empty Set

While this expression sounds a bit strange at first, the idea is very simple, as you can see by referring to example 14 above. The symbol used to indicate the empty set is $\varnothing$. So we might say: the empty set = $\{\quad\} = \varnothing$.

(1) $(A, R, T) \cap (B, C) = \{\quad\} = \varnothing$
Which letters occur both in the set with members A, R, and T and in the set with members B and C? Answer: none. So we say that the intersection is the empty set.
(2) $\{5, 7\} \cap \{0, 6, 12\} = \varnothing$
(3) $\{a, b\} \cup \{c\} = \{a, b, c\}$
(4) $\{a, b\} \cap \{c\} = \varnothing$
(5) $\{3, 7, 11\} \cap \{\quad\} = \varnothing$
Which numbers are both in the set with members 3, 7, and 11 and the empty set? Answer: none. Since the symbol {   }—braces with nothing listed between them—might be confusing, the empty

set is usually indicated by the symbol $\varnothing$. So $\{\quad\} \cap \{\quad\} = \{\quad\}$ would be written: $\varnothing \cap \varnothing = \varnothing$.

(6) $\varnothing \cap \{1, 2, 3\} =$

(7) $\varnothing \cup \{1, 2, 3\} = \{1, 2, 3\}$

(8) $\{(0, 8), (9, 1)\} \cap \{(9, 8)\} =$

(9) {Alice, Mary} $\cap$ {Bill} =

(10) $\{4, 8, 12\} \cup \{0, 5, 10\} = \{\_\_, \_\_, \_\_, \_\_, \_\_, \_\_\}$

(11) $\{4, 8, 12\} \cap \{0, 5, 10\} =$

5. Number of a Set

We are often concerned with the number of members in a set. Since we are referring to numbers, we can add and subtract. We write a capital N before the name of the set to indicate that we are concerned only with the number of members in the set.

(1) $N\,\{1, 4, 17\} = 3$

(2) $N\,\{a, b, x, y, z\} = 5$

(3) $N\,\{a, b, c\} + N\,\{x, y, z\} = 3 + 3 = 6$

(4) $N\,\{a, b, c\} + N\,\{a, b, c\} = 6$

(5) $\{a, b, c\} \cup \{a, b, c\} = \{a, b, c\}$

(6) $N\,\{4, 6, 8, 9\} + N\,\{6, 8, 11\} = 7$

(7) $\{4, 6, 8, 9\} \cup \{6, 8, 11\} = \{4, 6, 8, 9, 11\}$

(8) $\{4, 6, 8, 9\} \cap \{6, 8, 11\} = \{6, 8\}$

(9) $N\,\{x, y, z, a\} - N\,\{1, 2, 4\} = 1$

(10) $N\,\{0, 1, 2, 3\} + N\,\{0\} = 5$

(11) $N\,\{\quad\} = N\,\varnothing = 0$

6. Solution Sets for Mathematical Sentences

In the following examples let us agree to limit ourselves to the counting numbers and zero. Thus, for the open sentence:

$$\triangle + \square = 2$$

there are only three pairs of numbers that could be used to convert it into a true sentence:

(0, 2) . . . . 0 + 2 = 2
(1, 1) . . . . 1 + 1 = 2
(2, 0) . . . . 2 + 0 = 2

The *solution set* of this open sentence is the set whose members are (0, 2), (1, 1), and (2, 0).

In this case, where pairs of numbers are used, we must agree that the first member of the pair is associated with the first placeholder in the mathematical sentence, and that the second member is associated with the second placeholder. Such a pair is called an *ordered pair*. Since addition is commu-

tative, this seems rather trivial, but for the operations that are not commutative this idea becomes very important.

Here are additional examples of solution sets.

Examples	Open Sentence	Solution Set
(1)	$\triangle + \square = 2$	$\{(0,2),(1,1),(2,0)\}$
(2)	$\square \times 1 = \square$	$\{0, 1, 2, 3, 4, \ldots\}$
	because $\underline{0} \times 1 = \underline{0}$	$3 \times 1 = 3$
	$\underline{1} \times 1 = \underline{1}$	$4 \times 1 = 4$
	$2 \times 1 = 2$	etc.
(3)	$\triangle - \triangle = 0$	$\{0, 1, 2, 3, 4, \ldots\}$
(4)	$\square + \square = \square \times \square$	$\{0, 2\}$
	because $0 + 0 = 0 \times 0$	
	$2 + 2 = 2 \times 2$	

There are just two numbers in the solution set for example 4 because 0 and 2 are the only numbers whose doubles are the same as their squares.

(5)	$\square - \triangle = 6$	$\{(6,0),(7,1),(8,2),\ldots\}$
	because $6 - 0 = 6$	
	$7 - 1 = 6$	
	$8 - 2 = 6$ etc.	
(6)	$\square - 1 = \square$	$\emptyset$

In example 6, the solution set is the empty set because there is no number which is 1 less than itself:

$0 - 1 = 0$ is a false statement
$1 - 1 = 1$ is a false statement
$2 - 1 = 2$ is a false statement
$3 - 1 = 3$ is a false statement
etc.

(7)	$\square + \triangle = \triangle + \square$	All pairs of numbers
(8)	$\square + 9 = 24$	$\{15\}$
(9)	$10 - \triangle = 7$	$\{\_\_\_\_\}$
(10)	$3 \times \square - 1 = 14$	$\{\_\_\_\_\}$
(11)	$\triangle \times \square = 6$	$\{(\ ,\ ),(\ ,\ ),(\ ,\ ),(\ ,\ )\}$

7. Intersection of Solution Sets

Examples:			Intersection of a pair of solution sets
(1)	$\square - 3 = \triangle$	$\{(3,0),(4,1),(5,2),\ldots\}$	$\{(4,1)\}$
	$\square \times \triangle = 4$	$\{(1,4),(2,2),(4,1)\}$	
(2)	$\square + \triangle = 3$	$\{(0,3),(1,2),(\ ,\ ),(\ ,\ )\}$	$\{(\ ,\ ),(\ ,\ )\}$
	$\square \times \triangle = 2$	$\{(1,2),(\ ,\ )\}$	
(3)	$2 \times \square = \triangle$	$\{(0,0),(1,2),(2,4),\ldots\}$	$\{(1,2)\}$
	$\square + \triangle = 3$	$\{(3,0),(2,1),(1,2),(0,3)\}$	
(4)	$\square \times \triangle - 6 = 0$	$\{(1,6),(2,3),(\ ,\ ),(\ ,\ )\}$	$\{(2,3)\}$
	$\square = \triangle - 1$	$\{(1,2),(2,3),(3,4),(\ ,\ ),\ldots\}$	
(5)	$\square + \square = \triangle$	$\{(0,0),(1,2),(2,4),\ldots\}$	$\{(0,0)\}$
	$\square = \triangle + \triangle$	$\{(0,0),(2,1),(4,2),\ldots\}$	
(6)	$\square - \triangle = 2 \times \triangle$	$\{(0,0),(6,2),(\ ,\ ),\ldots\}$	$\{(0,0),(6,2),\ldots\}$
	$\square = 3 \times \triangle$	$\{(0,0),(6,2),(\ ,\ ),\ldots\}$	
(7)	$\square + 1 = \triangle$	$\{(0,1),(1,2),(\ ,\ ),\ldots\}$	$\emptyset$
	$\square \times \triangle = 5$	$\{(1,5),(\ ,\ )\}$	

8. Subsets of Sets

Examples:

Consider Set A = {M, N}. There are four subsets of Set A. They are

{M}, {N}, {M, N}, and ∅.

Consider Set B = {Carl}. There are two subsets of Set B. They are

{Carl} and ∅.

Consider Set C = {1, 2, 3}. There are eight subsets of Set C. They are

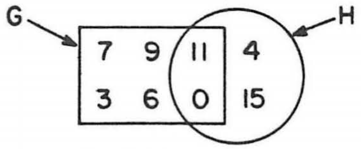
{1}, {2}, {3}, {1, 2}
{1, 3}, {2, 3}, {1, 2, 3}, and ∅.

Were you tempted to say that {N, M} is a subset of Set A? Or that {3, 2, 1} is a subset of Set C? Here order does not concern us, for {M, N} and {N, M} are the same set.

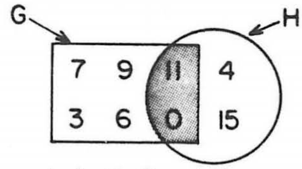
Notice that (a) every set is a subset of itself, and (b) the empty set is a subset of every set.

9. AND and OR

Many of our examples depended on the distinction between the meanings of *and* and *or* as mathematicians use these words. We can explore these ideas in diagrams. Suppose that we are talking about G, the set of numbers written in the rectangle, and H, the set of numbers written in the circle.

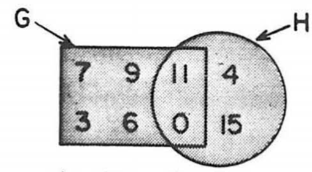


G = (7,9,11,3,6,0)
H = (11,4,0,15)



in both G and H

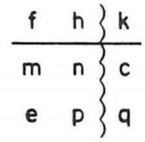
G ∩ H = (11,0)



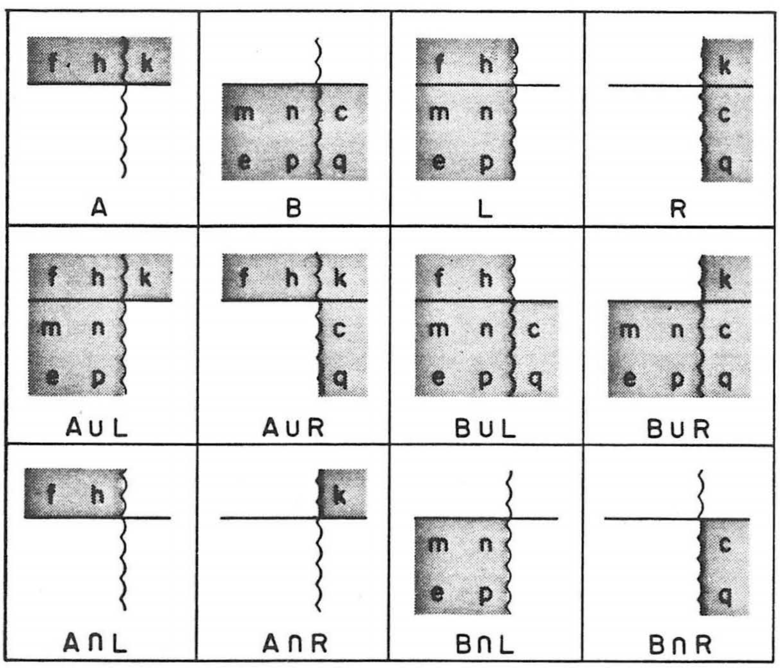
in either G or H

G ∪ H = (7,9,11,4,3,6,0,15)

Let us consider another diagram, involving nine letters.



A = the set of letters above the horizontal line = {f, h, k}
B = the set of letters below the horizontal line = {m, n, c, e, p, q,}
L = the set of letters to the left of the wavy line = {f, h, m, n, e, p}
R = the set of letters to the right of the wavy line = {k, c, q}



A ∪ L = {f, h, k, m, n, e, p}
A ∩ L = {f, h}
A ∪ R = {f, h, k, c, q}
A ∩ R = {k}
B ∪ L = {__________}
B ∩ L = {________}
B ∪ R = {____________}
B ∩ R = {_____}
A ∪ B = {f, h, k, m, n, c, e, p, q}
A ∩ B = ∅
L ∪ R = {______________}
L ∩ R =

This interpretation of *and* and *or* is a mathematical interpretation, but in many ways this usage conforms with conversational English. The idea is disarmingly simple. Once children grasp the concepts indicated by these two words, we have a way to ask many questions about a single example such as the diagram above.

10. AND—OR Games

Let us define *and* and *or* with a series of examples. How many of the following words contain the letter A? How many contain the letter T? How many contain the letter A *and* the letter T? How many contain the letter A *or* the letter T?

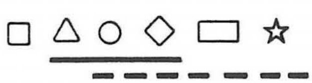
MAN	ATE
HIM	THE
CUT	ASK
BAT	BET

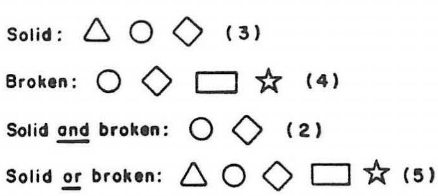
We shall list the groups we are interested in:

A: MAN, BAT, ATE, ASK (4)
T: CUT, BAT, ATE, THE, BET (5)
A *and* T: BAT, ATE (2)
A *or* T: MAN, CUT, BAT, ATE, ASK, THE, BET (7)

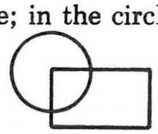
The group that contains A *and* T contains all the words that occur in *both* of the first two lists. The group that contains A *or* T contains all the words that occur at least once in *either* of the first two lists.

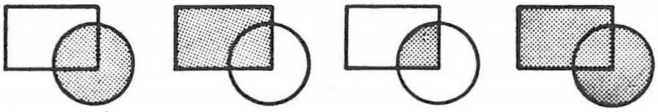
Another example: How many shapes below are underlined with the solid line? How many are underlined with the broken line? How many are underlined with the solid line *and* the broken line? How many are underlined with the solid line *or* the broken line?





Another example: Shade the area that is in the circle; in the rectangle; in the circle *and* in the rectangle; in the circle *or* in the rectangle.





In circle — In rectangle — In circle and rectangle — In circle or rectangle

Another example: List all whole-number multiples of 3 (including 0) that are less than 25. Call it List D. List all whole-number multiples of 2 (including 0) that are less than 25. Call it List F.

List D: 0, 3, 6, 9, 12, 15, 18, 21, 24
List F: 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 24

Now, list all the numbers that are in List D *and* List F:

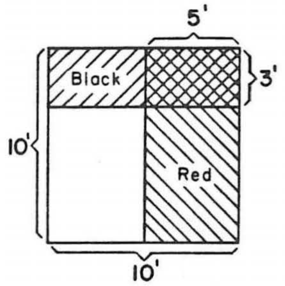
0, 6, 12, 18, 24

List all the numbers that are in List D *or* List F:

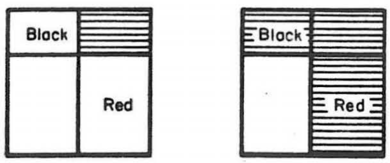
0, 2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24

Another example: List all the odd numbers less than 100. List all the whole-number multiples of 7 (including 0) that are less than 100. How many numbers are in the first list? (Fifty) How many numbers are in the second list? (Fifteen) How many numbers are in the first list *and* the second list? (Seven) How many are in the first list *or* the second list? (Fifty-eight)

Here is another example to help define *and* and *or* as these two words are used in mathematics. Below is a sketch of a floor that is 10 feet by 10 feet. A strip 3 feet wide was painted black along one side. Then a strip 5 feet wide was painted red along an adjacent side.



What is the area of the floor? (100 square feet)
What is the area that was painted black? (30 square feet)
What is the area that was painted red? (50 square feet)
What is the area that was painted black *and* red? (15 square feet)
What is the area that was painted black *or* red? (65 square feet)

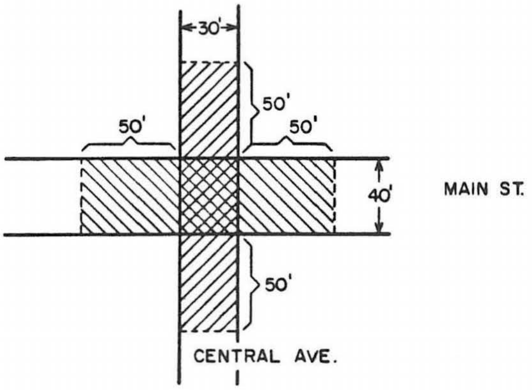


black AND red — black OR red

The word *and* suggests such ideas as "overlap," "both at once"—the *intersection* of areas or groups or lists.

The word *or* suggests such ideas as "in all," "either one or both"—the *union* of areas or groups or lists.

This language suggests another example. Ace Contractors were awarded a contract to replace the paving where Main Street crosses Central Avenue—to a distance of 50 feet from the corner in all four directions. Main Street was 40 feet wide and Central Avenue was 30 feet wide.



How much paving was laid on Main Street? (40×130, or 5,200, square feet)

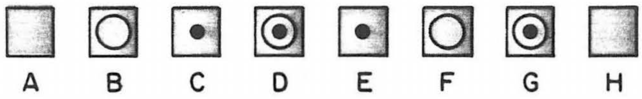
How much was repaved on Central Avenue? (30×140, or 4,200, square feet)

How much paving was replaced in the "intersection"? (30×40, or 1,200, square feet)

How much paving was replaced in all—how much paving is in the "union" of all the area covered by the contract? (8,200 square feet)

Why is the "union," or the total area, not $5{,}200 + 4{,}200$, or 9,400, square feet? (Because they paved the "intersection" only once)

Here is another example. Below are some blue and some green blocks:



How many blue blocks have dots? Obviously, the design labeled C is one that fits the conditions of the question. How about D? We are going to accept it also, because it would answer "yes" when asked, "Do you contain a dot?" even though it also contains a circle.

A further question: How many designs are without a dot *or* without a circle? Suppose that each design is asked a pair of questions: (1) Are you without a dot? (2) Are you without a circle?

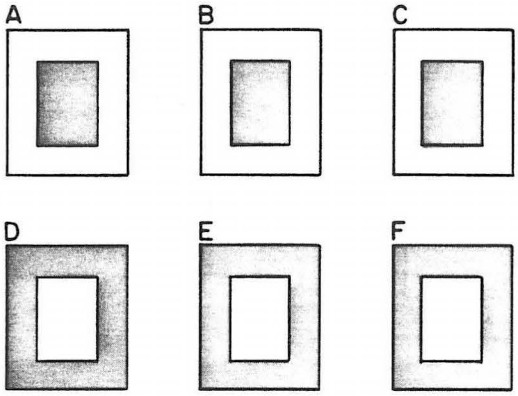
In answer to question 1, how many could answer "yes"? (A, B, F, H)

In answer to question 2, how many could answer "yes"? (A, C, E, H)

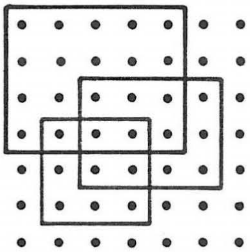
How many could answer "yes" to either one *or* the other (or both)? (A, B, C, E, F, H—six) So, six designs are without dots *or* without circles.

Had the original question been: "How many designs are without dots *and* without circles?" then we would find out how many could answer "yes" to *both* questions 1 and 2. Only two designs (A, H) could answer "yes" to both questions.

Prepare a set of playing cards that look like this:



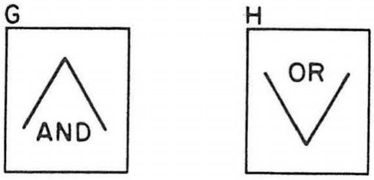
to be used with this playing board (which can be varied):



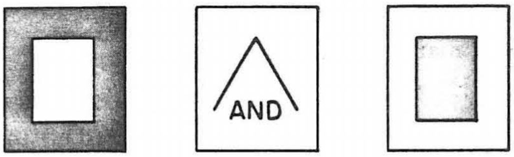
Since the interior of card A is shaded red, it represents the dots enclosed by (*in*) the red on the playing board. Since the exterior of card D is shaded red, it represents the dots *not in* the red on the playing board.

Card A: the number of dots *in* red ($4 \times 5 = 20$)
Card B: the number of dots *in* green ($3 \times 4 = 12$)
Card C: the number of dots *in* blue ($3 \times 3 = 9$)
Card D: the number of dots *not in* red ($49 - 20 = 29$)
Card E: the number of dots *not in* green ($49 - 12 = 37$)
Card F: the number of dots *not in* blue ($49 - 9 = 40$)

Two other cards (preferably several of each) are required:

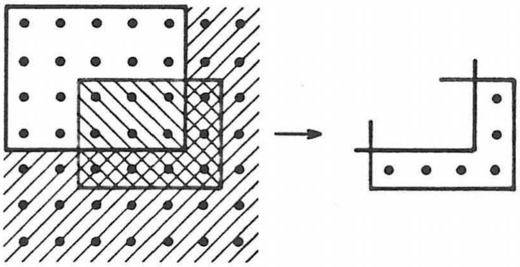


A player selects two colored cards and one of the *and/or* cards. He then places one colored card on each side of the *and* or the *or* card. It might look like this:



Not in Red   AND   in Green.

Now consider the playing board. How many dots could answer "yes" to both of the following questions? (1) Is it true that you are *not in* red? (2) Is it true that you are *in* green?



Only six dots could answer "yes" to both of these questions—and that is the player's score: 6.

Had the center card been *or* instead of *and*, the question would be: How many dots could answer "yes" to either question 1 or question 2? Then the player's score would be 35 $(49-14=35)$—all the dots in any part of the region shaded in the sketch above.

Players take turns and each accumulates a score. The first player to reach a predetermined total score wins the game. There are seventy-two possible combinations that can be summarized as follows:

$\vee$	R	G	B	$\cancel{R}$	$\cancel{G}$	$\cancel{B}$
R	20	26	26	49	42	43
G	26	12	17	35	49	44
B	26	17	9	32	41	49
$\cancel{R}$	49	35	32	29	43	46
$\cancel{G}$	42	49	41	43	37	45
$\cancel{B}$	43	44	49	46	45	40

$\wedge$	R	G	B	$\cancel{R}$	$\cancel{G}$	$\cancel{B}$
R	20	6	3	0	14	17
G	6	12	4	6	0	8
B	3	4	9	6	5	0
$\cancel{R}$	0	6	6	29	23	23
$\cancel{G}$	14	0	5	23	37	32
$\cancel{B}$	17	8	0	23	32	40

R stands for IN red ; $\cancel{R}$ stands for NOT IN red.
etc.

Pupils ought to play the game without these two charts so that each play must be thought out. After a very short time, the ideas of *and* and *or* will become clear.

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